IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 5 Ser. III (Sep. – Oct. 2020), PP 38-52
www.iosrjournals.org

On Generalized (r,s,t,u,v)-Numbers

Yüksel Soykan
Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey

Abstract. In this paper, we introduce the generalized (r,s,t,u,v) numbers sequences and we deal with, in detail, three special cases which we call them (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

2010 Mathematics Subject Classification. 11B39, 11B83.

Keywords. (r,s,t,u,v) numbers, Lucas (r,s,t,u,v) numbers, modified (r,s,t,u,v) numbers, generalized Pentanacci numbers.

Date of Submission: 03-10-2020 Date of Acceptance: 17-10-2020

---

1. Introduction

The generalized (r,s,t,u,v) sequence (the generalized Pentanacci sequence or 5-step Fibonacci sequence)

$$\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$$

is defined by the fifth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \qquad W_0 = a, W_1 = b, W_2 = c, W_3 = d, W_4 = e \tag{1.1}$$

where the initial values W_0, W_1, W_2, W_3, W_4 are arbitrary complex (or real) numbers and r, s, t, u, v are real numbers. Pentanacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [3,4,5]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-(n-1)} - \frac{t}{v}W_{-(n-2)} - \frac{s}{v}W_{-(n-3)} - \frac{r}{v}W_{-(n-4)} + \frac{1}{v}W_{-(n-5)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (1.1) holds for all integer n.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t, u and initial values.

Table 1. A few special case of generalized Pentanacci sequences.

Sequences (Numbers)	Notation	OEIS [6]	Ref
Pentanacci	$\{P_n\} = \{W_n(0,1,1,2,4;1,1,1,1,1)\}$	A001591	[8]
Pentanacci-Lucas	$\{Q_n\} = \{W_n(5,1,3,7,15;1,1,1,1,1)\}$	A074048	[8]
fifth order Pell	$\{P_n^{(5)}\} = \{W_n(0,1,2,5,13;2,1,1,1,1)\}$	A141448	[9]
fifth order Pell-Lucas	$\{Q_n^{(5)}\} = \{W_n(5,2,6,17,46;2,1,1,1,1)\}$		[9]
modified fifth-order Pell	$\{E_n^{(5)}\} = \{W_n(0,1,1,3,8;2,1,1,1,1)\}$		[9]
5-primes	$\{G_n\} = \{W_n(0,0,0,1,2;2,3,5,7,11)\}$		[10]
Lucas 5-primes	$\{H_n\} = \{W_n(5,2,10,41,150;2,3,5,7,11)\}$		[10]
modified 5-primes	$\{E_n\} = \{W_n(0,0,0,1,1;2,3,5,7,11)\}$		[10]

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

As $\{W_n\}$ is a fifth order recurrence sequence (difference equation), it's characteristic equation is

$$x^5 - rx^4 - sx^3 - tx^2 - ux - v = 0 \qquad (1.2)$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda$. Note that we have the following identities:

$$\begin{aligned}
\alpha+\beta+\gamma+\delta+\lambda &= r,\\
\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta &= -s,\\
\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta &= t,\\
\alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta &= -u\\
\alpha\beta\gamma\delta\lambda &= v.
\end{aligned}$$

Generalized Pentanacci numbers can be expressed, for all integers n, using Binet's formula.

THEOREM 1. *(Binet's formula of generalized (r,s,t,u,v) numbers (generalized Pentanacci numbers))*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}+\frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \qquad (1.3)\\
&\quad+\frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned}$$

where

$$\begin{aligned}
p_1 &= W_4-(\beta+\gamma+\delta+\lambda)W_3+(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_2-(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_1+(\beta\lambda\gamma\delta)W_0,\\
p_2 &= W_4-(\alpha+\gamma+\delta+\lambda)W_3+(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)W_2-(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)W_1+(\alpha\lambda\gamma\delta)W_0,\\
p_3 &= W_4-(\alpha+\beta+\delta+\lambda)W_3+(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)W_2-(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)W_1+(\alpha\beta\lambda\delta)W_0,\\
p_4 &= W_4-(\alpha+\beta+\gamma+\lambda)W_3+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)W_2-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)W_1+(\alpha\beta\lambda\gamma)W_0,\\
p_5 &= W_4-(\alpha+\beta+\gamma+\delta)W_3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)W_2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)W_1+(\alpha\beta\gamma\delta)W_0.
\end{aligned}$$

Usually, it is customary to choose r,s,t,u,v so that the Equ. (1.2) has at least one real (say α) solutions.

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n, (see [1], this result of Howard and Saidak [1] is even true in the case of higher-order recurrence relations).

(1.3) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n$$

where

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)},\\
A_2 &= \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)},\\
A_3 &= \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)},\\
A_4 &= \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)},\\
A_5 &= \frac{p_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}.
\end{aligned}$$

We have the following formula: for $n = 1, 2, 3, \ldots$ we have

$$W_{-n} = \frac{\alpha^{-n}X_1p_1 + \beta^{-n}X_2p_2 + \gamma^{-n}X_3p_3 + \delta^{-n}X_4p_4 + \lambda^{-n}X_5p_5}{\alpha^nX_1p_1 + \beta^nX_2p_2 + \gamma^nX_3p_3 + \delta^nX_4p_4 + \lambda^nX_5p_5}W_n$$

where

$$\begin{aligned}
X_1 &= (\delta-\gamma)(\lambda-\delta)(\lambda-\gamma)(\beta-\delta)(\beta-\gamma)(\beta-\lambda),\\
X_2 &= (\gamma-\delta)(\lambda-\delta)(\lambda-\gamma)(\alpha-\delta)(\alpha-\gamma)(\alpha-\lambda),\\
X_3 &= (\lambda-\delta)(\beta-\delta)(\beta-\lambda)(\alpha-\delta)(\alpha-\lambda)(\alpha-\beta),\\
X_4 &= (\gamma-\lambda)(\beta-\gamma)(\beta-\lambda)(\alpha-\gamma)(\alpha-\lambda)(\alpha-\beta),\\
X_5 &= (\delta-\gamma)(\beta-\delta)(\beta-\gamma)(\alpha-\delta)(\alpha-\gamma)(\alpha-\beta).
\end{aligned}$$

We can also give Binet's formula of the generalized (r, s, t, u, v) numbers (the generalized Pentanacci numbers) for the negative subscripts as follows: for $n = 1, 2, 3, \ldots$ we have

$$\begin{aligned}
W_{-n} &= \frac{\alpha^4 - r\alpha^3 - s\alpha^2 - t\alpha - u}{v(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}p_1\alpha^{1-n} + \frac{\beta^4 - r\beta^3 - s\beta^2 - t\beta - u}{v(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}p_2\beta^{1-n}\\
&+ \frac{\gamma^4 - r\gamma^3 - s\gamma^2 - t\gamma - u}{v(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}p_3\gamma^{1-n} + \frac{\delta^4 - r\delta^3 - s\delta^2 - t\delta - u}{v(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}p_4\delta^{1-n}\\
&+ \frac{\lambda^4 - r\lambda^3 - s\lambda^2 - t\lambda - u}{v(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}p_5\lambda^{1-n}.
\end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_nx^n$ of the sequence W_n. The following lemma is a special case of a well known formula of generating functions of the generalized m-step Fibonacci numbers which can be found in the literature (see for example [11]). For completeness, we include the proof.

LEMMA 2. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_nx^n$ is the ordinary generating function of the generalized (r, s, t, u, v) sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_nx^n$ is given by*

$$\sum_{n=0}^{\infty} W_nx^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3 + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)x^4}{1 - rx - sx^2 - tx^3 - ux^4 - vx^5}. \tag{1.4}$$

Proof. Using the definition of generalized (r,s,t,u,v) numbers, and substracting $rx\sum_{n=0}^{\infty}W_nx^n$, $sx^2\sum_{n=0}^{\infty}W_nx^n$, $tx^3\sum_{n=0}^{\infty}W_nx^n$, $ux^4\sum_{n=0}^{\infty}W_nx^n$ and $vx^5\sum_{n=0}^{\infty}W_nx^n$ from $\sum_{n=0}^{\infty}W_nx^n$ we obtain

$$\begin{aligned}(1-rx-sx^2-tx^3-ux^4-vx^5)\sum_{n=0}^{\infty}W_nx^n &= \sum_{n=0}^{\infty}W_nx^n-rx\sum_{n=0}^{\infty}W_nx^n-sx^2\sum_{n=0}^{\infty}W_nx^n\\ &\quad -tx^3\sum_{n=0}^{\infty}W_nx^n-ux^4\sum_{n=0}^{\infty}W_nx^n-vx^5\sum_{n=0}^{\infty}W_nx^n\\ &= \sum_{n=0}^{\infty}W_nx^n-r\sum_{n=0}^{\infty}W_nx^{n+1}-s\sum_{n=0}^{\infty}W_nx^{n+2}\\ &\quad -t\sum_{n=0}^{\infty}W_nx^{n+3}-u\sum_{n=0}^{\infty}W_nx^{n+4}-v\sum_{n=0}^{\infty}W_nx^{n+5}\\ &= \sum_{n=0}^{\infty}W_nx^n-r\sum_{n=1}^{\infty}W_{n-1}x^n-s\sum_{n=2}^{\infty}W_{n-2}x^n\\ &\quad -t\sum_{n=3}^{\infty}W_{n-3}x^n-u\sum_{n=4}^{\infty}W_{n-4}x^n-v\sum_{n=5}^{\infty}W_{n-5}x^n\end{aligned}$$

and so

$$\begin{aligned}&(1-rx-sx^2-tx^3-ux^4-vx^5)\sum_{n=0}^{\infty}W_nx^n\\ =\;&(W_0+W_1x+W_2x^2+W_3x^3+W_4x^4)-r(W_0x+W_1x^2+W_2x^3+W_3x^4)\\ &-s(W_0x^2+W_1x^3+W_2x^4)-t(W_0x^3+W_1x^4)-uW_0x^4\\ &+\sum_{n=5}^{\infty}(W_n-2W_{n-1}-sW_{n-2}-tW_{n-3}-uW_{n-4}-vW_{n-5})x^n\\ =\;&W_0+(W_1-rW_0)x+(W_2-rW_1-sW_0)x^2+(W_3-rW_2-sW_1-tW_0)x^3\\ &+(W_4-rW_3-sW_2-tW_1-uW_0)x^4.\end{aligned}$$

Rearranging above equation, we obtain (1.4). □

We next find Binet formula of generalized (r,s,t,u,v) numbers $\{W_n\}$ by the use of generating function for W_n.

THEOREM 3. *(Binet's formula of generalized (r,s,t,u,v) numbers)*

$$\begin{aligned}W_n &= \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}+\frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}+\frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \quad (1.5)\\ &\quad +\frac{q_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}+\frac{q_5\lambda^n}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}\end{aligned}$$

where

$$\begin{aligned}q_1 &= W_0\alpha^4+(W_1-rW_0)\alpha^3+(W_2-rW_1-sW_0)\alpha^2+(W_3-rW_2-sW_1-tW_0)\alpha+(W_4-rW_3-sW_2-tW_1-vW_0),\\ q_2 &= W_0\beta^4+(W_1-rW_0)\beta^3+(W_2-rW_1-sW_0)\beta^2+(W_3-rW_2-sW_1-tW_0)\beta+(W_4-rW_3-sW_2-tW_1-vW_0),\\ q_3 &= W_0\gamma^4+(W_1-rW_0)\gamma^3+(W_2-rW_1-sW_0)\gamma^2+(W_3-rW_2-sW_1-tW_0)\gamma+(W_4-rW_3-sW_2-tW_1-vW_0),\\ q_4 &= W_0\delta^4+(W_1-rW_0)\delta^3+(W_2-rW_1-sW_0)\delta^2+(W_3-rW_2-sW_1-tW_0)\delta+(W_4-rW_3-s3W_2-tW_1-vW_0),\\ q_5 &= W_0\lambda^4+(W_1-rW_0)\lambda^3+(W_2-rW_1-sW_0)\lambda^2+(W_3-rW_2-sW_1-tW_0)\lambda+(W_4-rW_3-sW_2-tW_1-vW_0).\end{aligned}$$

Proof. Let

$$h(x) = 1 - rx - sx^2 - tx^3 - ux^4 - vx^5.$$

Then for some $\alpha, \beta, \gamma, \delta$ and λ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)$$

i.e.,

$$1 - rx - sx^2 - tx^3 - ux^4 - vx^5 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x) \tag{1.6}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$ and $\frac{1}{\lambda}$ are the roots of $h(x)$. This gives $\alpha, \beta, \gamma, \delta$ and λ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{r}{x} - \frac{s}{x^2} - \frac{t}{x^3} - \frac{u}{x^4} - \frac{v}{x^5} = 0.$$

This implies $x^5 - rx^4 - sx^3 - tx^2 - ux - v = 0$. Now, by (1.4) and (1.6), it follows that

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3 + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)x^4}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)}.$$

Then we write

$$\begin{aligned} & \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3 + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)x^4}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x)} \\ = \; & \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} + \frac{B_4}{(1-\delta x)} + \frac{B_5}{(1-\lambda x)}. \end{aligned} \tag{1.7}$$

So

$$\begin{aligned} & W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 + (W_3 - rW_2 - sW_1 - tW_0)x^3 + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)x^4 \\ = \; & B_1(1-\beta x)(1-\gamma x)(1-\delta x)(1-\lambda x) + B_2(1-\alpha x)(1-\gamma x)(1-\delta x)(1-\lambda x) \\ & + B_3(1-\alpha x)(1-\beta x)(1-\delta x)(1-\lambda x) + B_4(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\lambda x) \\ & + B_5(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x). \end{aligned}$$

If we consider $x = \frac{1}{\alpha}$, we get

$W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2} + (W_3 - rW_2 - sW_1 - tW_0)\frac{1}{\alpha^3} + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)\frac{1}{\alpha^4}$
$= B_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})(1 - \frac{\delta}{\alpha})(1 - \frac{\lambda}{\alpha})$.

This gives

$$\begin{aligned} B_1 &= \frac{\alpha^4(W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2} + (W_3 - rW_2 - sW_1 - tW_0)\frac{1}{\alpha^3} + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)\frac{1}{\alpha^4})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} \\ &= \frac{W_0\alpha^4 + (W_1 - rW_0)\alpha^3 + (W_2 - rW_1 - sW_0)\alpha^2 + (W_3 - rW_2 - sW_1 - tW_0)\alpha + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} B_2 &= \frac{W_0\beta^4 + (W_1 - rW_0)\beta^3 + (W_2 - rW_1 - sW_0)\beta^2 + (W_3 - rW_2 - sW_1 - tW_0)\beta + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}, \\ B_3 &= \frac{W_0\gamma^4 + (W_1 - rW_0)\gamma^3 + (W_2 - rW_1 - sW_0)\gamma^2 + (W_3 - rW_2 - sW_1 - tW_0)\gamma + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}, \\ B_4 &= \frac{W_0\delta^4 + (W_1 - rW_0)\delta^3 + (W_2 - rW_1 - sW_0)\delta^2 + (W_3 - rW_2 - sW_1 - tW_0)\delta + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}, \\ B_5 &= \frac{W_0\lambda^4 + (W_1 - rW_0)\lambda^3 + (W_2 - rW_1 - sW_0)\lambda^2 + (W_3 - rW_2 - sW_1 - tW_0)\lambda + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}. \end{aligned}$$

Thus (1.7) can be written as

$$\sum_{n=0}^{\infty} W_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1} + B_4(1-\delta x)^{-1} + B_5(1-\lambda x)^{-1}.$$

This gives

$$\begin{aligned}\sum_{n=0}^{\infty} W_n x^n &= B_1\sum_{n=0}^{\infty}\alpha^n x^n + B_2\sum_{n=0}^{\infty}\beta^n x^n + B_3\sum_{n=0}^{\infty}\gamma^n x^n + B_4\sum_{n=0}^{\infty}\delta^n x^n + B_5\sum_{n=0}^{\infty}\lambda^n x^n \\ &= \sum_{n=0}^{\infty}(B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n + B_5\lambda^n)x^n.\end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$W_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n + B_4\delta^n + B_5\lambda^n$$

and then we get (1.5). □

Note that from (1.3) and (1.5) we have

$$\begin{aligned}
& W_4 - (\beta+\gamma+\delta+\lambda)W_3 + (\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_2 - (\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_1 + (\beta\lambda\gamma\delta)W_0 \\
= \; & W_0\alpha^4 + (W_1 - rW_0)\alpha^3 + (W_2 - rW_1 - sW_0)\alpha^2 + (W_3 - rW_2 - sW_1 - tW_0)\alpha + (W_4 - rW_3 - sW_2 - tW_1 - uW_0), \\
& W_4 - (\alpha+\gamma+\delta+\lambda)W_3 + (\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)W_2 - (\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)W_1 + (\alpha\lambda\gamma\delta)W_0 \\
= \; & W_0\beta^4 + (W_1 - rW_0)\beta^3 + (W_2 - rW_1 - sW_0)\beta^2 + (W_3 - rW_2 - sW_1 - tW_0)\beta + (W_4 - rW_3 - sW_2 - tW_1 - uW_0), \\
& W_4 - (\alpha+\beta+\delta+\lambda)W_3 + (\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)W_2 - (\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)W_1 + (\alpha\beta\lambda\delta)W_0 \\
= \; & W_0\gamma^4 + (W_1 - rW_0)\gamma^3 + (W_2 - rW_1 - sW_0)\gamma^2 + (W_3 - rW_2 - sW_1 - tW_0)\gamma + (W_4 - rW_3 - sW_2 - tW_1 - uW_0), \\
& W_4 - (\alpha+\beta+\gamma+\lambda)W_3 + (\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)W_2 - (\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)W_1 + (\alpha\beta\lambda\gamma)W_0 \\
= \; & W_0\delta^4 + (W_1 - rW_0)\delta^3 + (W_2 - rW_1 - sW_0)\delta^2 + (W_3 - rW_2 - sW_1 - tW_0)\delta + (W_4 - rW_3 - sW_2 - tW_1 - uW_0), \\
& W_4 - (\alpha+\beta+\gamma+\delta)W_3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)W_2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)W_1 + (\alpha\beta\gamma\delta)W_0 \\
= \; & W_0\lambda^4 + (W_1 - rW_0)\lambda^3 + (W_2 - rW_1 - sW_0)\lambda^2 + (W_3 - rW_2 - sW_1 - tW_0)\lambda + (W_4 - rW_3 - sW_2 - tW_1 - uW_0).
\end{aligned}$$

In this paper, we define and investigate, in detail, three special cases of the generalized (r,s,t,u,v) sequence $\{W_n\}$ which we call them (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) sequences. (r,s,t,u,v) sequence $\{G_n\}_{n\geq 0}$, Lucas (r,s,t,u,v) sequence $\{H_n\}_{n\geq 0}$ and modified (r,s,t,u,v) sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the fifth-order recurrence relations

$$\begin{aligned}
G_{n+5} &= rG_{n+4} + sG_{n+3} + tG_{n+2} + uG_{n+1} + vG_n, \\
G_0 &= 0, G_1 = 1, G_2 = r, G_3 = r^2 + s, G_4 = r^3 + 2sr + t, \\
H_{n+5} &= rH_{n+4} + sH_{n+3} + tH_{n+2} + uH_{n+1} + vH_n, \\
H_0 &= 5, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t, H_4 = r^4 + 4r^2s + 4tr + 2s^2 + 4u, \\
E_{n+5} &= rE_{n+4} + sE_{n+3} + tE_{n+2} + uE_{n+1} + vE_n, \\
E_0 &= 1, E_1 = r - 1, E_2 = -r + s + r^2, E_3 = r^3 - r^2 + 2sr - s + t, E_4 = r^4 - r^3 + 3r^2s - 2rs + 2tr + s^2 - t + u.
\end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{u}{v}G_{-(n-1)} - \frac{t}{v}G_{-(n-2)} - \frac{s}{v}G_{-(n-3)} - \frac{r}{v}G_{-(n-4)} + \frac{1}{v}G_{-(n-5)}, \quad (1.8)$$

$$H_{-n} = -\frac{u}{v}H_{-(n-1)} - \frac{t}{v}H_{-(n-2)} - \frac{s}{v}H_{-(n-3)} - \frac{r}{v}H_{-(n-4)} + \frac{1}{v}H_{-(n-5)}, \quad (1.9)$$

$$E_{-n} = -\frac{u}{v}E_{-(n-1)} - \frac{t}{v}E_{-(n-2)} - \frac{s}{v}E_{-(n-3)} - \frac{r}{v}E_{-(n-4)} + \frac{1}{v}E_{-(n-5)}, \quad (1.10)$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (1.8), (1.9) and (1.10) hold for all integers n.

Next, we present the first few values of the (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers with positive and negative subscripts:

Table 2. The first few values of the special fifth-order numbers with positive and negative subscripts.

n	0	1	2	3	4
G_n	0	1	r	r^2+s	$r^3+2sr+t$
G_{-n}		0	0	0	$\frac{1}{v}$
H_n	5	r	$2s+r^2$	$r^3+3sr+3t$	$r^4+4r^2s+4tr+2s^2+4u$
H_{-n}		$-\frac{u}{v}$	$\frac{1}{v^2}(u^2-2tv)$	$-\frac{1}{v^3}(u^3-3tuv+3sv^2)$	$\frac{1}{v^4}(2t^2v^2-4tu^2v+u^4+4suv^2-4rv^3)$
E_n	1	$r-1$	$-r+s+r^2$	$r^3-r^2+2sr-s+t$	$r^4-r^3+3r^2s-2rs+2tr+s^2-t+u$
E_{-n}		0	0	0	$-\frac{1}{v}$

Some special cases of (r,s,t,u,v) sequence $\{G_n(0,1,r,r^2+s,r^3+2sr+t;r,s,t,u,v)\}$ and Lucas (r,s,t,u,v) sequence $\{H_n(4,r,2s+r^2,r^3+3sr+3t,r^4+4r^2s+4tr+2s^2+4u;r,s,t,u,v)\}$ are as follows:

(1) $G_n(0,1,1,2,4;1,1,1,1,1) = P_n$, Pentanacci sequence,

(2) $H_n(5,1,3,7,15;1,1,1,1,1) = Q_n$, Pentanacci-Lucas sequence,

(3) $G_n(0,1,2,5,13;2,1,1,1,1) = P_n$, fifth-order Pell sequence,

(4) $H_n(5,2,6,17,46;2,1,1,1,1) = Q_n$, fifth-order Pell-Lucas sequence,

For all integers n, (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers (using initial conditions in (1.3) or (1.5)) can be expressed using Binet's formulas as

$$\begin{aligned}
G_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\
&\quad + \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \\
H_n &= \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n, \\
E_n &= \frac{(\alpha-1)\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{(\beta-1)\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{(\gamma-1)\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\
&\quad + \frac{(\delta-1)\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{(\lambda-1)\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)},
\end{aligned}$$

respectively. Note that for all n we have

$$E_n = G_{n+1} - G_n.$$

Lemma 2 gives the following results as particular examples (generating functions of (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers).

COROLLARY 4. *Generating functions of (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers are*

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-rx-sx^2-tx^3-ux^4-vx^5}, \\
\sum_{n=0}^{\infty} H_n x^n &= \frac{5-4rx-3sx^2-2tx^3-ux^4}{1-rx-sx^2-tx^3-ux^4-vx^5}, \\
\sum_{n=0}^{\infty} E_n x^n &= \frac{1-x}{1-rx-sx^2-tx^3-ux^4-vx^5},
\end{aligned}$$

respectively.

Proof. In Lemma 2, take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r, G_3 = r^2 + s, G_4 = r^3 + 2sr + t$, $W_n = H_n$ with $H_0 = 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t, H_4 = r^4 + 4r^2s + 4tr + 2s^2 + 4u$, and $W_n = E_n$ with $E_0 = 1, E_1 = r - 1, E_2 = -r + s + r^2, E_3 = r^3 - r^2 + 2sr - s + t, E_4 = r^4 - r^3 + 3r^2s - 2rs + 2tr + s^2 - t + u$, respectively. □

2. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized (r,s,t,u,v) sequence $\{W_n\}_{n\geq 0}$.

THEOREM 5 (Simson Formula of Generalized (r,s,t,u,v)-Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+4} & W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+3} & W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} & W_{n-3} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} & W_{n-4} \end{vmatrix} = v^n \begin{vmatrix} W_4 & W_3 & W_2 & W_1 & W_0 \\ W_3 & W_2 & W_1 & W_0 & W_{-1} \\ W_2 & W_1 & W_0 & W_{-1} & W_{-2} \\ W_1 & W_0 & W_{-1} & W_{-2} & W_{-3} \\ W_0 & W_{-1} & W_{-2} & W_{-3} & W_{-4} \end{vmatrix}. \tag{2.1}$$

Proof. (2.1) is given in Soykan [7]. □

The previous theorem gives the following results as particular examples.

COROLLARY 6. *For all integers n, Simson formula of (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers are given as*

$$\begin{vmatrix} G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} & G_{n-4} \end{vmatrix} = v^{n-1},$$

$$\begin{vmatrix} H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} \end{vmatrix} = v^{n-4} f(r,s,t,u,v),$$

$$\begin{vmatrix} E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \end{vmatrix} = v^{n-1}(r+s+t+u+v-1),$$

where

$f(r,s,t,u,v) = 256r^5v^3 - 192r^4suv^2 - 128r^4t^2v^2 + 144r^4tu^2v - 27r^4u^4 + 144r^3s^2tv^2 - 6r^3s^2u^2v - 80r^3st^2uv + 18r^3stu^3 + 1600r^3sv^3 + 16r^3t^4v - 4r^3t^3u^2 - 160r^3tuv^2 + 36r^3u^3v - 27r^2s^4v^2 + 18r^2s^3tuv - 4r^2s^3u^3 - 4r^2s^2t^3v + r^2s^2t^2u^2 - 1020r^2s^2uv^2 - 560r^2st^2v^2 + 746r^2stu^2v - 144r^2su^4 - 24r^2t^3uv + 6r^2t^2u^3 + 2000r^2tv^3 - 50r^2u^2v^2 + 630rs^3tv^2 - 24rs^3u^2v - 356rs^2t^2uv + 80rs^2tu^3 + 2250rs^2v^3 + 72rst^4v - 18rst^3u^2 - 2050rstuv^2 + 160rsu^3v - 900rt^3v^2 + 1020rt^2u^2v - 192rtu^4 + 2500ruv^3 - 108s^5v^2 + 72s^4tuv - 16s^4u^3 - 16s^3t^3v + 4s^3t^2u^2 - 900s^3uv^2 + 825s^2t^2v^2 + 560s^2tu^2v - 128s^2u^4 - 630st^3uv + 144st^2u^3 + 3750stv^3 - 2000su^2v^2 + 108t^5v - 27t^4u^2 - 2250t^2uv^2 + 1600tu^3v - 256u^5 + 3125v^4,$

respectively.

3. Some Identities

In this section, we obtain some identities of (r,s,t,u,v), Lucas (r,s,t,u,v) and modified (r,s,t,u,v) numbers. We can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

LEMMA 7. *The following equalities are true:*

(a): $v^3H_n = -(u^3 - 3tuv + 3sv^2)G_{n+4} + (ru^3 - 2tv^2 + u^2v + 3rsv^2 - 3rtuv)G_{n+3} + (3s^2v^2 + su^3 - 3tsuv - ru^2v - uv^2 + 2rtv^2)G_{n+2} + (-3t^2uv + tu^3 + 5stv^2 - su^2v + ruv^2 + 5v^3)G_{n+1} + (2t^2v^2 - 4tu^2v + u^4 + 4suv^2 - 4rv^3)G_n.$

(b): $v^2H_n = (u^2 - 2tv)G_{n+3} - (ru^2 + vu - 2rtv)G_{n+2} + (-su^2 + ruv + 5v^2 + 2stv)G_{n+1} - (-2t^2v + tu^2 - suv + 4rv^2)G_n - (u^3 - 3tuv + 3sv^2)G_{n-1}.$

(c): $vH_n = -uG_{n+2} + (5v + ru)G_{n+1} + (su - 4rv)G_n + (tu - 3sv)G_{n-1} + (u^2 - 2tv)G_{n-2}.$

(d): $H_n = 5G_{n+1} - 4rG_n - 3sG_{n-1} - 2tG_{n-2} - uG_{n-3}.$

(e): $H_n = rG_n + 2sG_{n-1} + 3tG_{n-2} + 4uG_{n-3} + 5vG_{n-4}.$

Proof. Note that all the identities hold for all integers n. We prove (d). To show d), writing

$$H_n = a \times G_{n+1} + b \times G_n + c \times G_{n-1} + d \times G_{n-2} + e \times G_{n-3}$$

and solving the system of equations

$$\begin{aligned} H_0 &= a \times G_1 + b \times G_0 + c \times G_{-1} + d \times G_{-2} + e \times G_{-3} \\ H_1 &= a \times G_2 + b \times G_1 + c \times G_0 + d \times G_{-1} + e \times G_{-2} \\ H_2 &= a \times G_3 + b \times G_2 + c \times G_1 + d \times G_0 + e \times G_{-1} \\ H_3 &= a \times G_4 + b \times G_3 + c \times G_2 + d \times G_1 + e \times G_0 \\ H_4 &= a \times G_5 + b \times G_4 + c \times G_3 + d \times G_2 + e \times G_1 \end{aligned}$$

we find that $a = 5, b = -4r, c = -3s, d = -2t, e = -u$. The other equalitities can be proved similarly. □

We present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

LEMMA 8. *The following equalities are true:*

(a): $v^2E_n = (u+v)G_{n+6} - (v+ru+rv)G_{n+5} - (su-rv+sv)G_{n+4} - (tu-sv+tv)G_{n+3} - (u^2+vu-tv)G_{n+2}$.

(b): $vE_n = -G_{n+5} + rG_{n+4} + sG_{n+3} + tG_{n+2} + (u+v)G_{n+1}$.

(c): $E_n = G_{n+1} - G_n$.

(d): $E_n = (r-1)G_n + sG_{n-1} + tG_{n-2} + uG_{n-3} + vG_{n-4}$.

(e): $(r+s+t+u+v-1)G_n = E_{n+4} - (r-1)E_{n+3} - (r+s-1)E_{n+2} - (r+s+t-1)E_{n+1} - (r+s+t+u-1)E_n$.

(f): $(r+s+t+u+v-1)G_n = E_{n+3} - (r-1)E_{n+2} - (r+s-1)E_{n+1} - (r+s+t-1)E_n + vE_{n-1}$.

(g): $(r+s+t+u+v-1)G_n = E_{n+2} - (r-1)E_{n+1} - (r+s-1)E_n + (u+v)E_{n-1} + vE_{n-2}$.

(h): $(r+s+t+u+v-1)G_n = E_{n+1} - (r-1)E_n + (t+u+v)E_{n-1} + (u+v)E_{n-2} + vE_{n-3}$.

Next, we give a basic relation between $\{H_n\}$ and $\{E_n\}$.

LEMMA 9. *The following equality is true:*

$(r+s+t+u+v-1)H_n = (r+2s+3t+4u+5v)E_n + (2s+3t+4u+5v-rs-2rt-3ru-4rv)E_{n-1} - (2rt-4u-5v-3t+3ru+st+4rv+2su+3sv)E_{n-2} - (3ru-5v-4u+4rv+2su+3sv+tu+2tv)E_{n-3} - v(4r+3s+2t+u-5)E_{n-4}$.

Note that all the identities in the above 3 lemmas can be proved by induction as well.

We now present a few special identities for the modified (r,s,t,u,v) sequence $\{E_n\}$.

THEOREM 10. *(Catalan's identity) For all integers n and m, the following identity holds*

$$E_{n+m}E_{n-m} - E_n^2 = G_{m+n+1}(G_{-m+n+1} - G_{-m+n}) + G_{m+n}(G_{-m+n} - G_{-m+n+1}) - (G_{n+1} - G_n)^2.$$

Proof. We use the identity

$$E_n = G_{n+1} - G_n. \quad \square$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the modified (r,s,t,u,v) sequnce.

COROLLARY 11. *(Cassini's identity) For all integers numbers n and m, the following identity holds*

$$E_{n+1}E_{n-1} - E_n^2 = (G_{n+2} - G_{n+1})(G_n - G_{n-1}) - (G_{n+1} - G_n)^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham's identities can also be obtained by using $E_n = G_{n+1} - G_n$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham's identities of modified (r,s,t,u,v) sequence $\{E_n\}$.

THEOREM 12. *Let n and m be any integers. Then the following identities are true:*

(a): *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = G_{m+2}\left(G_{n+1} - G_n\right) + G_{m+1}\left(G_n - G_{n+2}\right) + G_m\left(G_{n+2} - G_{n+1}\right).$$

(b): *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (G_{n+3} - G_{n+2})(G_{n+2} - G_{n+1})(G_n - G_{n-1})(G_{n-1} - G_{n-2}) - (G_{n+1} - G_n)^4.$$

(c): *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (G_{n+2} - G_{n+1})(G_{n+3} - G_{n+2})(G_{n+7} - G_{n+6}) - (G_{n+4} - G_{n+3})^3.$$

Proof. Use the identity $E_n = G_{n+1} - G_n$. □

4. Linear Sums

The following theorem presents sum formulas of generalized (r,s,t,u,v) numbers.

THEOREM 13. *For all integers m and j, we have*

$$\sum_{k=0}^{n} W_{mk+j} = \frac{\Lambda + \Psi_1}{\Omega} \tag{4.1}$$

where

$\Lambda = 6W_{mn+3m+j} + 6(1-H_m)W_{mn+2m+j} + 3(H_m^2 - H_{2m} - 2H_m + 2)W_{mn+m+j} + 6W_{mn-m+j}v^m - 6v^m\left(H_{-m} - 1\right)W_{mn+j}$,

$\Psi_1 = -6W_{3m+j} - 6(1-H_m)W_{2m+j} - 3(H_m^2 - H_{2m} - 2H_m + 2)W_{m+j} - 6W_{-m+j}v^m + (H_m^3 - 3H_m^2 - 3H_{2m}H_m + 2H_{3m} + 3H_{2m} + 6H_m - 6)W_j$,

$\Omega = H_m^3 - 3H_m^2 - 3H_{2m}H_m + 2H_{3m} + 3H_{2m} + 6H_m - 6v^m\left(H_{-m} - 1\right) - 6.$

Proof. Note that

$$\begin{aligned}
\sum_{k=0}^{n} W_{mk+j} &= W_{mn+j} + \sum_{k=0}^{n-1} W_{mk+j} = W_{mn+j} + \sum_{k=0}^{n-1}(A_1\alpha^{mk+j} + A_2\beta^{mk+j} + A_3\gamma^{mk+j} + A_4\delta^{mk+j} + A_5\lambda^{mk+j}) \\
&= W_{mn+j} + A_1\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right) + A_2\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right) + A_3\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right) \\
&\quad + A_4\delta^j\left(\frac{\delta^{mn}-1}{\delta^m-1}\right) + A_5\lambda^j\left(\frac{\lambda^{mn}-1}{\lambda^m-1}\right).
\end{aligned}$$

Simplifying the last equalities in the last two expression imply (4.1) as required. □

Note that (4.1) can be written in the following form:

$$\sum_{k=1}^{n} W_{mk+j} = \frac{\Lambda + \Psi_2}{\Omega}$$

where

$\Psi_2 = -6W_{3m+j} - 6(1-H_m)W_{2m+j} - 3(H_m^2 - H_{2m} - 2H_m + 2)W_{m+j} - 6W_{-m+j}v^m + 6v^m\left(H_{-m} - 1\right)W_j.$

As special cases of the above theorem, we have the following corollaries. Firstly, as special cases of the above theorem, we have the following corollary for the generalized Pentanacci numbers.

COROLLARY 14. *The following identities hold:*

(1) $m=1,\ j=0$.

(a): $\sum_{k=0}^{n} P_k = \frac{1}{4}(P_{n+3} - P_{n+1} + 2P_n + P_{n-1} - 1)$.

(b): $\sum_{k=0}^{n} Q_k = \frac{1}{4}(Q_{n+3} - Q_{n+1} + 2Q_n + Q_{n-1} + 5)$.

(2) $m=-1,\ j=0$.

(a): $\sum_{k=0}^{n} P_{-k} = \frac{1}{4}(-P_{-n+1} - 3P_{-n-1} - 2P_{-n-2} - P_{-n-3} + 1)$.

(b): $\sum_{k=0}^{n} Q_{-k} = \frac{1}{4}(-Q_{-n+1} - 3Q_{-n-1} - 2Q_{-n-2} - Q_{-n-3} + 15)$.

(3) $m=4,\ j=-6$.

(a): $\sum_{k=0}^{n} P_{4k-6} = \frac{1}{16}(P_{4n+6} - 14P_{4n+2} - 13P_{4n-2} + 2P_{4n-6} + P_{4n-10} + 1)$.

(b): $\sum_{k=0}^{n} Q_{4k-6} = \frac{1}{16}(Q_{4n+6} - 14Q_{4n+2} - 13Q_{4n-2} + 2Q_{4n-6} + Q_{4n-10} - 145)$.

(4) $m=-3,\ j=2$.

(a): $\sum_{k=0}^{n} P_{-3k+2} = \frac{1}{16}(-P_{-3n+5} + 6P_{-3n+2} - 6P_{-3n-1} - 2P_{-3n-4} - P_{-3n-7} + 20)$.

(b): $\sum_{k=0}^{n} Q_{-3k+2} = \frac{1}{16}(-Q_{-3n+5} + 6Q_{-3n+2} - 6Q_{-3n-1} - 2Q_{-3n-4} - Q_{-3n-7} + 52)$.

Secondly, as special cases of the above theorem, we have the following corollary for the generalized fifth-order Pell numbers.

COROLLARY 15. *The following identities hold:*

(1) $m=1,\ j=0$.

(a): $\sum_{k=0}^{n} P_k = \frac{1}{5}(P_{n+3} - P_{n+2} - 2P_{n+1} + 2P_n + P_{n-1} - 1)$.

(b): $\sum_{k=0}^{n} Q_k = \frac{1}{5}(Q_{n+3} - Q_{n+2} - 2Q_{n+1} + 2Q_n + Q_{n-1} + 9)$.

(c): $\sum_{k=0}^{n} E_k = \frac{1}{5}(E_{n+3} - E_{n+2} - 2E_{n+1} + 2E_n + E_{n-1})$.

(2) $m=-1,\ j=0$.

(a): $\sum_{k=0}^{n} P_{-k} = \frac{1}{30}(-6P_{-n+1} + 6P_{-n} - 18P_{-n-1} - 12P_{-n-2} - 6P_{-n-3} + 6)$.

(b): $\sum_{k=0}^{n} Q_{-k} = \frac{1}{5}(-Q_{-n+1} + Q_{-n} - 3Q_{-n-1} - 2Q_{-n-2} - Q_{-n-3} + 16)$.

(c): $\sum_{k=0}^{n} E_{-k} = \frac{1}{5}(-E_{-n+1} + E_{-n} - 3E_{-n-1} - 2E_{-n-2} - E_{-n-3})$.

(3) $m=4,\ j=-6$.

(a): $\sum_{k=0}^{n} P_{4k-6} = \frac{1}{75}(P_{4n+6} - 45P_{4n+2} - 58P_{4n-2} + 6P_{4n-6} + P_{4n-10} + 5)$.

(b): $\sum_{k=0}^{n} Q_{4k-6} = \frac{1}{75}(Q_{4n+6} - 45Q_{4n+2} - 58Q_{4n-2} + 6Q_{4n-6} + Q_{4n-10} - 640)$.

(c): $\sum_{k=0}^{n} E_{4k-6} = \frac{1}{75}(E_{4n+6} - 45E_{4n+2} - 58E_{4n-2} + 6E_{4n-6} + E_{4n-10} - 5)$.

(4) $m=-3,\ j=2$.

(a): $\sum_{k=0}^{n} P_{-3k+2} = \frac{1}{35}(-P_{-3n+5} + 16P_{-3n+2} - 6P_{-3n-1} - 2P_{-3n-4} - P_{-3n-7} + 74)$.

(b): $\sum_{k=0}^{n} Q_{-3k+2} = \frac{1}{35}(-Q_{-3n+5} + 16Q_{-3n+2} - 6Q_{-3n-1} - 2Q_{-3n-4} - Q_{-3n-7} + 219)$.

(c): $\sum_{k=0}^{n} E_{-3k+2} = \frac{1}{35}(-E_{-3n+5} + 16E_{-3n+2} - 6E_{-3n-1} - 2E_{-3n-4} - E_{-3n-7} + 45)$.

5. Matrices Related with Generalized (r,s,t,u,v) Numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix} \quad (5.1)$$

For matrix formulation (5.1), see [2]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ r & s & t & u & v \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix}.$$

We define the square matrix A of order 5 as:

$$A = \begin{pmatrix} r & s & t & u & v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = v$. From (1.1) we have

$$\begin{pmatrix} W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix}. \quad (5.2)$$

and from (5.1) (or using (5.2) and induction) we have

$$\begin{pmatrix} W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = G_n$ in (5.2) we have

$$\begin{pmatrix} G_{n+4} \\ G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} r & s & t & u & v \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{5.3}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} + vG_{n-3} & tG_n + uG_{n-1} + vG_{n-2} & uG_n + vG_{n-1} & vG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} + vG_{n-4} & tG_{n-1} + uG_{n-2} + vG_{n-3} & uG_{n-1} + vG_{n-2} & vG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} + vG_{n-5} & tG_{n-2} + uG_{n-3} + vG_{n-4} & uG_{n-2} + vG_{n-3} & vG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} + vG_{n-6} & tG_{n-3} + uG_{n-4} + vG_{n-5} & uG_{n-3} + vG_{n-4} & vG_{n-3} \\ G_{n-3} & sG_{n-4} + tG_{n-5} + uG_{n-6} + vG_{n-7} & tG_{n-4} + uG_{n-5} + vG_{n-6} & uG_{n-4} + vG_{n-5} & vG_{n-4} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} + vW_{n-3} & tW_n + uW_{n-1} + vW_{n-2} & uW_n + vW_{n-1} & vW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} + vW_{n-4} & tW_{n-1} + uW_{n-2} + vW_{n-3} & uW_{n-1} + vW_{n-2} & vW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} & tW_{n-2} + uW_{n-3} + vW_{n-4} & uW_{n-2} + vW_{n-3} & vW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} + vW_{n-6} & tW_{n-3} + uW_{n-4} + vW_{n-5} & uW_{n-3} + vW_{n-4} & vW_{n-3} \\ W_{n-3} & sW_{n-4} + tW_{n-5} + uW_{n-6} + vW_{n-7} & tW_{n-4} + uW_{n-5} + vW_{n-6} & uW_{n-4} + vW_{n-5} & vW_{n-4} \end{pmatrix}.$$

THEOREM 16. *For all integers $m, n \geq 0$, we have*

(a): $B_n = A^n$.

(b): $C_1 A^n = A^n C_1$.

(c): $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a): By expanding the vectors on the both sides of (5.3) to 5-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b): Using (a) and definition of C_1, (b) follows.

(c): We have $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

□

Some properties of matrix A^n can be given as

$$A^n = rA^{n-1} + sA^{n-2} + tA^{n-3} + uA^{n-4} + vA^{n-5}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = v^n$$

for all integers m and n.

THEOREM 17. *For $m, n \geq 0$ we have*

$$\begin{aligned} W_{n+m} &= W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2} + vG_{m-3}) \\ &\quad + W_{n-2}(tG_m + uG_{m-1} + vG_{m-2}) + W_{n-3}(uG_m + vG_{m-1}) + vW_{n-4}G_m. \end{aligned} \tag{5.4}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. □

REMARK 18. *By induction, it can be proved that for all integers $m, n \leq 0$, (5.4) holds. So for all integers m, n, (5.4) is true.*

COROLLARY 19. *For all integers m, n, we have*

$$\begin{aligned} G_{n+m} &= G_n G_{m+1} + G_{n-1}(sG_m + tG_{m-1} + uG_{m-2} + vG_{m-3}) \\ &\quad + G_{n-2}(tG_m + uG_{m-1} + vG_{m-2}) + G_{n-3}(uG_m + vG_{m-1}) + vG_{n-4}G_m, \\ H_{n+m} &= H_n G_{m+1} + H_{n-1}(sG_m + tG_{m-1} + uG_{m-2} + vG_{m-3}) \\ &\quad + H_{n-2}(tG_m + uG_{m-1} + vG_{m-2}) + H_{n-3}(uG_m + vG_{m-1}) + vH_{n-4}G_m, \\ E_{n+m} &= E_n G_{m+1} + E_{n-1}(sG_m + tG_{m-1} + uG_{m-2} + vG_{m-3}) \\ &\quad + E_{n-2}(tG_m + uG_{m-1} + vG_{m-2}) + E_{n-3}(uG_m + vG_{m-1}) + vE_{n-4}G_m. \end{aligned}$$

References

[1] Howard, F.T., Saidak, F., Zhou's Theory of Constructing Identities, Congress Numer. 200, 225-237, 2010.

[2] Kalman, D., Generalized Fibonacci Numbers By Matrix Methods, Fibonacci Quarterly, 20(1), 73-76, 1982.

[3] Melham, R. S., Some Analogs of the Identity $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$, Fibonacci Quarterly, 305-311, 1999.

[4] Natividad, L. R., On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order, International Journal of Mathematics and Computing, 3 (2), 2013.

[5] Rathore, G.P.S., Sikhwal, O., Choudhary, R., Formula for finding nth Term of Fibonacci-Like Sequence of Higher Order, International Journal of Mathematics And its Applications, 4 (2-D), 75-80, 2016.

[6] Sloane, N.J.A., The on-line encyclopedia of integer sequences, http://oeis.org/

[7] Soykan, Y., Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2), 45-56, 2019.

[8] Soykan, Y., On Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers, Asian Research Journal of Mathematics, 16(9), 102-121, 2020. DOI: 10.9734/ARJOM/2020/v16i930224

[9] Soykan, Y., Properties of Generalized Fifth-Order Pell Numbers, Asian Research Journal of Mathematics, 15(3), 1-18, 2019.

[10] Soykan, Y., A Study On Generalized 5-primes Numbers, Journal of Scientific Perspectives, 4(3), 185-202, 2020, DOI: https://doi.org/10.26900/jsp.4.017.

[11] Stanley, R. P., Generating Functions, Studies in Combinatorics, MAA Studies in Mathematics, Math. Assoc. of America, Washington, D.C., 17, 100-141, 1978.