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βp-regular spaces and βp-normal spaces in topology

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Abstract
The aim of this paper is to introduce and study some weak forms of regular spaces and weak forms of normal spaces , viz. βp- regular spaces , pβ-regular spaces , βp-normal spaces and pβ-normal spaces by using β-closed sets , β-open sets ,preclosed sets and preopen sets . Also , we studied some related functions like βgp-closed functions , βgp-continuous functions for preserving these regular spaces and normal spaces .
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I. Introduction

In 1982, A.S.Mashhour et al [20] introduced and studied the concepts of preopen sets and precontinuity in topological spaces and in 1983, M.E.Abd El–Monsef et al [1] have introduced the concepts of β-open sets and β-continuity in topology . Latter , these β-open sets are recalled as semipreopen sets , which were introduced by D.Andrejevic in [6]. Further these preopen sets and β-open sets (= semipreopen sets) have been studied by various authors in the literature see [5, 7,11,12,13,24,26, 28, 29]. In 1970 and 1998 , respectively Levine [15] and T.Noiri et al. [34] have defined and studied the concept of g-closed sets and gp-closed sets in topology,respectively. The aim of this paper is to introduce and study some weak forms of regular spaces and weak forms of normal spaces , viz. βp- regular spaces , pβ-regular spaces , βp-normal spaces and pβ-normal spaces by using β-closed sets and preopen sets. Also , we studied some related functions like βgp-closed functions , βgp-continuous functions for preserving these regular spaces and normal spaces .

II. Preliminaries

Throughout this paper X,Y will denote topological spaces on which no separation axioms assumed unless explicitly stated. Let f : X → Y represent a single valued function. Let A be a subset of X. The closure and interior of A are respectively denoted by Cl(A) and Int(A).

The following definitions and results are useful in the sequel.
Definition 2.1: A subset A of X is called
(i) semiopen (in short , s-open) set [16] if A ⊂ ClInt(A) .
(ii) preopen (in short , p-open) set [20] if A ⊂ IntCl(A).
(iii) β-open [1] **(**=semipreopen [6]) set if A ⊂ ClIntCl(A) .

The complement of a s-open (resp. p-open , β-open) set is called s-closed[9] (resp. p-closed [11] , β-closed [1]) set. The family of s-open (resp. p-open , β-open) sets of X is denoted by SO(X) (resp. PO(X) , βO(X)) .
Definition 2.2 : The intersection of all p-closed (resp. β-closed) sets containing a subset A of space X is called the p-closure [11] (resp. β-closure[2]) of A and is denoted by pCl(A) (resp. βCl(A)).
Definition 2.3 : The union of all p-open (resp. β-open) sets which are contained in A is called the p-interior[22] (resp. the β-interior [2]) of A and is denoted by pInt(A) (resp. βInt(A)).
Definition 2.4 [34]: A subset A of a space X is called gp-closed if pCl(A) ⊂ U whenever A ⊂ ,U and U is open set in X.
Definition 2.5[3]: A space X is said to be **β**-regular if for each closed set F and for each x ∈X-F , there exist two disjoint **β-open** sets U and V such that x∈U and F ⊂ V.
Definition 2.6 [18]: A space X is said to be **s**-regular if for each closed set F and for each x ∈X-F , there exist two disjoint **s-open** sets U and V such that x∈U and F ⊂ V.
Definition 2.7 [11]: A space X is said to be **p**-regular if for each closed set F and for each x ∈X-F , there exist two disjoint **p-open** sets U and V such that x∈U and F ⊂ V.

Definition 2.8 [10]: A space X is said to be **α**-regular if for each closed set F and for each x ∈X-F , there exist two disjoint **α-open** sets U and V such that x∈U and F ⊂ V.
Definition 2.9 [17]: A space X is said to be **s**-normal if for any pair of disjoint closed subsets A and B of X , there exist disjoint **s-open** sets U and V such that A⊂U and B ⊂ V.
Definition 2.10 [19]: A space X is said to be **β**-normal if for any pair of disjoint closed subsets A and B of X , there exist disjoint **β-open** sets U and V such that A⊂U and B ⊂ V.
Definition 2.11 [26&35]: A space X is said to be **p**-normal if for any pair of disjoint closed subsets A and B of X , there exist disjoint **p-open** sets U and V such that A⊂U and B ⊂ V.
Definition 2.12[23]: A space X is said to be **α**-normal if for any pair of disjoint closed subsets A and B of X , there exist disjoint **α-open** sets U and V such that A⊂U and B ⊂ V.
Definition 2.13 [8]: A space X is said to be submaximal if every dense set of X is open in X (i.e., every preopen set of X is open in X).
Definition 2.14[39]: A space X is said to be extremally disconnected (in brief , E.D.,) space if Cl(G) is open set for each open set G of X .
Definition 2.15[5]:A space X is called PS-space if every preopen subset in it is semiopen.
Definition 2.16[31]: A space X is said to be **βs**-normal if for any pair of disjoint β-closed subsets A and B of X , there exist disjoint **s-open** sets U and V such that A⊂U and B ⊂ V.
Definition 2.17 [31] : A space X is said to be βs-regular if for each **β**-closed set F and for each x∈ X-F , there exist **s**-open sets U and V such that F ⊂ U and x∈V.
Definition 2.18 [31] : A space X is said to be β*-regular if for each **β**-closed set F and for each x∈ X-F , there exist **β**-open sets U and V such that F ⊂ U and x∈V.
Definition 2.19 [31]: A space X is said to be strongly β*-regular if for each **β-c**losed set F and for each x∈ X-F , there exists disjoint open sets U and V such that F ⊂ U and x∈V.
Definition 2.20 [19]: A function f: X→Y is said to be **β**-irresolute if $f^{-1}(V)$ is is **β**-open set in X for each for each **β**-open set V in Y.
Definition 2.21[38] : A function f: X→Y is said to be preirresolute if $f^{-1}(V)$ is is p-open set in X for each for each p-open set V in Y.
Definition 2.22[22] : A function f: X→Y is said to be M-preopen if f(V) is is p-open set in Y for each for each p-open set V in X.
Definition 2.23[19] : A function f: X→Y is said to be M-β-closed if the image of each β-closed set of X is β-closed in Y.
Definition 2.24 [19] : A space X is called β-T_1 space if for each pair of distinct points x and y of X ,there exist β-open sets U and V such that x∈U & y∉U and y∈V & x∉V.
Definition 2.25 [14] : A space X is called pre-T_2 space if for each pair of distinct points x and y of X ,there exist disjoint p-open sets U and V such that x∈U and y∈V .
Lemma 2.26 [5] : For a space X the following conditions are equivalent :
(i) X is E.D.-space
(ii) every s-open subset is p-open
(iii) every s-open subset is α-open
(iv) every β-open subset is p-open
Lemma 2.27 [5] :For a space X the following conditions are equivalent :
(i) X is PS-space
(ii) every p-open subset is s-open
(iii) every β-open subset is s-open
(iv) every p-open subset is α-open
THEOREM 2.28 [13 & 32]: In an E.D.-space and submaximal-space X , then

$$\tau=\alpha O(X)=SO(X)=PO(X)=\beta O(X).$$

Lemma 2.29[20]: If A∈SO(X) and V∈PO(X) then A∩V is p-open in the subspace (A,τ_A).
Lemma 2.30[11]: If A∈αO(X) and V∈PO(X) then A∩V is p-open in the subspace (A,τ_A).

III. Properties of βp-regular spaces

We, define the following.
Definition 3.1 : A topological space X is said to be βp-regular if for each β -closed set F of X and each point x in X - F, there exist disjoint p-open sets U and V such that x ∈U and F ⊂ V.
Definition 3.2: A topological space X is said to be pβ-regular if for each p-closed set F of X and each point x in X - F, there exist disjoint β-open sets U and V such that x ∈U and F ⊂ V.

Clearly, (i)every βp-regular → p-regular, (ii) βp-regular →β*-regular ,(iii)strongly β*-regular →βp-regular ,(iv) strongly β*-regular → regular , (v)β*-regular →pβ-regular and (vi) βα-regular → βs-regular as well as βp-regular.
We,recall the following.

Lemma 3.3 [2]: If A is subset of X and B∈βO(X) such that A∩B=∅ then βCl(A) ∩B = ∅.
We , prove the following.
Theorem 3.4 : For a topological space X the following statements are equivalent ;
(a) X is **βp-**regular
(b) For each x∈X and for each **β**-open set U containing x there exists a p-open set V containing x such that x ∈V ⊂ pCl(V)⊂U.
(c) For each **β**-closed set F of X , ∩ {pCl(V)/F⊂V and V∈PO(X)} = F
(d) For each nonempty subset A of X and each U ∈βO(X) if A ∩U ≠ ∅ then there exists V∈PO(X) such that A∩V ≠∅ and pCl(V) ⊂ U
(e) For each nonempty subset A of X and each F∈βF(X) if A ∩F =∅ then there exists V,W∈PO(X) such that A∩V ≠∅ , F⊂W and V∩W =∅.

Proof : (a) ⇒(b) Let X be **βp-** regular space. Let x∈X and U be **β**-open set containing x implies X - U is **β**-closed such that x∉X- U. Therefore by (a) there exists two p-open sets V and W such that x∈V and X-U ⊂ W ⇒ X-W ⊂ U. Since V∩W = ∅ ⇒ pCl(V) ∩W =∅ ⇒ pCl(V) ⊂ X- W ⊂ U. Therefore , x∈ V ⊂pCl(V) ⊂ U.

(b) ⇒ (c) Let F be a **β**-closed subset of X and x∉F , then X - F is **β**-open set containing x. By (b) there exists p-open set U such that x∈U ⊂ pCl(U) ⊂ X-F implies F ⊂ X- pCl(U) ⊂ X-U i.e F ⊂ V ⊂ X-U where V = X-pCl(U) ∈PO(X) and x∉V that implies x ∉ pCl(V) implies x∉ ∩ { pCl(V) / F⊂V∈PO(X)}. Hence , ∩ { pCl(V) / F⊂V∈PO(X)} = F

(c) ⇒ (d) A be a subset of X and U∈βO(X) such that A ∩U ≠∅.

⇒ there exists x_0 ∈ X such that. x_0∈ A ∩ U. Therefore X-U is **β**-closed set not containing x_0 ⇒ x_0 ∉ **β**Cl(X-U). By (c) , there exists W ∈ PO(X) such that X-U ⊂ W ⇒ x_0 ∉ pCl(W). Put V = X-pCl(W) , then V is p-open set containing x_0. ⇒ A ∩V ≠ ∅ and pCl(V) ⊂ pCl(X-pCl(W)) ⊂ pCl(X-W).Therefore , pCl(V) ⊂ pCl(X-W) ⊂ U.

(d) ⇒ (e) Let A be a nonempty subset of X and F be **β**-closed set such that A ∩ F = ∅.Then X-F is **β-**open in X and A ∩ (X-F) ≠ ∅. Therefore by (d) , there exist V ∈ PO(X) such that A ∩V ≠ ∅ and pCl(V) ⊂ X –F. Put W = X- pCl(V) then W ∈ PO(X) such that F ⊂ W and W∩V =∅.

(f) ⇒ (a) Let x ∈ X be arbitrary and F be **β-**closed set not containing x. Let A= X\F be a nonempty **β-**open set containing x then by (e) ,there exist disjoint p-open sets V and W such that F ⊂ W and A ∩ V ≠ ∅ ⇒ x∈V. Thus X is a **βp-**regular.
Theorem 3.5 : In a topological space X following statements are equivalent;
(a) X is **βp-**regular
(b) for each **β-**open set U of X containing x there exists p-open set V such that x ∈ V ⊂ pCl(V) ⊂ U

Proof : (a) ⇒ (b) Let x ∈ X and U be **β-**open set of X containing x ⇒ X - U is **β-**closed set not containing x. As X is **βp-** regular , there exist disjoint p-open sets V and W such that x∈V and X-U ⊂ W ⇒ X-W ⊂ U. As V∩W = ∅ ⇒ pCl (V) ∩W =∅ ⇒ pCl(V) ⊂ X-W ⊂ U. Hence x∈ V ⊂ pCl(V) ⊂ U.
(b)⇒ (a) Let for each x∈ X, F be **β-**closed set not containing x , therefore X-F is **β-**open set containing x hence from (b) there exists p-open set V such that x∈ V ⊂ pCl(V) ⊂ X-F . Let U = X – pCl(V) then U is p-open set such that F ⊂ U , x∈ V and U ∩ V = ∅. Thus there exists disjoint p-open sets U and V such that x∈V and F ⊂ U. Therefore X is **βp-** regular.

In [31] the folloing are proved :

Lemma 3.6 : For a space X , the following are true :
(i) If X is β*-regular space then it is β-regular.
(ii) If X is strongly β*- regular space then it is β*-regular space.

Theorem 3.7 : For a topological space X the following statements are equivalent ;

(a) X is p**β**-regular

(b) For each $x \in X$ and for each **p**-open set U containing x there exists a β-open set V containing x such that $x \in V \subset \beta Cl(V) \subset U$.

(c) For each **p**-closed set F of X , $\cap$ {βCl(V)/F⊂V and V∈βO(X)} = F

(d) For each nonempty subset A of X and each U ∈**P**O(X) if $A \cap U \neq \varnothing$ then there exists V∈βO(X) such that $A \cap V \neq \varnothing$ and $\beta Cl(V) \subset U$

(e) For each nonempty subset A of X and each F∈**P**F(X) if $A \cap F = \varnothing$ then there exists V,W∈βO(X) such that $A \cap V \neq \varnothing$, $F \subset W$ and $V \cap W = \varnothing$.

The routine proof of the Theorem is omitted.

We,prove the following.

Theorem 3.8 : β-T_1 space and βp-regular space is pre-T_2 space.

Proof : Let X be β-T_1 space and βp-regular space. As X is β-T_1 space, every singleton set {x} is β-closed set for all $x \in X$. X being βp-regular and {x} is a β-closed subset of X and y be any point of X-{x}, then $x \neq y$.By definition of βp-regularity, there exist disjoint p-open sets G and H such that $\{x\} \subset G$ and $y \in H$.This implies that $x \in G$ and $y \in H$. Therefore , X is pre-T_2 space.

We,prove some subspace theorems in the following.

Theorem 3.9 : If X be a βp-regular space and G∈SO(X) , then G is βp-regular as subspace.

Proof : Let F be a β-closed set of G and $x \in$ (G-F) then there exists a β-closed set H of X such that $F = G \cap H$ and $x \notin H$. Since X is βp-regular ,therefore for each β-closed set H of X and $x \notin H$ there exist p-open sets U_x and V_H of X such that $x \in U_x$ and $H \subset V_H$ with $U_x \cap V_H = \varnothing$. Now , put $A = U_x \cap G$ and $B = V_H \cap G$ then A and B are p-open subsets of G by Lemma-2.29 such that $x \in A$ and $F \subset B$ with $A \cap B = \varnothing$. This shows that G is βp-regular space.

Similarly , one can prove the following in view of Lemma-2.30.

Theorem 3.10 : If X be a βp-regular space and G∈αO(X) , then G is βp-regular as subspace.

Next , we prove some preservation theorems in the following.

Theorem 3.11 : If $f : X \to Y$ is a M-preopen , β-irresolute bijection and X is βp-regular space , then Y is βp-regular .

Proof : Let F be any β-closed subset of Y and $y \in Y$ with $y \notin F$. Since f is β-irresolute, $f^{-1}(F)$ is β-closed set in X. Again, f is bijective, let f (x) = y , then $x \notin f^{-1}(F)$.Since X is βp-regular , there exist disjoint p-open sets U and V such that $x \in U$ and $f^{-1}(F) \subset V$. Since f is M-preopen bijection , we have $y \in f(U)$ and $F \subset f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \varnothing$. Hence, Y is βp-regular space.

We , define the following.

Definition 3.12 : A function $f : X \to Y$ is said to be always-β-closed if the image of each β-closed subset of X is β-closed set in Y.

Now, we prove the following.

Theorem 3.13 : If $f : X \to Y$ is an always β-closed , preirresolute injection and Y is βp-regular space , then X is βp-regular .

Proof : Let F be any β-closed set of X and $x \notin F$. Since f is an always β-closed injection, f(F) is β-closed set in Y and $f(x) \notin f(F)$. Since Y is βp-regular space and so there exist disjoint p-open sets U and V in Y such that f(x) ∈ U and f(F) ⊂ V . By hypothesis, $f^{-1}(U)$ and $f^{-1}(V)$ are p-open sets in X with x ∈ $f^{-1}(U)$, F ⊂ $f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence, X is βp-regular space.

We, define the following.

Definition 3.14 : A function f : X → Y is said to be βp-continuous if the inverse image of each β-open set of Y is p-open set in X.

Next, we give the following.

Theorem 3.15 : If f : X → Y is an always β-closed , βp-continuous injection and Y is β-regular space , then X is βp-regular .

Implication Result 3.16 : In view of Lemma-2.26 , we have :
(i)Every βs-regular space →βp-regular space,
(ii)Every βs-regular space →βα-regular space,
(iii)Every β*-regular space → βp-regular space

Implication Result 3.17 : In view of Lemma-2.27 , we have :
(i)Every βp-regular space →βs-regular space,
(ii)Every βp-regular space →βα-regular space,
(iii)Every β*-regular space → βs-regular space

Implication Result 3.18 : In view of Th.-2.28 , we have :

Regular space = α-regular space = s-regular space = p-regular space =β-regular space

IV. Properties of βp-normal spaces.

We , define the following.
Definition 4.1 : A function f : X→ Y is said to be βgp-continuous if for each β-closed set F of Y , f^{-1} (F) is gp-closed set in X.
It is obvious that a function f : X → Y is βgp-continuous if and only if $f^{-1}(V)$ is gp-open in X for each β-open set V of Y.
Definition 4.2: A function f : X→ Y is said to be βgp-closed if for each β-closed set F of X , f(F) is gp-closed set in Y.

We, recall the following .

Definition 4.3 : A function f : X→ Y is said to be :
(i)pre-gp-continuous [37] if the inverse image of each p-closed set F of Y is gp-closed in X.
(ii)pre-gp-closed [34] if the image of each p-closed set of X is gp-closed in Y.
Clearly, (i) every pre-βgp-continuous function is pre-gp-continuous ,
(ii) every pre-βgp-closed function is pre-gp-closed , since in both cases every p-closed set is β-closed set.

We ,prove the following.

Theorem 4.4 : A surjective function f : X→Y is βgp-closed if and only if for each subset B of Y and each β-open set U of X containing $f^{-1}(B)$, there exists a gp-open set V of Y such that B ⊂ V and $f^{-1}(V)$ ⊂ U.
Proof : Suppose f is βgp-closed function. Let B be any subset of Y and U be any β-open set in X containing $f^{-1}(B)$. Put V = Y – f(X-U). Then , V is gp-open set in Y such that B ⊂ V and $f^{-1}(V)$ ⊂ U.
Conversely, let F be any β-closed set of X . Put B = Y – f(F) , then we have $f^{-1}(B)$ ⊂ X –F and X-F is β-open in X. There exists a gp-open set V of Y such that B =Y-f(F) ⊂ V and $f^{-1}(V)$ ⊂ X –F .Therefore , we obtain that f(F) = Y-V and hence f(F) is gp-closed set in Y .This shows that f is βgp-closed function.
We,define the following.
Definition 4.5 : A function f : X→Y is said to be strongly β-closed if the image of each β-closed set of X is closed in Y.

Definition 4.6 [25] : A function f : X→Y is said to be always gp-closed if the image of each gs-closed set of X is gs-closed in Y.

Theorem 4.7: Let f : X→Y and g : Y→ Z be two functions . Then the composition function gof : X→ Z is βgp-closed if f and g satisfy one of the following conditions :

(i) f is βgp-closed and g is always gp-closed .

(ii) f is strongly β-closed and g is gp-closed .

Proof : (i)Let H be any β-closed set in X and f is βgp-closed , then f(H) is gp-closed set in Y.Again , g is always gp-closed function and f(H) is gp-closed set in Y , then gof (H) is gp-closed set in Z . This shows that gof is βgp-closed function.

(ii)Let F be any β-closed set in X and f is strongly β-closed function, then f(F) is closed set in Y. Again, g is gp-closed function and f(F) is closed set in Y, then gof (F) is gp-closed set in Z. This shows that gof is βgp-closed function.

We, define the following.

Definition 4.8 : A function f : X→ Y is said to be gpβ-closed if for each gp-closed set F of X , f(F) is β-closed set in Y.

Theorem 4.9 : Let f : X→Y and g : Y→ Z be two functions . Then ,

(i) if f is β-losed and g is βgp-closed , then gof is gp-closed .

(ii) if f is βgp-closed and g is strongly gp- closed , then gof is strongly β-closed.

(iii) if f is M-β-closed and g is βgp-closed , then gof is βgp-closed .

(iv) if f is alays gp-closed and g is gpβ-closed , then gof is gpβ-closed.

Proof : (i). Let H be a closed set in X , then f(H) be β-closed set in Y since f is β-closed function. Again , g is βgp-closed and f(H) is β-closed set in Y then g(f(H))= gof(H) is gp-closed set in Z .Thus, gof is gp-closed function.

(ii)Let F be any β-closed set in X and f is βgp-closed function, then f(F) is gp-closed set in Y. Again , g is strongly gp-closed and f(F) is gp-closed set in Y , then gof(F) is closed set in Z. This shows that gof is strongly β-closed function.

(iii)Let H be any β-closed set in X and f is M-β-closed function then f(H) is β-closed set in Y. Again, g is βgp-closed function and f(H) is β-closed set in Y , then gof (H) is gp-closed set in Z. Therefore , gof is βgp-closed function.

(iv)Let H be any gp-closed set in X and f is always gp-closed function , then f(H) be gp-closed set in Y . Again , g is gpβ-closed function and f(H) is gp-closed set in Y, then gof (H) is β-closed set in Z . This shows that gof is gpβ-closed function.

Theorem 4.10 : Let f : X→Y and g : Y→ Z be two functions and let the composition function gof : X→ Z is βgp-closed.Then, the following hold :

(i) if f is β-irresolute surjection , then g is βgp-closed.

(ii) if g is gp-irresolute injection , then f is βgp-closed .

Proof : (i) Let F be a β-closed set in Y . Since f is β-irresolute surjective , $f^{-1}(F)$ is β-closed set in X and (gof)($f^{-1}(B)$) = g(F) is gp-closed set in Z . This shows that g is βgp-closed function.

(ii)Let H be a β-closed set in X .Then, gof (H) is gp-closed set in Z . Again , g is gp-irresolute injective , g^{-1}(gof(H)) = f(H) is gp-closed set in Y . This shows that f is βgp-closed function.

We , define the following.

Definition 4.11: A space X is said to be βp-normal if for any pair of disjoint β-closed sets A and B of X ,there exist disjoint p-open sets U and V such that A ⊂ U and B ⊂ V.

Definition 4.12: A space X is said to be pβ-normal if for any pair of disjoint p-closed sets A and B of X ,there exist disjoint β-open sets U and V such that A ⊂ U and B ⊂ V.

We, recall the following.

Definition 4.13[30] : A space X is said to be β*-normal if for any pair of disjoint β-closed sets A and B of X ,there exist disjoint β-open sets U and V such that A ⊂ U and B ⊂ V.

We define the following.

Definition 4.14: A space X is said to be strongly β*-normal if for any pair of disjoint β-closed sets A and B of X ,there exist disjoint open sets U and V such that A ⊂ U and B ⊂ V.

Clearly, (i)every βp-normal → p-normal, (ii) βp-normal →β*-normal ,(iii)strongly β*-normal →βp-normal ,(iv) strongly β*-normal → normal , (v)β*-normal →pβ-normal and (vi) βα-normal → βs-normal as well as βp-normal.

We , recall the following.

Lemma 4.15 [34] : A subset A of a space X is gp-open iff $F \subset pInt(A)$ whenever $F \subset A$ and F is closed set in X.
Lemma 4.16 [4] : If X is submaximal and E.D. space , then every β-open set in X is open set.
We, characterize the βp-normal spaces in the following.
Theorem 4.17 : The following statements are equivalent for a submaximal and E.D., space X :
(i) X is βp-normal space,
(ii) For any pair of disjoint β-closed sets A , B of X , there exist disjoint gp-open sets U , V such that $A \subset U$ and $B \subset V$,
(iii) For any β-closed set A and any β-open set V containing A , there exists a gp-open set U such that $A \subset U \subset pCl(U) \subset V$.
Proof : (i) → (ii) . Obvious , since every p-open set is gp-open set.
(ii)→(iii) . Let A be any β-closed set and V be any β-open set containing A . Since A and X-V are disjoint β-closed sets of X , there exist gp-open sets U and W of X such that $A \subset U$ and $X\text{-}V \subset W$ and $U \cap W = \square$. By Lemmas – 4.12 and 4.13 , we have $X - V \subset pInt(W)$. Since $U \cap pInt(W) = \square$, we have $pCl(U) \cap pInt(W) = \square$ and hence $pCl(U) \subset X\text{-}pInt(W) \subset V$. Thus , we obtain that $A \subset U \subset pCl(U) \subset V$.
(iii)→ (i). Let A and B be any disjoint β-closed sets of X . Since X-B is β-open aet containing A, there exists a gp-open set G such that $A \subset G \subset pCl(G) \subset X\text{-}B$. Then by Lemmas-4.12 and 13 , $A \subset pInt(G)$. Put U = pInt (G) and V = X-pCl(G). Then , U and V are disjoint p-open sets such that $A \subset U$ and $B \subset V$. Therefore, X is βp-normal space.
We, characterize the βp-normal spaces in the following.

Theorem 4.18 : The following statements are equivalent for a submaximal and E.D., space X :
(i) X is pβ-normal space,
(ii) For any pair of disjoint p-closed sets A , B of X , there exist disjoint gβ-open sets U , V such that $A \subset U$ and $B \subset V$,
(iii) For any p-closed set A and any p-open set V containing A , there exists a gβ-open set U such that $A \subset U \subset \beta Cl(U) \subset V$.
We prove the following .

Theorem 4.19 : Every β-T_1 space and βp-normal space is βp-regular space.
Proof : Let X be β-T_1 space and βp-normal space . Let F be any β-closed set in X and $x \notin F \Rightarrow x \in X\text{-}F$. As X is β-T_1 space, {x} is β-closed set for all $x \in X$. Thus , F and {x} are two disjoint β-closed sets of X .Since X is βp-normal space , there exist disjoint p-open sets G and H in X such that $\{x\} \subset G$ and $F \subset H$ i.e., $x \in G$ and $F \subset H$, this shows that X is βp-regular space.
Now , we prove some subspace theorem for βp-normality in the following

Theorem 4.20: Let X be βp-normal space and $Y \in \alpha O(X)$,then Y is βp-normal space.
Proof : Let X be βp-normal space and Y be an α-open subset of X. Let A_Y and B_Y be disjoint β-closed subsets of Y.Therefore , $A_Y = Y \cap A$ and $B_Y = Y \cap B$ where A and B are disjoint β-closed subsets of X.As X is βp-normal ,there exist disjoint p-open subsets G and H of X such that $A \subset G$ and $B \subset H$ which implies that : $Y \cap A \subset Y \cap G$, $Y \cap B \subset Y \cap H$ where $Y \cap G$ and $Y \cap H$ are p-open subsets of Y by Lemma- 2.30 with $(Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = \emptyset$. Therefore ,Y is βp-normal space.

Similarly , one can prove the following in view of Lemma 2.29 .
Theorem 4.21 : Let X be βp-normal space and $Y \in SO(X)$,then Y is βp-normal space.
Theorem 4.22 : If $f : X \to Y$ is a β-irresolute , βgp-closed surjection and X is βp-normal, then Y is βp-normal space.
Proof : Let A and B be any disjoint β-closed sets of Y. Then, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint β-closed sets of X since f is β-irresolute function . Since X is βp-normal , there exist disjoint p-open sets U and V in X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By Th.4.4 , there exist gp-open sets G and H of Y such that $A \subset G$, $B \subset H$, $f^{-1}(G) \subset U$ and $f^{-1}(H) \subset V$. Then , we have $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and hence $G \cap H = \emptyset$. It follows from Th.4.14 that space Y is βp-normal.
Implication Result 4.23 : In view of Lemma-2.26 , we have :
(i)Every βs-normal space →βp-normal space,
(ii)Every βs-normal space →βα-normal space,
(iii)Every β*-normal space → βp-normal space

Implication Result 4.24 : In view of Lemma-2.27 , we have :
(i)Every βp-normal space →βs-normal space,
(ii)Every βp-normal space →βα-normal space,
(iii)Every β*-normal space → βs-normal space

Implication Result 4.25: In view of Th.-2.28 , we have :

Normal space = α-normal space = s-normal space = p-normal space =β-normal space

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