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Note on the Crucial Test of General Relativity Due to Curved Space-time

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Abstract:*Light follows the curvature of space-time, hence when it passes around a massive object, it is bent. The speed of light depends on the gravitational potential and this bending can be viewed as a consequence. The gravitational attraction can be viewed as the motion of undisturbed objects in a curved geometry.*

Background*: All the observations demonstrate that the light from stars passing close to the sun is slightly bent, so that stars appeared slightly out of position. A watcher can see various types of image of a single light source, if the light were deflected around a mass.*

Materials and Methods**:** *Manifold refers to the various types of space or surface such as the plane is two dimensional manifolds. A manifold (or sometimes differential manifold) is one of the most fundamental concepts in mathematics and physics [2, 12]. Consider P be any point on the manifold M. Now, the open sub set U(P) which is the neighborhood of P can be explained by the N number of real quantities (x^1, x^2, x^3, ... x^N), which are the coordinates of N-dimensional Euclidean space and here the distance between two points P_A and P_B be*

$$\Delta l_{AB} = \left[\left(X_A^1 - X_B^1\right)^2 + \ldots\ldots\ldots + \left(X_A^N - X_B^N\right)^2\right]^{\frac{1}{2}}$$

Results*: Light is bent when it passes through the strong gravitational field and the amount of bending is one of the predictions of Einstein's General Theory of relativity and this is visible when a distribution of matter (such as a cluster of galaxies) between a distant light source and an watcher, that is capable of bending the light from the source as the light travels towards the observer and hence we can conclude bending of light is one of the crucial test of General theory of Relativity[1, 19, 44].*

Conclusion:*In order to get the path of a light pulse, there have been put the line element $ds = 0$ and the result shows that the deflection in the path of light due to the relativistic field of a heavy mass like sun is twice that predicted by the Newtonian theory [25].This treatment in General theory of Relativity can be verified by observations at the times of eclipse on the apparent positions of the stars. Hence, this results the General theory of Relativity and the study of gravitational phenomenon's with the help of this theory gives small deviations from those obtained from the special theory and these deviations have been verified by experimental results.*

Key Word**: *Housdorf space, Minkowskian metric, Gravitational potential, Gravitational field, Geodesic, Metric tensor,None inertial field.***

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I. Introduction

All the observations demonstrate that the light from stars passing close to the sun is slightly bent, so that stars appeared slightly out of position. A watcher can see various types of image of a single light source, if the light were deflected around a mass [2].On the basis of General theory of Relativity, deviation of light path passing close to heavy gravitational mass is visible to the observer and which is not visible in Newtonian theory or Minkowski flat space . In presence of matter space-time is curved otherwise it is flat and for flat space no deviation of light will be occurred which is measured by space-time metric.

II. Manifolds

Manifold refers to the various types of space or surface such as the plane is two dimensional manifolds. A manifold (or sometimes differential manifold) is one of the most fundamental concepts in mathematics and physics [2, 12]. Consider P be any point on the manifold M. Now, the open sub set U(P) which is the neighborhood of P can be explained by the N number of real quantities (x1, x2, x3, ... xN), which are the coordinates of N-dimensional Euclidean space and here the distance between two points PA and PB be

$$\Delta l_{AB} = \left[\left(x_A^1 - x_B^1\right)^2 + \ldots\ldots. + \left(x_A^N - x_B^N\right)^2\right]^{\frac{1}{2}} \quad (1)$$

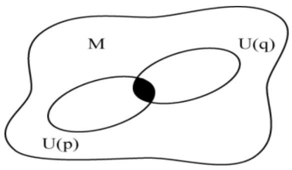

Fig.1

Another point q and its neighborhood, open set U(q) can be explained by the N-number of real quantities (y^1, y^2, ... y^N). Now the points of the common region of these two neighborhoods can be expressed through any of them coordinate systems. In this case, the coordinate transformation is

$$y^i = y^i (x^1, x^2, x^3, \ldots. x^N) \; [i = 1, 2, 3, \ldots. N].$$

So, we can define the manifold as a connected Housdorf space which is locally Euclidean.

III. Space-time Metric

The special relativistic line element (metric),when Cartesian coordinates *(ct, x, y, z)* are used, is given by $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = c^2dt^2 - dx^2 - dy^2 - dz^2$. (2)

Where $\eta_{\mu\nu}$ is the flat-space Minkowskian metric tensor. In natural units: c= 1

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (3)$$

If one goes form one inertial coordinate frame in to another by Lorentz transformation, the metric (2) does not change [12].

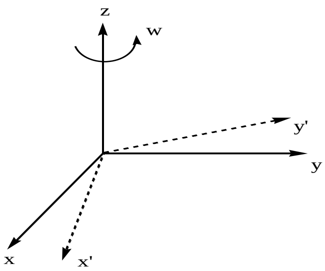

Fig.2

Suppose that one goes from an inertial frame into a uniformly rotating (i.e., non-inertial frame). The rotation is about z-axis. Then the transformation equations are

$$\left.\begin{array}{c} x = x'\cos\omega t - y'\sin\omega t \\ x = x'\cos\omega t - y'\sin\omega t \\ z = z' \end{array}\right]. \quad (4)$$

Whereω is the angular velocity of rotation. From (4) we have

$$dx = dx'\cos\omega t - x'\omega\sin\omega t\, dt - \sin\omega t dy' - y'\omega\cos\omega t\, dt \quad (5)$$

$$dy = \sin\omega t\, dx' + x'\omega\cos\omega t\, dt + \cos\omega t\, dy' - y'\omega\sin\omega t\, dt \quad (6)$$

$$dz = dz'. (7)$$

Putting these values in (2) we have

$$ds^2 = [c^2 - \omega^2(x'^2 + y'^2)]dt^2 + 2\omega dt(y'dx' - x'dy') - (dx' + dy' + dz')^2. \quad (8)$$

The non-inertial coordinate frames are used, the line element will have following expression

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad (9)$$

We can use non-Cartesian coordinates and in that case, the coordinates x^1, x^2 and x^3 describe curvilinear coordinates. The metric tensor $g_{\mu\nu}$ describes the non-inertial field of forces are equivalent to gravitational fields. So, the space-time metric in general relativity has the more general form given by (9). Here$g_{\mu\nu}(x)$represents the gravitational potential (field) [5]. Using tensor transformation we can show that

$$\bar{g}_{\mu\nu}(\bar{x})d\bar{x}^{\mu}d\bar{x}^{\nu} = g_{\mu\nu}dx^{\mu}dx^{\nu} = ds^2.$$

Hence, the form of (9) is the amount of space-time curvature.

IV. Bending of Light Ray

The equation of orbit of a particle in the presence of a gravitating mass M with Gravitational potential $\varphi$is given by [1, 2].

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{L^2} + 3GMu^2. \quad (10)$$

With

$$r^2\frac{d\varphi}{ds} = L \quad (11)$$

For the track of light ray,$ds = 0$;

Equation (11) implies that L= ∝. Here the track of the light ray in the neighborhood of a gravitating mass M is given by

$$\frac{d^2u}{d\varphi^2} + u = 3GMu^2. \quad (12)$$

In the approximation that the gravitational field is completely neglected, i.e., for $r\rightarrow\propto$ we can neglect the term $3GMu^2$ from equ. (10) we have

$$\frac{d^2u}{d\varphi^2} + u = 0. \quad (13)$$

This equation has a solution

$$u = A\cos\varphi + B\sin\varphi, \quad (14)$$

where, A and B are constants.

Therefore,

$$\frac{du}{d\varphi} = -A\cos\varphi + B\sin\varphi. \quad (15)$$

The boundary conditions are

$$\varphi = \frac{\pi}{2}, \qquad u = \frac{1}{r_{max}} = \frac{1}{R(constant)}, \qquad \frac{du}{d\varphi} = 0.$$

From (14) and (15) we have

$$A = 0 \text{ and } \frac{1}{R} = B$$

Putting the values of A and B in equation (14), we have

$$u = \frac{\sin\varphi}{R}. \quad (16)$$

At $\varphi = \frac{\pi}{2}$ equation $y = r\sin\varphi$reduced to $y = R$.

Now from (10) we have

$$\frac{d^2u}{d\varphi^2} + u = \frac{3GM}{R^2}\sin^2\varphi. \quad (17)$$

Here the complementary function

$$u_c = \frac{\sin\varphi}{R}. \quad (18)$$

And the particular integral is

$$u_p = \frac{1}{1+D^2}\left(\frac{3GM}{R^2}\sin^2\varphi\right)$$

$$= \frac{3GM}{2R^2}\frac{(3+\cos 2\varphi)}{3}$$

$$= \frac{GM}{2R^2}(3 + 2\cos^2\varphi - 1)$$

$$= \frac{GM}{R^2}(1 + \cos^2\varphi). \quad (19)$$

The general solution of (17) is

$$u = \frac{\sin\varphi}{R} + \frac{GM}{R^2}(1 + \cos^2\varphi). \quad (20)$$

Multiplying it by $\frac{R}{u}$ we have

$$R = \frac{\sin\varphi}{u} + \frac{GM}{Ru}(1+\cos^2\varphi)$$

$$\Rightarrow \quad y = R - \frac{GM}{R}\frac{r^2(r\cos\varphi)^2}{r}$$

$$\Rightarrow y = R - \frac{GM}{R}\frac{(2x^2+2y^2)}{\sqrt{(x^2+y^2)}}. \tag{21}$$

Again

$$\frac{1}{r} = u = \frac{1}{R}\sin\varphi$$

$\Rightarrow R = r\sin\varphi$

Therefore,

$$y = R. \tag{22}$$

Now it is clear from equation (21) and (22) that the 2nd term on the R.H.S of equation (21) measures the very slight deviation (Geodesic path) from the straight line path $y = R$. The asymptotes to (21) can be found by taking x very large as compared to y, so that asymptotes to eqn. (21) are

$$y = R - \frac{GM}{R}\left(\pm\ \frac{2x^2}{x}\right)$$

$$= R - \frac{GM}{R}\ (\pm\ 2x). \tag{23}$$

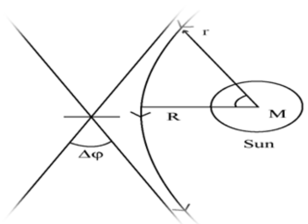

Fig.3

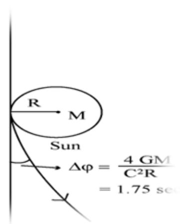

Fig.4

V. Result

Now

$$y = \frac{2\ GMx}{R}\ + R \tag{24}$$

and

$$y = -\frac{2\ GMx}{R}\ + R \tag{25}$$

Here slopes

$$m_1 = \frac{2GM}{R},\ \ m_2 = -\frac{2GM}{R}$$

If $\Delta\varphi$ be the angle between the equations (24) and (25), then we have

$$tan\Delta\varphi = \frac{\frac{2\,GM}{R} - \left(-\frac{2\,GM}{R}\right)}{1 + \frac{2\,GM}{R}\left(-\frac{2\,GM}{R}\right)}$$

$$= \frac{4GM}{R}.$$

Threfore,

$$\tan\Delta\varphi \approx \Delta\varphi = \frac{4GM}{R} = \frac{4GM}{Rc^2}\ ; \quad \text{if } c \neq 1 [\Delta\varphi \text{ is very small.}] \tag{26}$$

where$R$ is the closet distance of the light ray from the center of the body [25].

Thus

$$\Delta\varphi = \frac{4GM}{Rc^2}. \tag{27}$$

This equation represents the total deflection of a light ray passing near a heavy mass M in Fig. 3.
For a light ray grazing the surface of the sun, Fig. 4. Now, we have

$$\Delta\varphi = \frac{4 \times 6.66 \times 10^{-11} \times 1.99 \times 10^{30}}{(3 \times 10^8)^2 \times 6.97 \times 10^8}\ \text{Radians}$$

$$= \frac{8.36 \times 10^{-6}}{4.85 \times 10^{-6}}\ \text{arc-sec}$$

= 1.75 arc-sec.

i.e.,

$\Delta\varphi = 1.75''$.

The observed value is $(1.75 \pm 0.10)''$.

VI. Conclusion

Light is bent when it passes through the strong gravitational field and the amount of bending is one of the predictions of Einstein's General Theory of relativity and this is visible when a distribution of matter (such as a cluster of galaxies) between a distant light source and an watcher, that is capable of bending the light from the source as the light travels towards the observer and hence we can conclude bending of light is one of the crucial test of General theory of Relativity[1, 19, 44].In order to get the path of a light pulse, there have been put the line element $ds = 0$ and the result shows that the deflection in the path of light due to the relativistic field of a heavy mass like sun is twice that predicted by the Newtonian theory [25].This treatment in General theory of Relativity can be verified by observations at the times of eclipse on the apparent positions of the stars. Hence, this results the General theory of Relativity and the study of gravitational phenomenon's with the help of this theory gives small deviations from those obtained from the special theory and these deviations have been verified by experimental results.

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