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Some Identities Satisfied By Ternary Semirings

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Abstract*: In this paper we study about the ternary semirings satisfying some identities and introduce the concept of IMP in ternary semirings. By using these identities we characterize various properties of ternary semirings.*

Keywords: *Ternary Semiring, Singular, Band, Rectangular band, Zeroid, Mono-ternarysemiring, IMP, Periodic, Quasi-seperative, Weakly –seperative.*

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I. Introduction:

A Ternary semiring is an algebraic system with operations addition and ternary multiplication. The concept of ternary algebraic system was invested by D.H.Lehmer [2].After that he investigated certain algebraic systems which turn out to be ternary groups. The notion of ternary semigroups was studied by S.Banach He Showed an example that a ternary semigroup does not necessarily reduce an ordinary semigroup. S.Kar [1,3]and T.K.Dutta [1,3]introduced and studied some properties of ternary semirings which is generalization of ternary rings. Our main purpose of this paper is to study about the notion of some concepts of ternary semirings.

II. Prelimanaries:

In this section we study about the preliminary definitions regarding binary operation in semigroups and ternary multiplication regarding to ternary semigroups.

Definition2.1: A non empty set T together with a binary operation called addition and a ternary multiplication denoted by [ ] is called ternary semiring if T is an additive commutative semigroup satisfying the following conditions.

(i) [[abc]de] =[a[bcd]e]=[ab[cde]]
(ii) [(a+b)cd]=[acd]+ [bcd]
(iii) [a(b + c)d]=[abd]+[acd]
(iv) [ab(c+d)] = [abc] +[abd] all a,b,c,d ,e in T.

Note :for our convenience we write abc instead of [abc]
Note: Any semiring can be reduced to ternary semiring.

Examples:
(i)Let T be an semigroup of all mxn matrices over the set of all non negative rational Examples: numbers.Then T is ternary semiring with matrix multiplication as ternary operation.
(ii)The set T consist of a single element '0' with binary operation defined by 0 + 0 = 0 and ternary operation 0.0.0 =0 is ternary semiring.This ternary semiring is called null ternary semiring (or) Zero ternary semiring.
(iv) The set T={0,1,2,3,4} is a ternary semiring with respect to addition modulo 5 and multiplication modulo 5 as ternary operation defined below.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	2	2
4	4	0	1	2	3

X_5	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Definition2.2:A ternary semigroup is called commutative

if abc=acb=bac=bca=cab=cba for all a,b,c in T.

Definition 2.3: An element x in a ternary semigroup (T,.) is called
- (i) Left cancellative if abx=aby implies x = y for all a,b, x,y in T.
- (ii) Lateral cancellative if axb =ayb implies x=y for all a,b, x,y in T.
- (iii) Right cancellative if xab =yab implies x=y for all a,b, x,y in T

Definition 2.4.: A ternary semigroup (T,.) is called two sided cancellative if it is both right and left cancellative.
Definition2.5:A ternary semigroup (T,.) is called cancellative if it is left,right and lateral cancellative.
Definition 2.6: An element 'a' of semiring T is said to be additive idempotent element provided a + a =a
Definition 2.7: An element 'a' ternary semiring T is said to be multiplicatively idempotent if $a^3 = a$
Definition2.8: A ternary semigroup (T,.) is band if every element in T is an idempotent.
Definition 2.9:A commutative band is called semilattice.
Definition 2.10:A semigroup (S,.) is said to be rectangular band if a b a = a for all a,b in S.
Definition 2.11:A ternary semigroup (T,.) is called rectangular band if ababa=a for all a,b in T.
Definition 2.12:A semigroup (S,.) is called weakly seperative if $a^2 = a\, b = b^2$ implies a = b for all a,b in S.
Definition 2.13: A ternary semigroup (T,.) is called weakly seperative if $a^3 = aba = b^3$ implies a = b for all a,b in T.
Definition 2.14: A ternary semigroup (T,.) is called quasi -seperative if $a^3 = aba = bab = b^3$ implies that a = b for all a,b in T.
Definition 2.15: A semigroup (S,.) is said to be left (right) singular if a b = a (a b = b) for all a,b in S.
Definition 2.16: A ternary semigroup (T,.) is said to be left singular if it satisfies the identity $ab^2 = a$ for all a,b in T.
Definition 2.17: A ternary semigroup (T,.) is said to be lateral singular if it satisfies the identity bab = a for all a,b in T.
Definition 2.18: A ternary semigroup (T,.) is said to be right singular if it satisfies the identity $b^2a = a$ for all a,b in T.
Definition 2.19: A ternary semiring (T,.) is said to be two sided singular, if it is both left and right singular.
Definition 2.20:A ternary semiring (T,.) is said to be singular if it is left, lateral and right singular.
Definition 2.21: A ternary semiring (T,+,.) is said to be mono-ternary semiring if $a + b = ab^2$ (or) $a + b = a^2b$ for all a,b in T.
Definition 2.22: The set Z of ternary semiring T is said to be zeroid of T provided Z= {a $\in$ T:a + b = b (or) b + a =b for some b$\in$T}.

III. Ternary semiring satisfying the identities:

This section contains results on Ternary semiring satisfying the some identities by using the properties like singular, band, rectangular band quasi -seperative etc.
Theorem 3.1: Suppose (T,+,.) is a ternary semiring satisfying the condition a +aba =b for all a,b in T in which multiplicative identity is also an additive Identity .Then (T,+) is singular.
Proof: As given (T,+,.) is a ternary semiring which satisfying the condition a + aba = b for all a,b in T .Let 'e' be the multiplicative identity which is also an additive identity.
To prove (T,+) is singular,
Consider a +aba = b
$\Rightarrow$ a +aba +a = b + a
$\Rightarrow$ a +a (ba+ee) = b + a
$\Rightarrow$ a + aba = b + a
$\Rightarrow$ b = b + a
Therefore (T,+) is Left singular Ternary semigroup
Similarly a +aba =b $\Rightarrow$ a + a + aba =a +b
$\Rightarrow$ a +a (ee+ba) =a + b
$\Rightarrow$ a + aba = a + b
$\Rightarrow$ b = a+b
Therefore (T,+) is Right Singular ternary semigroup.
Hence (T, +) is Singular ternary semigroup.
Theorem 3.2: Assume that (T,+,.) is a ternary semiring satisfying the condition a +aba =b for all a,b in T in which multiplicative identity which is also an additive Identity .If (T,.) is singular ternary semigroup then (T, +) is semilattice.
Proof: Given that (T,+,.) is a ternary semiring satisfying the condition a +aba =b for all a,b in T.Let 'e' be the multiplicative identity which is also an additive identity and also given (T,.) is singular ternary semigroup

To prove (T, +) is semilattice .i.e. (T,+) is commutative band
Consider $a + aba = b$
$\Rightarrow a + aba + a = b + a$
$\Rightarrow a + a(ba + ee) = b + a$
$\Rightarrow a + aba = b + a$
$\Rightarrow a + b = b + a$
Therefore (T,+) is commutative
Again Consider $a + aba = b$
$\Rightarrow a + aba + b = b + b$
$\Rightarrow a + (aea + ee)b = b + b$
$\Rightarrow a + aab = b + b$
$\Rightarrow a + a^2b = b + b$
$\Rightarrow a + b = b + b$
$\Rightarrow a + aba = b + b$
$\Rightarrow b = b + b$ for all b in T
Therefore (T,+) is band
Hence (T,+) is semilattice.

Theorem3.3: Let (T,+,.) be a ternary semiring satisfying the condition $a + aba = b$ for all a,b in T. If T contains multiplicative identity which is also an additive Identity .Then (T,.) is quasi seperative.
Proof: As given (T,+,.) is a ternary semiring which satisfying the condition $a + aba = b$ for all a,b in T .Let 'e' be the multiplicative identity which is also an additive identity.To prove (T,.) is quasi seperative .i.e.
For $a^3 = aba = bab = b^3 \Rightarrow a = b$ for all a,b in T
Let $a^3 = bab$
$= b(ab + ee)$
$= bab + bee$
$= bab + b$
$a^3 = a$
Similarly $b^3 = aba$
$= a(ee + ba)$
$= aee + aba$
$= a + aba$
$b^3 = b$
$\Rightarrow a^3 = a$ and $b^3 = b$
If $a^3 = aba = bab = b^3 \Rightarrow a = aba = bab = b$
$\Rightarrow a = b$
Hence (T,.) is quasi seperative.

Theorem 3.4: Suppose (T,+,.) is a ternary semiring satisfying the condition $a + aba = b$ for all a,b in T.If (T,.) is semilattice then(T,.) is singular.
Proof: Given that (T,+,.) is a ternary semiring satisfying the condition
$a + aba = b$ for all a,b in T and (T,.) is semilattice
To prove (T,.) singular
Consider $a + aba = b$
$\Rightarrow (a + aba)a^2 = ba^2$
$\Rightarrow a^3 + aba^3 = ba^2$
$\Rightarrow a + aba = ba^2$
$\Rightarrow b = ba^2$
Therefore (T,.) is left Singular.
Similarly $a + aba = b$
$\Rightarrow a^2(a + aba) = a^2b$
$\Rightarrow a^3 + a^3ba = a^2b$
$\Rightarrow a + aba = a^2b$
$\Rightarrow b = a^2b$
(T,.) is right Singular.
Again consider $a + aba = b$
$\Rightarrow a(a + aba)a = aba$
$\Rightarrow a^3 + a^2ba = aba$

$\Rightarrow a + a^2baa = aba$
$\Rightarrow a + a^2aba = aba$
$\Rightarrow a + a^3 ba = aba$
$\Rightarrow a + aba = aba$
$b = aba$
Therefore (T,.) is lateral Singular.
Hence (T,.) is singular ternary semigroup

Theorem 3.5: Assume that (T,+,.) is a ternary semiring in which multiplicative identity is also an additive identity. The necessary and sufficient is that (T,.) is right singular if and only if $a + b^2 a=a$ for all a, b in T
Proof: Given that (T,+,.) is a ternary semiring. Let 'e' be the multiplicative identity which is also an additive identity.
Suppose T satisfies the identity $a + b^2 a=a$ for all a,b in T.
We have to prove (T,.) is right singular
For this consider $a + b^2 a = a$
$\Rightarrow (ee + b^2)a = a$
$\Rightarrow b^2a = a$
$\Rightarrow$ (T,.) is right singular.
Conversely assume that (T,.) is right singular we have to prove (T,+,.) satisfies the condition $a + b^2 a=a$ for all a,b in T
Consider $a + b^2 a = (ee + b^2)a$
$= b^2a$
$a + b^2a = a$
Hence this proves the necessary and sufficient condition.

Theorem 3.6.: Suppose (T,+,.) is a ternary semiring satisfying the condition $a + b^2a=a$ for all a,b in T in which multiplicative identity is also an additive identity. If (T,.) is band and (T,+) is singular then (T,+) is weakly seperative.
Proof: Given that (T,+,.) be a ternary semiring satisfying the identity $a + b^2 a=a$ for all a,b in T. Let 'e' be the multiplicative identity which is also additive identity .Also given (T,.) is band and (T,+) is singular.
To prove (T,+) is weakly seperative .i.e. $a +a = a+b = b+b \Rightarrow a=b$ for all a,b in T
Consider $a + a = a + b$
$\Rightarrow b^2 (a + a) = b^2 (a + b)$
$\Rightarrow b^2 a+ b^2a = b^2a + b^3$
$\Rightarrow a + b^2a = b^2a + b^3$
(By theorem 3.5 (T,.) is right singular)
$\Rightarrow a = b^2(a + b)$
$\Rightarrow a = b^2(b)$
$\Rightarrow a = b^3$
$\Rightarrow a = b$
Similarly $b + b = a + b$
$\Rightarrow a^2(b + b)= a^2(a + b)$
$\Rightarrow a^2b + a^2b = a^2(a)$
$\Rightarrow b + a^2 b = a^3$
(By theorem 3.6 (T,.) is right singular)
$\Rightarrow (ee +a^2) b =a^3$
$\Rightarrow a^2b = a$
$\Rightarrow b = a$
Therefore $a + a =a +b =b + b \Rightarrow a = b$
Hence (T, +) is weakly seperative.
Definition 3.7: A ternary semigroup (T,.) is said to be antiregular if ababa = b for all a, b in T
Theorem 3.8: Suppose (T,+,.) is an Antiregular ternary semiring in which (T,.) is singular. Then (T,.) is band
Proof: Given that (T,+,.) is an Antiregular ternary semiring and (T,.) is singular .
To prove (T,.) is band
Consider $a^3= a.a.a$
$= a.a.(babab)$
$= a(ababa)b$

= a(b)b

= ab^2

a^3 = a (Since (T,.) is singular)

Therefore (T,.) is band.

Theorem 3.9: Suppose (T,+,.) is a commutative ternary semiring satisfying the condition bab + a = a for all a,b in T .If T contains a multiplicative identity which is also additive identity. Then (T,+,.) is idempotent ternary semiring.

Proof: As given (T, +,.) is a commutative ternary semiring satisfying the condition bab + a =a for all a,b in T. Let 'e' be the multiplicative identity which is also additive identity.To show (T,+,.) is Idempotent Ternary semiring.

Consider bab + a = a

⟹ (beb + ee)a = a

⟹ (bb)a = a

⟹ bab = a for all a,b in T

⟹ bab + a = a

⟹ a + a = a for all a in T

Therefore (T,+) is a band

Now bab = a for all a,b in T

⟹ (T,.) is lateral singular

a^3= a. a.a

= a(bab)a

= (aba)ba

= b.ba (Since (T,.) is lateral singular)

= bab (Since T is commutative)

a^3= a for all a in T

Therefore (T,.) is band.

Hence (T,+,.) is Idempotent ternary semiring.

Theorem 3.10: If (T,+,.) is a commutative ternary semiring contains multiplicative identity which is also an additive identity then the necessary and sufficient condition is that (T,.) is singular if and only if T satisfies bab + a=a for all a,b in T.

Proof: Given that (T,+,.) is a commutative ternary semiring. Let 'e' be the multiplicative identity which is also additive identity.

Suppose T satisfies the condition bab + a =a for all a,b in T .To prove (T,.) is singular,

Consider bab + a = a

⟹ (beb +ee)a = a

⟹ (bb)a = a

⟹ b^2a = a

⟹ (T,.) is right singular

Again Consider bab + a = a

⟹ abb + a = a

⟹a(bb +ee) = a

⟹ abb = a

⟹ ab^2= a

Therefore (T,.) is left singular

Similarly bab + a= a

⟹ (beb + ee)a = a

⟹ bba = a

⟹ bab = a

Therefore (T,.) is lateral singular

Hence (T,.) is singular ternary semigroup.

Conversely assume that (T,.) is singular ternary semigroup ,we have to prove T satisfies the condition bab + a = a for all a, b in T.

For this let bab + a = (beb + ee) a

= bba

= b^2a

bab + a = a(Since (T,.) is singular)

Hence the necessary and sufficient condition is proved

Theorem 3.11: Assume that (T,+,.) is a mono- ternary semiring satisfying the condition bab + a = a for all a, b in T. If (T,.) is commutative then (T,+) is rectangular band.

Proof: Given that (T,+,.) is a mono- ternary semiring satisfying the condition

bab + a = a for all a,b in T .i.e $a + b = a^2b$ (or) $a + b = b^2a$ for all a,b in T

To prove (T,+) is rectangular band

Consider $a + b + a = b^2a + a$

$= bba + a$

$= bab + a$

$= a$

$\Rightarrow a + b + a = a$ for all a,b in T

Hence (T,+,.) is rectangular band .

IV. Ternary Semiring Satisfying Imp Property:

In this section we discuss about ternary semirings which satisfy the Integral Multiple Property (IMP).T.vasanthi [7] studied the structure of semirings and ordered semirings with IMP. In this section we study the properties of ternary semirings with IMP

Definition 4.1: A Ternary semiring (T,+,.) is said to satisfy the Integral multiple property (IMP) if $a^3 = na$ for all a in T where n is the positive integer which depends on a.

Definition 4.2: A Ternary semigroup is said to be semi simple if $x \in TxTxTxT$ for every x in T.

Definition 4.3: A ternary semigroup is said to be periodic if every one of its element is periodic.

Example: The following are the examples of ternary semiring with IMP where '+' and '.' Are given by

$$a \oplus b = \begin{cases} a + b & if \quad a + b \leq m + r - 1 \\ m + t & if \quad a + b \geq m + r \end{cases}$$

Where t< r is given by a + b – m ≡ t(mod r)

$$a \otimes b = \begin{cases} abn & if \quad abn \leq m + r - 1 \\ m + t & if \quad abn \geq m + r \end{cases}$$

Where t < r is given by abn – m ≡ t(mod r), where m is the index of the element and r is the period of the element

T={1,2,3,4,5,6,7} m=1 r=7 be the index and period of respective element

Define '+' and '.' On T by the following tables

+	1	2	3	4	5	6	7
1	2	3	4	5	6	7	1
2	3	4	5	6	7	1	2
3	4	5	6	7	1	2	3
4	5	6	7	1	2	3	4
5	6	7	1	2	3	4	5
6	7	1	2	3	4	5	6
7	1	2	3	4	5	6	7

.	1	2	3	4	5	6	7
1	2	4	6	1	3	5	7
2	4	1	5	2	6	3	7
3	6	5	4	3	2	1	7
4	1	2	3	4	5	6	7
5	3	6	2	5	1	4	7
6	5	3	1	6	4	2	7
7	7	7	7	7	7	7	7

Theorem..4.4: Suppose (T,+,.) is a ternary semiring with IMP property in which (T,.) is semi simple and contains multiplicative identity. Then (T,+) is periodic

Proof Given that (T,+,.) is a ternary semiring with IMP .i.e. a^3=na for a in T and (T,.) is semi simple.To prove (T,+) is periodic

Consider a = 1.a.1.a.1.a.1
= a.a.a
= a^3
a = na (Since T satisfies IMP)

Hence (T,+) is periodic.

Theorem 4.5: Assume that (T,+,.) is a ternary semiring with IMP .If (T,.) is regular and also contains multiplicative identity ,then (T,+) is periodic

Proof: Given that (T,+,.) is a ternary semiring with IMP.i.e. a^3=na for a in T and(T,.) is regular we have to prove (T,+) is periodic for this consider

a = a.1.a.1.a
= a.a.a
= a^3
a = na (Since T satisfies IMP)

Hence (T, +) is periodic.

Theorem 4.6: If (T,+,) is a ternary semiring with IMP. If (T,.) is rectangular band. Then (T,+) is periodic.

Proof: Given that (T,+,.) is ternary semiring with IMP.i.e. a^3=na for a in T.

(T,.) is rectangular band we have to prove (T,+) is periodic.

Since (T,.) is rectangular band x= xaxax for all a,x in T

Put a = x

x = x.x.x.x.x
= (x.x.x).x.x
=x^3.x.x
= nx.x.x (Since T satisfies IMP)
= n x^3
= n(nx)
x = n^2x

Hence (T,+) is periodic.

Definition 4.7: A ternary semiring is said to be Zeroid if a + b= b(or) b + a=b for all a,b in T .

Theorem 4.8: Let (T,+,.) be a ternary semiring with IMP in which (T,+) is Zeroid. If (T, +) is cancellative then (T,.) is band.

Proof: Given that (T,+,.) is ternary semiring with IMP i.e. a^3=na for a in T.Since (T,+) is Zeroid then a + b = b (or) b +a = a for some a,b in T.

Suppose a + b = b
⟹a + (a + b) = b
⟹ a + a + b = b
⟹2a + b = b
⟹ 2a + b = a + b

Continuing like this na + b = a + b

This implies na = a (since (T,+) is right cancellative) ----- I

Similarly consider b + a = b
⟹(b + a) + a = b
⟹ b + 2a = b
⟹b + 2a = b + a

Continuing like this b + na = b + a

This implies na = a ------ II

T satisfies IMP

a^3 = na ----------- III

From I,II and III a^3 = na =a

⟹a^3 = a which proves (T,.) is band

Hence theorem is proved.

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