IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 4 Ser. V (Jul.–Aug. 2020), PP 24-28
www.iosrjournals.org

New subclass of harmonic univalent functions defined by derivative operator

N. D. Sangle, G. M. Birajdar, S. A. Morye
Department of Mathematics, Annasaheb Dange College of Engineering & Technology, Ashta. Sangli, (M.S.) India 416301
Department of Mathematics, Shivaji University, Kolhapur (M.S) India 416004
Government College of Arts and Science, Aurangabad,(M.S.) India 431001

Abstract
The purpose of the present paper is to investigate new subclass of harmonic univalent function in the unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$ by using derivative operator. Also, we obtain coefficient inequalities and distortion theorems for this subclass.
2000 Mathematics Subject Classification: *30C45, 30C50*
Keywords: *Harmonic, Univalent, Coefficient inequalities, Derivative operator.*

Date of Submission: 13-08-2020 Date of Acceptance: 29-08-2020

I. Introduction

A continuous complex valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbb{C}$, we can write $f = h + \bar{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f.

A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that

$$|h'(z)| > |g'(z)|, \qquad z \in D$$

Let SH denote the class of functions $f = h + \bar{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f \in S_H$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, \; |b_1| < 1 \; . \qquad (1.1)$$

Clunie and Sheil-Small [2] investigated the class SH as well as its geometric subclasses and established some coefficient bounds. Since then, there have been several related papers on SH and its subclasses. In fact, by introducing new subclasses Avci and Zlotkiewicz [1], Darus and Sangle [3], Dixit and Porwal [4], Jahangiri [6], Silverman [9], Silverman and Silvia [10], Yalcin et al.[12], etc. presented a systematic and unified study of harmonic univalent functions. Furthermore we refer to Duren [5], Ponnusamy and Rasila [8] and references therein for basic results on the subjects.

For $f = h + \bar{g}$ given by (1.1), Al-Shaqsi and Darus [7] introduced the operator D_λ^n as:

$$D_\lambda^n f(z) = D_\lambda^n h(z) + (-1)^n \overline{D_\lambda^n g(z)}, (n, \; \lambda \in N_0 = N \cup \{0\}, \; z \in U) \qquad (1.2)$$

where $D_\lambda^n h(z) = z + \sum_{k=2}^{\infty} k^n C(\lambda, k) a_k z^k$, $D_\lambda^n g(z) = \sum_{k=1}^{\infty} k^n C(\lambda, k) b_k z^k$ and $C(\lambda, k) = \binom{k+\lambda-1}{\lambda}$.

Now for $0 \le \alpha < 1, n \in N_0$ and $z \in U$, suppose that $S_H(n, \alpha, \lambda)$ denote the family of harmonic univalent functions f of the form (1.1) such that

$$Re\left\{\frac{D_\lambda^n h(z) + D_\lambda^n g(z)}{z}\right\} > \alpha \qquad (1.3)$$

where$D_\lambda^n f(z)$ is defined by Al-Shaqsi and Darus M.[8].
Further let the subclass $\overline{S_H}(n,\alpha,\lambda)$ consisting harmonic functions $f=h+\bar{g}$ in $S_H(n,\alpha,\lambda)$so that h and g are of the form

$$h(z)=z-\sum_{k=2}^{\infty}a_kz^k\,,\;\; g(z)=-\sum_{k=1}^{\infty}b_kz^k. \qquad (1.4)$$

II. Main Results

We begin by proving some sharp coefficient inequalities contained in the following theorem.
Theorem 2.1. Let the function $f=h+\bar{g}$ be such that h and g are given by (1.1).
Furthermore

$$\sum_{k=2}^{\infty}k^nC(\lambda,k)|a_k|z^k+\sum_{k=1}^{\infty}k^nC(\lambda,k)|b_k|z^k\le 1-\alpha \qquad (2.1)$$

where $0\le\alpha<1, n,\ \lambda\in N_0$. Then f is harmonic univalent, sense-preserving in U and $f\in S_H(n,\alpha,\lambda)$.

Proof: If $z_1\neq z_2$ then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right|\ge 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty}b_k(z_1^k-z_2^k)}{z_1-z_2+\sum_{k=2}^{\infty}a_k(z_1^k-z_2^k)}\right|$$

$$>1-\frac{\sum_{k=1}^{\infty}|b_k|}{1-\sum_{k=2}^{\infty}|a_k|}$$

$$\ge 1-\frac{\sum_{k=1}^{\infty}\frac{k^nC(\lambda,k)}{(1-\alpha)}|b_k|}{1-\sum_{k=2}^{\infty}\frac{k^nC(\lambda,k)}{(1-\alpha)}|a_k|}$$

$$\ge 0.$$

Hence f is univalent in U.

$f$is sense preserving in U. This is because

$$|h'(z)|\ge 1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1}$$

$$>1-\sum_{k=2}^{\infty}k|a_k|$$

$$\ge 1-\sum_{k=2}^{\infty}\frac{k^nC(\lambda,k)}{(1-\alpha)}|a_k|$$

$$\ge\sum_{k=1}^{\infty}\frac{k^nC(\lambda,k)}{(1-\alpha)}|b_k|$$

$$\ge\sum_{k=1}^{\infty}k|b_k||z|^{k-1}$$

$$\ge |g'(z)|.$$

Now, we show that $f\in S_H(n,\alpha,\lambda)$. Using the fact that $Re(w)\ge\alpha$ if and only if $|1-\alpha+w|>|1+\alpha-w|$.

It suffices to show that,

$$\left|(1-\alpha)+\frac{D_\lambda^n h(z)+D_\lambda^n g(z)}{z}\right|-\left|(1+\alpha)-\frac{D_\lambda^n h(z)+D_\lambda^n g(z)}{z}\right|>0 \qquad (2.2)$$

Substituting for $D_\lambda^n\ h(z)$ and $D_\lambda^n\ g(z)$in (2.2), we have

$$= \left|(2-\alpha) + \sum_{k=2}^{\infty} k^n C(\lambda,k) a_k z^{k-1} + \sum_{k=1}^{\infty} k^n C(\lambda,k) b_k z^{k-1}\right| - \left|\alpha - \sum_{k=2}^{\infty} k^n C(\lambda,k) a_k z^{k-1} - \sum_{k=1}^{\infty} k^n C(\lambda,k) b_k z^{k-1}\right|$$

$$\geq 2(1-\alpha)\left\{1 - \sum_{k=2}^{\infty} \frac{k^n C(\lambda,k)}{(1-\alpha)} |a_k||z|^{k-1} - \sum_{k=1}^{\infty} \frac{k^n C(\lambda,k)}{(1-\alpha)} |b_k||z|^{k-1}\right\}$$

$$\geq 2(1-\alpha)\left\{1 - \sum_{k=2}^{\infty} \frac{k^n C(\lambda,k)}{(1-\alpha)} |a_k| - \sum_{k=1}^{\infty} \frac{k^n C(\lambda,k)}{(1-\alpha)} |b_k|\right\} \geq 0.$$

The harmonic mappings

$$f(z) = z + \sum_{k=2}^{\infty} \frac{(1-\alpha)}{k^n C(\lambda,k)} x_k z^k + \sum_{k=1}^{\infty} \frac{(1-\alpha)}{k^n C(\lambda,k)} \overline{y_k z^k}$$

Where $\sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1$, show that coefficient bound given by (2.1) is sharp.

In the following theorem, it is proved that the condition (2.1) is also necessary for functions $f = h + \bar{g}$where h and g are of the form (1.4).

Theorem 2.2. Let $f = h + \bar{g}$ be given by (1.4). Then $f \in \overline{S_H}(n,\alpha,\lambda)$ if and only if

$$\sum_{k=2}^{\infty} \frac{k^n C(\lambda,k)}{(1-\alpha)} |a_k| + \sum_{k=1}^{\infty} \frac{k^n C(\lambda,k)}{(1-\alpha)} |b_k| \leq 1 \qquad (2.3)$$

where $0 \leq \alpha < 1, n \in N_0$.

Proof: The if part follows from Theorem 2.1. For the only if part, we show that $f \in \overline{S_H}(n,\alpha,\lambda)$ if the condition (2.3) holds, we notice that the condition

$$Re\left\{\frac{D_\lambda^n\ h(z) + D_\lambda^n\ g(z)}{z}\right\} > \alpha$$

is equivalent to

$$Re\{1 - \sum_{k=2}^{\infty} k^n C(\lambda,k)|a_k|z^{k-1} + \sum_{k=1}^{\infty} k^n C(\lambda,k)|b_k|z^{k-1}\} > \alpha\ .$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, we must have

$$1 - \sum_{k=2}^{\infty} k^n C(\lambda,k)|a_k| + \sum_{k=1}^{\infty} k^n C(\lambda,k)|b_k| \geq \alpha$$

which is precisely the assertion (2.3).
Next, we determine the extreme points of closed convex hulls of $\overline{S_H}(n,\alpha,\lambda)$ denoted by clco$\overline{S_H}(n,\alpha,\lambda)$.

Theorem 2.3: Let f be given by (1.4). Then $f \in \overline{S_H}(n,\alpha,\lambda)$ if and only if

$$f(z) = \sum_{k=1}^{\infty} (x_k h_k(z) + y_k g_k(z))$$

where $h_1(z) = z$,

$h_k(z) = z - \frac{(1-\alpha)}{k^n C(\lambda,k)} z^k, (k = 2,3,4,\dots)$and $g_k(z) = z - \frac{(1-\alpha)}{k^n C(\lambda,k)} \bar{z}^k, (k = 1,2,3,4,\dots),$

$x_k \geq 0, y_k \geq 0,\ \sum_{k=1}^{\infty}(x_k + y_k) = 1$.

In particular the extreme points of $\overline{S_H}(n,\alpha,\lambda)$ are $\{h_k\}$and $\{g_k\}$.

The following theorem gives the bounds for functions in $\overline{S_H}(n,\alpha,\lambda)$which yields a covering result for this class.

Theorem 2.4:Let $f \in \overline{S_H}(n,\alpha,\lambda)$.Then for $0 \leq |z| = r < 1$, we have

$$|f(z)| \leq (1 + |b_1|r) + \frac{1}{2^n(1+\lambda)}(1 - |b_1| - \alpha)r^2, \qquad |z| = r < 1$$

and

$$|f(z)| \geq (1 - |b_1|r) - \frac{1}{2^n(1+\lambda)}(1 - |b_1| - \alpha)r^2, \qquad |z| = r < 1$$

Proof: Let $f \in \overline{S_H}(n,\alpha,\lambda)$. Taking the absolute value of f(z) ,we have

$$\begin{aligned}|f(z)| &\leq (1 + |b_1|r) + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \\ &\leq (1 + |b_1|r) + \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2 \\ &\leq (1 + |b_1|r) + \frac{1}{2^n(1+\lambda)}\sum_{k=2}^{\infty} k^n C(\lambda,k)(|a_k| + |b_k|)r^2 \\ &\leq (1 + |b_1|r) + \frac{1}{2^n(1+\lambda)}(1 - |b_1| - \alpha)r^2\end{aligned}$$

and

$$\begin{aligned}|f(z)| &\geq (1 - |b_1|r) - \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \\ &\geq (1 - |b_1|r) - \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^2 \\ &\geq (1 - |b_1|r) - \frac{1}{2^n(1+\lambda)}\sum_{k=2}^{\infty} k^n C(\lambda,k)(|a_k| + |b_k|)r^2 \\ &\geq (1 - |b_1|r) - \frac{1}{2^n(1+\lambda)}(1 - |b_1| - \alpha)r^2\end{aligned}$$

The functions $z + |b_1|\bar{z} + \frac{1}{2^n(1+\lambda)}(1 - |b_1| - \alpha)\bar{z}^2$ and $z - |b_1|z - \frac{1}{2^n(1+\lambda)}(1 - |b_1| - \alpha)z^2$ for $|b_1| \leq (1-\alpha)$ show that the bounds given in the Theorem 2.4 are sharp.

References:

[1]. Avci Y. and Zlotkiewicz E., (1990), On harmonic univalent mappings, Ann. Univ. Mariae Curie- Sklodowska Sect.A 44, 1-7.

[2]. Clunie J. and Sheil-Small, T. (1984), Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser.AI Math., 9, 3-25.

[3]. Darus M., Sangle N.D. (2011), On certain class of harmonic univalent functions defined by generalized derivative operator, Int. J. Open Problems Compt. Math., Vol.4, No. 2,83-96.

[4]. Dixit K.K. and Porwal Saurabh (2010), A subclass of harmonic univalent functions with positive coefficients, Tamkang J. Math., 41(3), 261-269.

[5]. Duren P.(2004), Harmonic Mappings in the Plane, Cambridge Tracts in Mathematics, Vol.156, Cambridge University Press, Cambridg.

[6]. Jahangiri J.M.(1999), Harmonic functions starlike in the unit disc, J. Math. Anal. Appl., 235, 470-477.

[7]. Al-Shaqsi, K. and Darus, M. (2008), An operator defined by convolution involving the polylogarithms functions, Journal of Math. and Stat. 4 (1), 46{50.

[8]. Ponnusamy S.and Rasila A. (2007), Planar harmonic and quasi-conformal mappings, Mathematics Newsletters, 17(3), 85-101.

[9]. Silverman H. (1998), Harmonic univalent function with negative coefficients, J.Math. Anal. Appl., 220, 283-289.

[10]. Silverman H. and Silvia E.M. (1999), Subclasses of harmonic univalent functions, New Zealand J. Math., 28, 275-284.

[11]. Salagean G.S. (1983), Subclasses of univalent functions, Complex Analysis-Fifth Romanian Finish Seminar, Bucharest, 1(1983) 362-372.

[12]. Yalcin S., Ozturk M. and Yamankaradeniz M. (2003), A subclass of harmonic univalent functions with negative coefficients, Appl. Math., Comput., 142 , 469-476.

N. D. Sangle, et. al. "New subclass of harmonic univalent functions defined by derivative operator." *IOSR Journal of Mathematics (IOSR-JM)*, 16(4), (2020): pp. 24-28.