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A Study on Some Properties of Fuzzy Soft Topological Space

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Abstract
In this paper we discussed the basic definitions of soft set, fuzzy soft set, fuzzy soft topological space, fuzzy soft continuous mapping and the composition of the mapping. Then we studied fuzzy soft separation axioms (T_0, T_1, T_2) *with some of the properties.*

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I. Introduction

A lot of real life problems in since,economic, environmentsengineering,medical,etc,cannot be solved by using classical mathematical methods,and this methods are not enough to meet the new requirements, therefore some kinds of theories were given like fuzzy set theory,soft set theory and fuzzy soft set theory and its applications and theyhave been developed to solve these problems. The notion of fuzzy set was introduce by Zadeh [3] in his classical paper of 1965.In 1968, Chang [2] gave the definition of fuzzy topology, which is family of fuzzy sets.1999 ,Molodtsov [5] initiated a novel concept of soft set theory,which is a completely new approach for modelling regueness and uncertainty .Applications of soft set theory in other disciplines and real life problems are now catching momentum. In 2014,Abdulkadir ,Vildan and Halis[1]studied an introduction to fuzzy soft topological spaces ,Roy .S.Samanta T.K in 2012 [6] discussed A note on fuzzy soft topological spaces.Sabir ,Bashir, 2011 [8]gives some properties of soft topological spaces .Sabir Hussain ,2017 [7] studied On some properties of fuzzy almost soft continuous mappings. In the present paper we discussed some of definitions of soft sets, fuzzy soft sets and fuzzy soft topological space. Then we studied fuzzy soft mapping (image and inverse image) .then we discussed the fuzzy soft axioms.

II. Preliminaries

In this section, we present several preliminary definitions which are necessary in theprocess of defining our main results. For the sake of consistency, the following notationsare used throughout the whole paper:
U: the initial universe,
E: the possible parameters for U,
P(U) : the power set of U,
I^U:the set of all fuzzy subsets of U,
$(U: E)$: the universal set U and the parameter set E.

Definition 2.1 [3] A fuzzy set A in U is a set of ordered pairs

$$A = \{(x, \mu_A(x) : x \in U\}$$

Where$\mu_A : U \to [0,1] = I$ is a mapping and $\mu_A(x)$ (or A(x)) states the degree of belonging of x in A.

Definition 2.2 [5] LetA ⊆ E. A pair (F, A) is called a soft set over U where F is a mapping given by$F: A \longrightarrow P(U)$

Definition 2.3[5] Let A ⊆ E. f_A is defined to be a fuzzy soft set $U_E$if $f: A \longrightarrow I^U$ is a mapping given by$f(e) = \mu_f^e$ such that

$$f^e = \begin{cases} \mu_f^e = \bar{O} & if\ e \in E - A \\ \mu_f^e \neq \bar{O} & if\ e \in A \end{cases}$$

$where\ \bar{O}(e) = 0\ for\ each\ u \in U$

Definition 2.4[6] The complement of a fuzzy soft setf_A is a fuzzy soft set onU_E .which is denoted by f_A^c furthermore , $f: A \longrightarrow I^U$ is defined as follows:

$$f^c = \begin{cases} \mu_f^e = 1 - \mu_f^e & if\ e \in A \\ \mu_f^e = \bar{1} & if\ \ e \in\ E - A \end{cases}$$

$where\ \bar{1}(e) = 1\ for\ each\ u \in U$

Definition 2.5[6] The fuzzy soft set f_Φ on U_E is defined as a null fuzzy soft set denoted by Φ .
Moreover $\Phi(e) = \bar{O}$ for every $e \in E$

Definition 2.6[6] The fuzzy soft set f_{A} on U_E is defined to be an absolute fuzzy soft set denoted by U_E. Moreover $U(e) = f(e) = \bar{1}$for every $e \in E$

Definition 2.7:[4] Two fuzzy soft sets f_A and g_B over a common universe U_E, we say that f_A is a fuzzy soft subset of g_B if :

1- $A \subseteq B$

2- $\forall\, a \in A \ \ , f(a) \leq g(a)$

And it can be written as$f_A \subseteq g_B$

Definition 2.8:[4] Two fuzzy soft sets f_A and g_B over a common universe U_E, we say that f_A is equal to g_B if $f_A \subseteq g_B$ and $g_B \subseteq f_A$

Definition 2.9:[4] Let the fuzzy soft sets f_A , $g_B \in U_E$ then the union of f_A and g_B is also a fuzzy set h_C , defined by $h_C = f_A(e) \vee g_B(e)$, $\forall\, e \in E$, where $C = A \cup B$. Here we write $h_C = f_A \cup g_B$

Definition 2.10:[4] Let the fuzzy soft sets f_A , $g_B \in U_E$ then the intersection of f_A and g_B is also a fuzzy set h_C , defined by $h_C = f_A(e) \wedge g_B(e)$, $\forall\, e \in E$, where $C = A \cap B$. Here we write $h_C = f_A \cap g_B$

Definition 2.11:[4]Let a fuzzy soft sets f_A over U_E. Then the complement of $f_A$is denoted by f_A^c and is defined by $f_A^c(e) = 1 - f_A$, $\forall e \in E$

III. Fuzzy Soft Topological Space

Definition 3. 1: Let U be a set and τ be a family of a fuzzy subsets of U. τ is called a fuzzy topology on U if it is satisfy the following conditions:

1- $0,1 \in \tau$

2- If $G_j \in \tau \ for\ each\ j \in J\ then\ \vee G_j \in \tau$

3- IF G,H $\in \tau\ \ then\ G \wedge H \in \tau$

The pair (U, τ) is called a fuzzy topological space.The members of τ are called fuzzy open sets and a fuzzy set A in U is said to be closed iff 1-A is a fuzzy open set in U.

Remark: Every topological space is a fuzzy topological space but not conversely.

Example:$let\ U = \{a, b, c\}\ be\ a\ set\ and\ let\ A = \{(a, 0), (b, 0.4), (c, 1)\}$ be a fuzzy set in U. Let $\tau = \{0, A, 1\}$ .Then (U, τ) is a fuzzy topological space which is not a topological space.

Definition 3. 2 : Ifτ is a fuzzy soft topology on (U, E), the triple (U, τ, E) is said to be a fuzzy soft topological space. Also each member of τ is called a fuzzy soft open set in (U, τ, E) .A fuzzy soft subset of (U, τ, E) is called a fuzzy soft closed set if it is complement is member of τ.

Proposition 3. 1 : let(U, τ, E) be a fuzzy soft topological space over U.then the collection $\tau_\alpha = \{F(x): (F, E) \in$ $\tau$for each $\alpha \in E$,defines a fuzzy topology on U.

Proof: By definition, for any $\alpha \in E$ we have $\tau_\alpha = \{F(x): (F, E) \in \tau\}$,now

1) $\phi, \widetilde{U} \in \tau$ implies that $0, U \in \tau_\alpha$.

2) let $\{F_i(\alpha): i \in I\}$ be a collection of sets in τ_α ,since $(F_i , E) \in \tau$, $for\ all$ i$\in I$ so that $\bigcup_{i \in I}(F_i , E) \in \tau$ thus $\bigcup_{i \in I} F_i(\alpha) \in \tau_\alpha$

3)let $F(\alpha), G(\alpha) \in \tau_\alpha$ for some $(F, E), (G, E) \in \tau$,since $(F, E) \cap (G, E) \in \tau$ so $F(\alpha) \cap G(\alpha) \in \tau_\alpha$ .thus $\tau_\alpha$is a topology on U for each $\alpha \in E$.

Proposition (3.1) shows that corresponding to each parameter $\alpha \in E$,we have a topology τ_α on U .thus a soft topology on U gives a parameterized family of topologies on U .

Example: Let $U = \{x_1, x_2, x_3\}$, $E = \{\alpha_1, \alpha_2\}\ and\ \tau = \{\phi, \widetilde{U}, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\}$ where $(F_1, E), (F_2, E), (F_3, E), (F_4, E)\ and\ (F_5, E)$ are soft sets over U and it is defined as follow :

$F_1(\alpha_1) = \{x_2\}$, $F_1(\alpha_2) = \{x_1\}$

$F_2(\alpha_1) = \{x_2, x_3\}$, $F_2(\alpha_2) = \{x_1, x_2\}$

$F_3(\alpha_1) = \{x_1, x_2\}$, $F_3(\alpha_2) = \widetilde{U}$

$F_4(\alpha_1) = \{x_1, x_2\}$, $F_4(\alpha_2) = \{x_1, x_3\}$

$F_5(\alpha_1) = \{x_2\}$, $F_5(\alpha_2) = \{x_1, x_2\}$

Then τ defines a soft topology on U, hence (U, τ, E) is a soft topological space over U . It can easily seen that:

$$\tau_{\alpha_1} = \left\{\phi, \widetilde{U}, \{x_2\}, \{x_2, x_3\}, \{x_1, x_2\}\right\}$$

and

$\tau_{\alpha_2} = \left\{\phi, \widetilde{U}, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}\right\}$

Are topologies on U .

Now the soft closed sets are :$\widetilde{U}, \phi, \left\{\{x_1, x_3\}, \{x_2, x_3\}\right\}, \left\{\{x_1\}, \{x_2\}\right\}, \left\{\{x_3\}, \phi\right\}, \left\{\{x_3\}, \{x_2\}\right\}, \left\{\{x_1, x_3\}, \{x_3\}\right\}$

Definition 3.3:$Let\ \varphi : X \to Y\ and\ \Psi: E \to F$ be two mappings, where E and F are paremeter sets for the sets $X$and$Y$, respectively.thenφ_Ψ is called a fuzzy soft mapping from $\widetilde{(X, E)}$in to $\widetilde{(Y, F)}$ and denoted by$\varphi_\Psi: \widetilde{(X, E)} \to \widetilde{(Y, F)}$.

Definition 3. 4: $Let f_A$ and g_B be two fuzzy soft sets over X and Y, respectively and let φ_Ψ be a fuzzy soft mapping from $\widetilde{(X,E)}$ in to $\widetilde{(Y,F)}$.

1) The image of f_A under the fuzzy soft mapping φ_Ψ, denoted by $\varphi_\Psi(f_A)$ and is defined as,

$$\varphi_\Psi(f_A)_k(y) = \begin{cases} V_{\varphi(x)=y} V_{\Psi(e)=k} f_{A(e)(x);} if\ \varphi^{-1}(y) \neq \emptyset, \Psi^{-1}(k) \neq \emptyset; \\ 0, \qquad otherwise \end{cases}$$

For all $k \in F$, $for\ all\ y \in Y$.

2)The inverse image of g_B under the fuzzy soft mapping φ_Ψ, denoted by $\varphi_\Psi{}^{-1}(g_B)$ and defined as,

$$\varphi_\Psi{}^{-1}(g_B)(e)(x) = g_B(\Psi(e))(\varphi(x)), for\ all\ e \in E, for\ all\ x \in X$$

Example: let $X = \{a, b, c\}$, $Y = \{x, y, z\}$, $E = \{e_1, e_2, e_3, e_4\}$, $F = \{f_1, f_2, f_3\}$ and $\widetilde{(X,E)}$, $\widetilde{(Y,F)}$ of fuzzy soft sets ,

Let $\varphi : X \to Y$ and $\Psi: E \to F$ be mappings defined as: $(a) = z$, $\varphi(b) = y$, $\varphi(c) = y$, $\Psi(e_1) = f_1$, $\Psi(e_2) = f_1$, $\Psi(e_3) = f_3$, $\Psi(e_4) = f_2$

Choose two fuzzy soft sets in $\widetilde{(X,E)}$ and $\widetilde{(Y,F)}$, respectively as:

$$(K, N) = \{e_1 = \{a_{0.5}, b_0, c_{0.8}\}, e_2 = \{a_{0.1}, b_{0.9}, c_{0.5}\}, e_4 = \{a_{0.4}, b_{0.3}, c_{0.6}\}\}$$
$$(L, M) = \{f_1 = \{x_{0.3}, y_{0.5}, z_{0.1}\}, f_2 = \{x_{0.9}, y_{0.1}, z_{0.5}\}, \}f_3 = \{x_{0.7}, y_{0.5}, z_{0.6}\}$$

Then the fuzzy set image of (K, N) under $\varphi_\Psi: \widetilde{(X,E)} \to \widetilde{(Y,F)}$ is obtained as:

$$\varphi_\Psi(K,N)(f_1)(x) = \vee_{s\in\varphi^{-1}(x)} \left(\vee_{\propto\in\Psi^{-1}(f_1)\cap N} K(\propto)\right)(s)$$
$$= 0, \quad (as\ \varphi^{-1}(x) = \emptyset,)$$
$$\varphi_\Psi(K,N)(f_1)(y) = \vee_{s\in\varphi^{-1}(y)} \left(\vee_{\propto\in\Psi^{-1}(f_1)\cap N} K(\propto)\right)(s)$$
$$= \vee_{s\in\{b,c\}} (\vee_{\propto\in\{e_1,e_2\}} K(\propto))(s)$$
$$= \vee_{s\in\{b,c\}} (K(e_1) \vee K(e_2)(s)$$
$$= \vee_{s\in\{b,c\}} (\{a_{0.5}, b_{0.9}, c_{0.8}\})(s)$$
$$= \vee (0.9, 0.8) = 0.9,$$
$$\varphi_\Psi(K,N)(f_1)(z) = 0.5$$

By similar calculations, consequently, we get

$$\varphi_\psi((K,N), M) = \{f_1 = \{x_0, y_{0.9}, z_{0.5}\}, f_2 = \{x_0, y_{0.6}, z_{0.4}\}, f_3 = \{x_0, y_0, z_0\}\}$$

Next for $\psi(e_i) \in M, i = 1,2,4, we\ calculate$

By similar calculations, consequently, we get

$$\varphi_\psi{}^{-1}(L,M) = \{\{e_1 = \{a_{0.1}, b_{0.5}, c_{0.5}\}, e_2 = \{a_{0.1}, b_{0.5}, c_{0.5}\}, \quad e_3 = \{a_{0.6}, b_{0.5}, c_{0.5}\}, e_4 = \{a_{0.5}, b_{0.1}, c_{0.1}\}\}\}$$

Definition 3.5: If φ and ψ are injective , surjective , then the fuzzy soft mapping φ_Ψ is injective,surjective .if φ_ψ is both injective and surjective,then it is called bijective.

Definition 3.6: Let φ_ψ be a fuzzy soft mapping from $\widetilde{(X,E)}$ in to $\widetilde{(Y,F)}$ and $\varphi^*_{\psi^*}$ be a fuzzy soft mapping from $\widetilde{(Y,F)}$ in to $\widetilde{(Z,K)}$ .Then the composition of these mappings from $\widetilde{(X,E)}$ in to $\widetilde{(Z,K)}$ is defined as follows:

Proposition3.2: let (X, τ_1, E_1), (Y, τ_2, E_2) and (Z, τ_3, E_3) be soft topology space. If $(\psi_1, \varphi_1): (X, \tau_1, E_1) \to (Y, \tau_2, E_2)$ and $(\psi_2, \varphi_2): (Y, \tau_2, E_2) \to (Z, \tau_3, E_3)$ are soft continuous functions, then the composition $(\psi_2, \varphi_2) \circ (\psi_1, \varphi_1) = (\psi_2 \circ \psi_1, \varphi_2 \circ \varphi_1)$ is also soft continuous .

Proof : since (ψ_1, φ_1) is soft continuous for each $e_1 \in E_1$ and $e_2 = \psi(e_1) \in E_2$, $\varphi_1: (\boldsymbol{X}, \boldsymbol{\mathcal{T}_1}(e_1)) \to (\boldsymbol{Y}, \boldsymbol{\mathcal{T}_2}(e_2))$ is continuous

Since (ψ_2, φ_2) is soft continuous for each $e_2 \in E_2$ and $e_3 = \psi(e_2) \in E_3$, $\varphi_2: (\boldsymbol{X}, \boldsymbol{\mathcal{T}_2}(e_2)) \to (\boldsymbol{Y}, \boldsymbol{\mathcal{T}_3}(e_3))$ is continuous. Hence $(\varphi_2 \circ \varphi_1)$ is continuous ,then $(\psi_2, \varphi_2) \circ (\psi_1, \varphi_1)$ is soft continuous.

Theorem 3.1 : $Let\ \varphi : X \to X\ and\ \psi: E \to E$ be the identity mappings,then $I = \varphi_\psi$ is called identity fuzzy soft function and this function is fuzzy soft continuous.

Definition 3.7 : let (U, τ_1, E) and (V, τ_2, E) be two fuzzy soft topological space and $f: (U, \tau_1, E) \to (V, \tau_2, E)$ be a mapping, for each $(G, E) \in \tau_2$, $if\ f^{-1}(G, E) \in \tau_1$, then $f: (U, \tau_1, E) \to (V, \tau_2, E)$ is said to be a fuzzy soft set continuous mapping of a fuzzy soft topological spaces.

Proposition 3.3 : If mapping $f: (U, \tau_1, E) \to (V, \tau_2, E)$ is a fuzzy soft continuous mapping , then $\forall \alpha \in E$, $f: (U, \tau_{1\alpha}) \to (V, \tau_{2\alpha})$ is a fuzzy continuous mapping.

Proof: let $A \in \tau_{2\alpha}$ then there exists a fuzzy soft open set (G, E) over V such that $A = G(\alpha)$. $since\ f: (U, \tau_1, E) \to (V, \tau_2, E)$ is a fuzzy soft continuous mapping , $f^{-1}(G, E)$ is a fuzzy soft open set over U and $f^{-1}(G, E)(\alpha) = f^{-1}G(\alpha) = f^{-1}(A)$ is a fuzzy soft open set .this implies that is a fuzzy continuous mapping.

Definition 3. 8: Two soft pointes $e_K, e_H\ in\ U_E$ are distinct, written $e_K \neq e_H$ if there corresponding soft sets $(K, E)\ and\ (H, E)$ are disjoint.

IV. Fuzzy Soft Sepatation Axioms:

Definition 4. 1:

$\boldsymbol{T_0} -\ suppose\ that\ e_K, e_H \in U_E$ be two soft points($e_K \neq e_H$),where$(U, \tau, E)$is a fuzzy soft topological space over U, if $\exists(F, E)$and(G, E) two fuzzy open sets s.t:$e_K, \in (F, E), e_H \notin (F, E)\ or\ e_H \in (G, E), e_K \notin (G, E)$ then$(U, \tau, E)$is said to a fuzzy soft T_{0-}space.

Definition 4. 2:

$\boldsymbol{T_1} -\ suppose\ that\ e_K, e_H \in U_E$ be two soft points ($e_K \neq e_H$),where $(U, \tau, E)$is a fuzzy soft topological space over U , if $\exists(F, E)$and(G, E) two fuzzy open sets s.t: $e_K, \in (F, E), e_H \notin (F, E)\ and\ e_H \in (G, E), e_K \notin (G, E)$ then$(U, \tau, E)$is said to a fuzzy soft T_1 − space.

Definition 4. 3:

$\boldsymbol{T_2} -\ suppose\ that\ e_K, e_H \in U_E$ be two soft points ($e_K \neq e_H$),where $(U, \tau, E)$is a fuzzy soft topological space over U , if $\exists(F, E)$and(G, E) two fuzzy open sets s.t:$e_K, \in (F, E)$and $e_H \in (G, E)$, and$(F, E) \cap (G, E) = \emptyset_E$,then$(U, \tau, E)$is said to a fuzzy soft T_2 − space.

PROPOSITION 4.1 :

(i) Every soft T_1 –space is a soft T_0 –space.

(ii) Every soft T_2 –space is a soft T_1 –space.

Proof: $suppose\ that\ e_K, e_H \in U_E\ be\ two\ soft\ points, (e_K \neq e_H), where\ (U, \tau, E)$is a fuzzy soft topological space over U

(i) If (U, τ, E) is a soft T_1 –space then , $\exists(F, E)$and(G, E) two fuzzy open sets s.t: $e_K, \in (F, E)$and$e_H \notin (F, E)\ and\ e_H \in (G, E)\ and\ e_K \notin (G, E)$.obviously then we have $e_K \in (F, E)$and$e_H \notin (F, E)\ or\ e_H \in (G, E)\ and\ e_K \notin (G, E)$thus (U, τ, E) is a soft T_0 –space.

(ii) If (U, τ, E) is a soft T_2 –space then , $\exists(F, E)$and(G, E) two fuzzy open sets s.t: $e_K \in (F, E)\ and\ e_H \in (G, E), (e_K \neq e_H))\ and\ (F, E) \cap (G, E) = \emptyset_E$ since $(F, E) \cap (G, E) = \emptyset_E$, there fore$e_K \notin (G, E)\ and\ e_H \notin (F, E)$ .thus it follows that(U, τ, E) is a soft T_1 –space.

Remark : every soft T_1 –space is a soft T_0 –space and every soft T_2 –space is a soft T_1 –space .

Example:let $U = \{u_1, u_2\}, E = \{e_1, e_2\}\ and\ \ \ \tau = \{\emptyset, \widetilde{U}, (F_1, E), (F_2, E), (F_3, E)\}$where

$F_1(e_1)$ =U , $F_1(e_2) = \{u_2\}$

$F_2(e_1) = \{u_1\}$, $F_2(e_2) = U$

$F_3(e_1) = \{u_1\}$, $F_3(e_2) = \{u_2\}$

Then (U, τ, E) is a soft topological space over U. Also (U, τ, E) is a soft T_1 –space over U but not a soft T_2 –space because $h_1, h_2 \in U$ and there do not exit any soft open sets $(F, E)\ and\ (G, E)$ in U such that $h_1 \in (F, E)$, $h_2 \in (G, E)$ and $(F, E) \cap (G, E) = \emptyset$

Now consider the following soft topology on U.

$\tau = \{\emptyset, \widetilde{U}, (F_1, E)\}$, where

$F_1(e_1) = U,\ F_1(e_2) = \{u_2\}$

Then (U, τ, E) is a soft topological space over U. Also (U, τ, E) is a soft T_0 –space over U but not a soft T_1 –space because $h_1, h_2 \in U$ but there do not exit soft open sets $(F, E)\ and\ (G, E)$ such that $h_1 \in (F, E)$, $h_2 \notin (F, E)$ and$h_2 \in (G, E)$, $h_1 \notin (G, E)$.

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