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Asymptotics approximations and expansions of Integrals with a matrix argument

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Abstract:
Asymptotic approximations are derived for integrals that depends on a Matrix. A matrix version analogous to Watson's lemma for scalar functions are obtained.
***Key Word**: Special functions; Asymptotics; Asymptotic Approximations; Asymptotic Expansions; Matrix; Gamma function of Matrices; Jordan Canonical For Watson’s lemma.*
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I. Introduction

In the past two decades generalization and extensions of scalar special functions to Matrix special functions have been developed. The Gamma matrix function, whose eigenvalues are all in the right open half-plane is introduced and studied in L. J_odar, J. Cort_es [9] for matrices in $\mathbb{C}^{r\times r}$. Hermite matrix polynomials are introduced by L. J_odar et al [8] and some of their properties are given in E. Defez, L. J_odar [3]. Other classical orthogonal polynomials as Laguerre and Chebyshev have been extended to orthogonal matrix polynomials, and some results have been investigated in L. J_odar, J. Sastre [11] and E. Defez, L. J_odar [4]. Relations between the Beta, Gamma and the Hypergeometric matrix function are given in L. J_odar, J. G. Cort_es [10] and R. S. Batahan [1]. These special functions of matrices have become an important tool in both theory and applications. The order of presentation in this article is as follows. In section 2 we provide basic necessary notation, definitions and auxiliary theorems that need to be cited in the sequel. In section 3 we extend the asymptotic methods of integration by parts and Watson's lemma to integrals with a matrix argument.

II. Preliminaries

This In this section we elaborate on some necessary language that is adopted from L. Jodar, J. Sastre [11] and N. J. Higham [7]. We also record some basic theorems from asymptotic analysis that can be found in e.g. W. Wasow [13] and A. Erdelyi [7] Denote by $\lambda_1, \cdots \lambda_n$ the distinct eigenvalues of a matrix $P \in \mathbb{C}^{r\times r}$. The spectrum $\sigma(P)$ of $P \in \mathbb{C}^{r\times r}$, denotes the set of all the eigenvalues of P. We put $\gamma(P)$ and $\varrho(P)$ the real numbers

$$\gamma(P) = \max\{\mathrm{Re}(\lambda): \lambda \in \sigma(P)\},\ \ \varrho(P) = \min\{\mathrm{Re}(\lambda): \lambda \in \sigma(P)\} \qquad (2.1)$$

A holomorphic function $f(\lambda)$ at a point was defined as a regular analytic function in a neighborhood of the point, see e.g. W. Wasow [13]. It is called holomorphic in a set if it is holomorphic at every point of the set. A matrix is called holomorphic if every entry of it is a holomorphic function. If $f(\lambda)$ and $g(\lambda)$ are homomorphic functions of the complex variable λ, which are defined in an open set of the complex plane, and P is matrix in $\mathbb{C}^{r\times r}$ with $\sigma(P) \subset \Omega$, then from the properties of the matrix functional calculus, see N. Dunford, J. Schwartz [5], it follows that

$$f(P)g(P) = g(P)f(P) \qquad (2.2)$$

Definition 2.1 A set of complex numbers is called positive stable if all the elements of the set have positive real part and a square matrix P is called positive stable if $\sigma(P)$ is positive stable.
If P is a positive stable matrix in $\mathbb{C}^{r\times r}$, than $\Gamma(P)$ is well defined, see L. Jodar, J. G. Cortes [9]

$$\Gamma(P) = \int_0^\infty e^{-t}t^{P-I}dt, \qquad t^{P-I} = \exp\big((P-I)\ln t\big) \qquad (2.3)$$

If f(P) is well defined and T is an invertible matrix in $\mathbb{C}^{r\times r}$, then

$$f(TPT^{-1}) = Tf(P)T^{-1} \qquad (2.4)$$

It is a standard result that for any matrix $P \in \mathbb{C}^{r\times r}$ there exist a nonsingular matrix $T \in \mathbb{C}^{r\times r}$ such that

$$T^{-1}PT = J = \mathrm{diag}(J_1, J_2, \ldots, J_s) \qquad (2.5)$$

Where

$$J_k = J_k(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda_k & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_k \end{bmatrix} \in \mathbb{C}^{m_k \times m_k} \qquad (2.6)$$

Definition 2.2 The function f is said to be defined on $\sigma(P)$ if the values

$$f^{(j)}(\lambda_i), \quad 0 \le j \le r_i - 1 , 1 \le i \le n$$

exist. These are called the values of the function f on $\sigma(P)$.

The following of f(P) requires only the values of f on $\sigma(P)$, it does not require any other information about f see N. J. Higham [7]. Ti is well knows that if f(P) is well defined and T is an invertible matrix in $\mathbb{C}^{r\times r}$, then

$$f(T^{-1}PT) = T^{-1}f(P)T \qquad (2.7)$$

Lemma 2.1 (matrix function via Jordan canonical form). Let f be defined on $\sigma(P)$, $P \in \mathbb{C}^{r\times r}$ and let P have the Jordan canonical form (2.5) subject to (2.6).Then

$$f(P) = Tf(J)T^{-1} = T\mathrm{diag}\big(f(J_1), f(j_2), \cdots, f(J_s)\big)T^{-1} \qquad (2.8)$$

where

$$f(J_k) = \begin{bmatrix} f(\lambda_k) & f^{(1)}(\lambda_k) & \cdots & \frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ 0 & f(\lambda_k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & f^{(1)}(\lambda_k) \\ 0 & \cdots & 0 & f(\lambda_k) \end{bmatrix} \in \mathbb{C}^{m_k\times m_k} \qquad (2.9)$$

Proof See [7]

The symbols $\mathcal{O}$, o and $\sim$, due to Bachmann and Landau (1927), which are also used by e.g. F. W. J. Olver [12] and A. Erdelyi [6], . Concerning the definition and elementary properties of asymptotic series we refer to. W. Wasow [13], A. Erdelyi [6] . Watson's lemma is a useful technique for obtaining the asymptotic behavior, as $z \to \infty$, of integrals of the form

$$\int_0^\infty e^{-zt} f(t)\, dt,$$

see e.g. G.N. Watson [14], and e.g. R. Wong [15] , C. M. Bender, S. A. Orszag [2].

Lemma 2.2 (Watson's lemma.) Let $f(t)$ be a complex valued function of a real variable t such that

(i) $f(t)$ is continuous on $(0, \infty)$,

(ii)

$$f(t) \sim \sum_{n=0}^\infty a_n t^{\xi_n - 1}, \text{ as } t \to 0$$

With

$$0 < \xi_0 < \xi_1 < \xi_2 < \cdots$$

(iii) for some fixed $c > 0$

$$f(t) = \mathcal{O}\{e^{ct}\} \text{ for } t \to \infty$$

Then we have

$$\int_0^\infty e^{-zt} f(t)\, dt = \sum_{n=0}^\infty \frac{a_n \Gamma(\xi_n)}{z^{\xi_n}} \text{ as } |z| \to \infty \qquad (2.10)$$

Where

$$|\arg(z)| \le \frac{\pi}{2} - \delta < \frac{\pi}{2} \text{ for some } \delta \text{ such that } 0 < \delta < \frac{\pi}{2}$$

III. Asymptotics approximations and expansions of integrals with a matrix argument

The Gamma function of a matrix with matrix arguments is expressed in terms of integrals with integrands being matrix functions. Therefore, it behooves us first to extend some important asymptotic theorems that apply to integrals with scalar integrands to integrals with matrix integrands. Hence, the method of integration by parts and Watson's lemma will be developed for

$$\varphi(P) = \int_a^b g(t) e^{-Pt}\, dt$$

Consider a positive stable matrix $P \in \mathbb{C}^{r\times r}$ and suppose we have

$$\varphi(P) = \int_a^b g(t) e^{-Pt}\, dt \qquad (3.1)$$

subject to:

(i) $g(t) \in C^\infty[0, \infty)$.

(ii) For each arbitrary nonnegative integer υ we have $g^{(\upsilon)} = \mathcal{O}\{e^{ct}\}$.

(iii) c is a real constant which is independent of υ.

The integral (3.1) converges when $\varrho(P) > c$. Integration by parts of the equation (3.1) produces

$$\varphi(P) = -g(t)P^{-1}e^{-Pt}|_a^b + \int_a^b g^{(t)}(t)\, P^{-1}\, e^{-Pt}\, dt$$

$$= -g(b)P^{-1}e^{-Pb} + g(a)P^{-1}e^{-Pa} + \int_a^b g^{(1)}(t)\, P^{-1}\, e^{-Pt}\, dt$$

Repeated integrations by parts produce

$$\varphi(P) = -g(b)P^{-1}e^{-Pb} + g(a)P^{-1}e^{-Pa} - g^{(1)}(b)\, P^{-2}\, e^{-Pb} + g^{(1)}(a)\, P^{-2}\, e^{-Pa} - g^{(2)}(b)\, P^{-3}\, e^{-Pb} + g^{(2)}(a)\, P^{-3}\, e^{-Pa} - \cdots + \int_a^b g^{(\upsilon)}(t)\, P^{-\upsilon}\, e^{-Pt}\, dt$$

when $b \to \infty$ we have

$$g^{(i)}(b)\, P^{-i-1}\, e^{-Pb} \to 0 \quad \text{for all} \quad 0 < i < \upsilon.$$

Therefore

$$\varphi(P) = g(a)P^{-1}e^{-Pa} + g^{(2)}(a)\, P^{-2}\, e^{-Pa} + g^{(3)}(a)\, P^{-3}\, e^{-Pa} + \cdots + \varepsilon_\upsilon(P)$$

where

$$\varepsilon_\upsilon(P) = \int_a^\infty g^{(\upsilon)}(t)\, P^{-\upsilon}\, e^{-Pt}\, dt.$$

With the assumed conditions,

$$\varepsilon_\upsilon(P) = \int_a^\infty \mathcal{O}\{e^{ct}\}\, P^{-\upsilon}\, e^{-Pt}\, dt$$

$$= P^{-\upsilon}\mathcal{O}\left\{\int_a^\infty e^{ct}\, e^{-Pt}\, dt\right\}$$

$$= P^{-\upsilon}\mathcal{O}\left\{\int_a^\infty e^{(-P+cI)}\, dt\right\}$$

$$= \mathcal{O}\left\{P^{-\upsilon}(-P + cI)^{-1}e^{(-P+cI)a}\right\}$$

Therefore when $a = 0$ and $P \to \infty$ we have,

$$\varphi(P) \sim \sum_{s=0}^{\infty} P^{-(s+1)} g^{(s)}(0) \qquad (3.2)$$

For example consider an integral of the form

$$\Gamma(P, z) = \int_z^\infty e^{-t}\, t^{P-I} dt,$$

where P is a positive stable matrix. Integration once by parts gives

$$\Gamma(P, z) = e^{-z} z^{P-I} + (P - I)\Gamma(P - I, z) \qquad (3.3)$$

Repeated application of (3.3) leads to

$$\Gamma(P, z) = e^{-z} z^{P-I}\left\{1 + \frac{P - I}{z} + \frac{(P - I)(P - 2I)}{z^2} + \cdots + \frac{(P - I)(P - 2I) \cdots (P - (n - 1)I)}{z^{n-1}}\right\} + \varepsilon_n(z)$$

where $n \in \mathbb{N}$, and

$$\varepsilon_n(z) = (P - I)(P - 2I) \cdots (P - nI) \int_z^\infty e^{-t}\, t^{P-(n+1)I} dt$$

If $n \geq \gamma(P - I)$, and we obtain

$$|\varepsilon_n(z)| \leq |(P - I)(P - 2I) \cdots (P - nI)|\, e^{-z} z^{P-(n+1)I}$$

Accordingly, for fixed P and large z

$$\Gamma(P, z) \sim e^{-z} z^{P-I} \sum_{s=0}^{\infty} \frac{(P-I)(P-2I)\cdots(P-nI)}{z^s} \qquad (3.4)$$

Proposition 3.1 Let $f(t)$ be a complex valued function of a real variable t such that

(i) $f(t)$ is continuous on $(0, \infty)$,

(ii) $f(t) \sim \sum_{n=0}^{\infty} a_n t^{\xi_n - 1}$ as $t \longrightarrow 0$ with $0 < \xi_0 < \xi_1 < \xi_2 < \cdots$;
without loss of generality assume $a_0 \neq 0$,

(iii) for some fixed $c > 0$, $f(t) = \mathcal{O}\{e^{ct}\}$ as $t \to \infty$.

Then we have

$$\frac{d^{\upsilon}}{d\lambda_k^{\upsilon}}\left(\int_0^{\infty} e^{-\lambda_k t} f(t)dt\right) \sim \sum_{n=0}^{\infty} \frac{d_n \Gamma(\xi_n+\upsilon)}{\lambda_k^{\xi_n+\upsilon}} \text{ as } \mathrm{Re}\lambda_k \to \infty \quad (3.5)$$

where $d_n = (-1)^{\upsilon} a_n$ and

$$|\arg(\lambda_k)| \le \frac{\pi}{2} - \delta < \frac{\pi}{2} \text{ for some } \delta \text{ such that } 0 < \delta < \frac{\pi}{2}$$

Proof By the conditions (i); (ii) and (iii) we have

a. $t^{\upsilon} f(t)$ is continuous on $(0, \infty)$.

b. $t^{\upsilon} f(t) \sim \sum_{n=0}^{\infty} b_n t^{\xi_n+\upsilon-1}$ as $t \to o$.

c. For any fixed υ there exist a constant $c_1 > 0$ such that$t^{\upsilon} f(t) = \mathcal{O}\{e^{ct}\}$

Apply Watson's lemma to $t^{\upsilon} f(t)$ to get the desired result.

The following lemma is a matrix version analogous to Watson's lemma for scalar functions. See e.g. [12].

Lemma 3.2 Suppose Q is a positive stable matrix in $\mathbb{C}^{r\times r}$ and suppose also that $f(t)I \in \mathbb{C}^{r\times r}$, where f(t) is a function of a real variable t such that

(i) f(t) is continuous on $(0, \infty)$.

(ii)

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{\xi_n-1} \text{ as } t \to o \quad (3.6)$$

with

$$0 < \xi_0 < \xi_1 < \xi_2 < \cdots$$

(iii) for some fixed $c > 0$, $f(t) = \mathcal{O}\{e^{ct}\}$ as $t \to \infty$.

(iv) $Q = P$ has the Jordan canonical form subject to (2.5) , (2.6)

with $|\arg(\lambda_k)| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$ for some δ such that $0 < \delta < \frac{\pi}{2}$.

Then we have

$$T^{-1}\left[\int_0^{\infty} e^{-\lambda_k t} f(t)dt\right]T \sim \Phi_Q \text{ as } \mathrm{Re}\,\lambda_k \to \infty \quad (3.7)$$

where $\Phi_Q = \mathrm{diag}(\Phi_1, \Phi_2, \cdots, \Phi_s)$ is a square block diagonal matrix in $\mathbb{C}^{r\times r}$ with blocks $\mathbb{C}^{m_k\times m_k}$

$$\Phi_k = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{a_n\Gamma(\xi_n)}{\lambda_k^{\xi_n}} & \sum_{n=0}^{\infty} \frac{d_n\Gamma(\xi_n+1)}{\lambda_k^{\xi_n+1}} & \cdots & \sum_{n=0}^{\infty} \frac{d_n\Gamma(\xi_n+m_{k-1})}{\lambda_k^{\xi_n+m_{k-1}}(m_{k-1})!} \\ 0 & \sum_{n=0}^{\infty} \frac{a_n\Gamma(\xi_n)}{\lambda_k^{\xi_n}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sum_{n=0}^{\infty} \frac{d_n\Gamma(\xi_n+1)}{\lambda_k^{\xi_n+1}} \\ 0 & \cdots & 0 & \sum_{n=0}^{\infty} \frac{a_n\Gamma(\xi_n)}{\lambda_k^{\xi_n}} \end{bmatrix} \quad (3.8)$$

or we can write Φ_k as

$$\Phi_k = D_k + E_k = D_k\left[I_{m_k} + D_k^{-1}E_k\right]$$

where

$$D_k = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{a_n\Gamma(\xi_n)}{\lambda_k^{\xi_n}} & 0 & \cdots & \sum_{n=0}^{\infty} \frac{d_n\Gamma(\xi_n+m_{k-1})}{\lambda_k^{\xi_n+m_{k-1}}(m_{k-1})!} \\ 0 & \sum_{n=0}^{\infty} \frac{a_n\Gamma(\xi_n)}{\lambda_k^{\xi_n}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sum_{n=0}^{\infty} \frac{a_n\Gamma(\xi_n)}{\lambda_k^{\xi_n}} \end{bmatrix} \in \mathbb{C}^{m_k\times m_k}$$

and

$$E_k = \begin{bmatrix} 0 & \sum_{n=0}^{\infty} \frac{d_n \Gamma(\xi_n + 1)}{\lambda_k^{\xi_n + 1}} & \cdots & \sum_{n=0}^{\infty} \frac{d_n \Gamma(\xi_n + r_{k-1})}{\lambda_k^{\xi_n + r_{k-1}} (r_{k-1})!} \\ \vdots & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \sum_{n=0}^{\infty} \frac{d_n \Gamma(\xi_n + 1)}{\lambda_k^{\xi_n + 1}} \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{C}^{m_k \times m_k}$$

Proof By the conditions (i), (ii), (iii) and (iv) the integral

$$\int_0^{\infty} e^{-Qt} f(t) dt$$

converges for all $\operatorname{Re} \lambda_k > c$, and

$$T^{-1}\left[\int_0^{\infty} e^{-Qt} f(t) dt\right] T = \int_0^{\infty} e^{-Jt} f(t) dt .$$

Conditions (i) and (ii) imply that

$$\left| f(t) - \sum_{n=0}^{N-1} a_n t^{\xi_N - 1} \right| \leq M_N e^{ct} \left| t^{\xi_N - 1} \right| \quad t > 0$$

for every $N \geq 0$ with some constant M_N. Thus we have

$$\left| \int_0^{\infty} e^{-\lambda_k t} f(t) - \sum_{n=0}^{N-1} a_n \int_0^{\infty} e^{-\lambda_k t}\, t^{\xi_n - 1} dt \right| \leq M_N \int_0^{\infty} e^{-(\lambda_k - c)t} \left| t^{\xi_N - 1} \right| dt$$

Hence we have

$$\left| \int_0^{\infty} e^{-\lambda_k t} f(\lambda_k) - \sum_{n=0}^{N-1} \frac{a_n \Gamma(\xi_n)}{\lambda_k^{\xi_n}} \right| \leq M_N \int_0^{\infty} e^{-(\lambda_k - c)t} \left| t^{\xi_N - 1} \right| dt$$
$$= \frac{M_N \Gamma(\xi_N)}{(\lambda_k - c)^{\xi_N}}$$

that is

$$\int_0^{\infty} e^{-\lambda_k t} f(\lambda_k) = \sum_{n=0}^{N-1} \frac{a_n \Gamma(\xi_n)}{\lambda_k^{\xi_n}} + \mathcal{O}\left\{ \lambda_k^{-\xi_n} \right\} \qquad (3.9)$$

we have

$$\int_0^{\infty} e^{-\lambda_k t} f(t) = \sum_{n=0}^{\infty} \frac{a_n \Gamma(\xi_n)}{\lambda_k^{\xi_n}} \quad \text{as } \operatorname{Re} \lambda_k \to \infty \qquad (3.10)$$

Let

$$h(J_k) = \int_0^{\infty} e^{-\lambda_k t} f(t)\, dt$$

by the lemma 2.1 we have

$$h(J_k) = \begin{bmatrix} h(\lambda_k) & h^{(1)}(\lambda_k) & \cdots & \frac{h^{(m_k - 1)}(\lambda_k)}{(m_k - 1)!} \\ 0 & h(\lambda_k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & h^{(1)}(\lambda_k) \\ 0 & \cdots & 0 & h(\lambda_k) \end{bmatrix} \in \mathbb{C}^{m_k \times m_k} \qquad (3.11)$$

By the proposition 3.1 we get $h(J_k) \sim \Phi_k$. Therefore,

$$T^{-1}\left[\int_0^{\infty} e^{-Qt} f(t) dt\right] T = \int_0^{\infty} e^{-Jt} f(t) dt \sim \Phi_k \quad \text{as} \quad \operatorname{Re} \lambda_k \to \infty ,\ k \in \mathbb{N}.$$

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