IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 16, Issue 5 Ser. V (Sep. – Oct. 2020), PP 44-66
www.iosrjournals.org

Construction of Two Infinite Classes of Strongly Regular Graphs Using Magic Squares

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Dedicated to French mathematician Philippe de La Hire

Abstract. *We say that a regular graph G of order n and degree* $r \geq 1$ *(which is not the complete graph) is strongly regular if there exist non-negative integers* τ *and* θ *such that* $|S_i \cap S_j| = \tau$ *for any two adjacent vertices i and j and* $|S_i \cap S_j| = \theta$ *for any two distinct non-adjacent vertices i and j, where* S_k *denotes the neighborhood of the vertex k. Using a method for constructing the magic and semi-magic squares of order* $2k + 1$*, we have created two infinite classes of strongly regular graphs (i) strongly regular graph of order* $n = (2k + 1)^2$ *and degree* $r = 8k$ *with* $\tau = 2k + 5$ *and* $\theta = 12$ *and (ii) strongly regular graph of order* $n = (2k + 1)^2$ *and degree* $r = 6k$ *with* $\tau = 2k + 1$ *and* $\theta = 6$ *for* $k \geq 2$*.*

Date of Submission: 26-10-2020 Date of Acceptance: 05-11-2020

I. Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{1, 2, \ldots, n\}$. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix A and is denoted by $\sigma(G)$. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph K_n) is strongly regular if there exist non-negative integers τ and θ such that $|S_i \cap S_j| = \tau$ for any two adjacent vertices i and j, and $|S_i \cap S_j| = \theta$ for any two distinct non-adjacent vertices i and j, where $S_k \subseteq V(G)$ denotes the neighborhood of the vertex k. We know that a regular connected graph G is strongly regular if and only if it has exactly three distinct eigenvalues [2]. Let $\lambda_1 = r$, λ_2 and λ_3 denote the distinct eigenvalues of a connected strongly regular graph G. Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r, λ_2 and λ_3. Further, let $\overline{r} = (n - 1) - r$, $\overline{\lambda}_2 = -\lambda_3 - 1$ and $\overline{\lambda}_3 = -\lambda_2 - 1$ denote the distinct eigenvalues of the strongly regular graph $\overline{G}$, where $\overline{G}$ denotes the complement of G. Then $\overline{\tau} = n - 2r - 2 + \theta$ and $\overline{\theta} = n - 2r + \tau$, where $\tau = \tau(G)$ and $\theta = \theta(G)$.

Remark 1. If G is a disconnected strongly regular graph of degree r then $G = mK_{r+1}$, where mH denotes the m-fold union of the graph H.

Remark 2. We also know that a strongly regular graph $G = \overline{mK_{r+1}}$ if and only if $\theta = r$. Since $\lambda_2\lambda_3 = -(r - \theta)$ it follows that $G = \overline{mK_{r+1}}$ if and only if $\lambda_2 = 0$.

Remark 3. (i) A strongly regular graph G of order $n = 4k + 1$ and degree $r = 2k$ with $\tau = k - 1$ and $\theta = k$ is called a conference graph; (ii) a strongly regular graph is a conference graph if and only if $m_2 = m_3$ and (iii) if $m_2 = m_3$ then G is an integral[1] graph.

Remark 4. The line graph of the complete bipartite graph $K_{n,n}$ is called a lattice graph and is denoted[2] by L(n). It is a strongly regular graph of order n^2 and degree $2(n - 1)$ with $\tau = n - 2$ and $\theta = 2$.

Let $X = X[x_{ij}]$ be a square matrix of order n with all distinct x_{ij} which belong to the set $\{1, 2, \ldots, n^2\}$. Let G[X] be a graph obtained from the matrix $X[x_{ij}]$ in the following way: (i) the vertex set of the graph G[X] is $V(G[X]) = \{x_{ij} \; i, j = 1, 2, \ldots, n\}$ and (ii) The neighborhood of the vertex x_{ij} is $S_{x_{ij}} = S_{x_{i,-j}} \cup S_{x_{-i,j}}$ where

(1) $$S_{x_{i,-j}} = \{x_{i1}, x_{i2}, \ldots, x_{i,j-1}, x_{i,j+1}, \ldots, x_{in}\},$$

(2) $$S_{x_{-i,j}} = \{x_{1j}, x_{2j}, \ldots, x_{i-1,j}, x_{i+1,j}, \ldots, x_{nj}\}$$

for[3] any $i, j = 1, 2, \ldots, n$. Since $|S_{x_{ij}}| = |S_{x_{i,-j}}| + |S_{x_{-i,j}}| = (n - 1) + (n - 1)$ we note that G[X] is a regular graph of order n^2 and degree $r = 2(n - 1)$. Let x_{st} be adjacent to x_{ij}. Then x_{st} belongs to the i-th row or to the j-th column. Without loss of generality we may assume that x_{st} belongs to the i-th row. In this case we have $s = i$ and $t = j$. So we obtain

$$|S_{x_{ij}} \cap S_{x_{it}}| = |S_{x_{i,-j}} \cap S_{x_{i,-t}}| + |S_{x_{i,-j}} \cap S_{x_{-i,t}}| + |S_{x_{-i,j}} \cap S_{x_{i,-t}}| + |S_{x_{-i,j}} \cap S_{x_{-i,t}}|$$

We note $|S_{x_{i,-j}} \cap S_{x_{-i,t}}| = 0$ because $x_{it} \notin S_{x_{it}}$ and $|S_{x_{-i,j}} \cap S_{x_{i,-t}}| = 0$ because $x_{ij} \notin S_{x_{ij}}$. Next, we have $|S_{x_{-i,j}} \cap S_{x_{-i,t}}| = 0$ because $t = j$. In the view of this we get $|S_{x_{ij}} \cap S_{x_{it}}| =$
$|S_{x_{i,-j}} \cap S_{x_{i,-t}}|$. Since $x_{ij} \notin S_{x_{ij}}$ and $x_{it} \notin S_{x_{it}}$ we find that $|S_{x_{ij}} \cap S_{x_{it}}| = n - 2$ for any
two adjacent vertices x_{ij} and x_{st}.

Further, let us assume that x_{ij} and x_{st} are two distinct non-adjacent vertices of the graph G[X]. In this case x_{st} neither belongs to the i-th row of the matrix X nor belongs to the j-th column of the matrix X, which provides that $s = i$ and $t = j$. So we obtain

$$|S_{x_{ij}} \cap S_{x_{st}}| = |S_{x_{i,-j}} \cap S_{x_{s,-t}}| + |S_{x_{i,-j}} \cap S_{x_{-s,t}}| + |S_{x_{-i,j}} \cap S_{x_{s,-t}}| + |S_{x_{-i,j}} \cap S_{x_{-s,t}}|$$

We note $|S_{x_{i,-j}} \cap S_{x_{s,-t}}| = 0$ because $s = i$ and $|S_{x_{-i,j}} \cap S_{x_{-s,t}}| = 0$ because $t = j$. Since $x_{it} \in S_{x_{i,-j}}$ and $x_{it} \in S_{x_{-s,t}}$ we find that $|S_{x_{i,-j}} \cap S_{x_{-s,t}}| = 1$. Since $x_{sj} \in S_{x_{-i,j}}$ and
$x_{sj} \in S_{x_{s,-t}}$ we find that $|S_{x_{-i,j}} \cap S_{x_{s,-t}}| = 1$. Finally, we arrive at

$$|S_{x_{ij}} \cap S_{x_{st}}| = |S_{x_{i,-j}} \cap S_{x_{-s,t}}| + |S_{x_{-i,j}} \cap S_{x_{s,-t}}| = 1 + 1,$$

which provides[4] that $G[X]$ is a strongly regular graph of order n^2 and degree $r = 2(n-1)$ with $\tau = n-2$ and $\theta = 2$. Therefore, according to Remark 4 it follows that $G[X] = L(n)$ for $n \geq 2$.

II. Magic squares of order 2k + 1

Let $M_n = M_n[m_{ij}]$ be a square matrix of order n with all distinct m_{ij} which belong to the set $\{1, 2, \ldots, n^2\}$. The matrix M_n is called the magic square of order n if the sum of all elements in any row and column and both diagonals is the same. The matrix M_n is called the semi-magic square of order n if the sum of all elements in any row and column is the same. We shall now demonstrate how to construct a magic square of order 5 created by "the method of cyclic permutations" established by French mathematician Philippe de La Hire, as follows. Let $(\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)) = (2, 5, 4, 1, 3)$ be a fixed permutation of the numbers 1, 2, 3, 4, 5 and let $(\pi(0), \pi(5), \pi(10), \pi(15), \pi(20)) = (20, 0, 10, 5, 15)$ be a fixed permutation of the numbers 0, 5, 10, 15, 20. Using the method of cyclic permutations we obtain the following two matrices

2	5	4	1	3
4	1	3	2	5
3	2	5	4	1
5	4	1	3	2
1	3	2	5	4

20	0	10	5	15
5	15	20	0	10
0	10	5	15	20
15	20	0	10	5
10	5	15	20	0

K[5] [5] and L[5] [5]

Then the matrix $M_5[m_{ij}] = K_5[k_{ij}] + L_5[\ell_{ij}]$ is a magic square of order 5, where $K_5[k_{ij}] = K[5][5]$ and $L_5[\ell_{ij}] = L[5][5]$.

We now proceed to obtain a new method for creating the semi-magic squares of order $2k+1$ for $k \geq 2$, which is based on "the method of cyclic permutations", as follows.

First, let us assume that $(\pi(1), \pi(2), \ldots, \pi(2k+1))$ is a fixed permutation of the numbers $1, 2, \ldots, 2k+1$. Let

$\pi(1)$	$\pi(2)$	...	$\pi(k)$	$\pi(k+1)$	$\pi(k+2)$	...	$\pi(2k)$	$\pi(2k+1)$
$\pi(k+1)$	$\pi(k+2)$	...	$\pi(2k)$	$\pi(2k+1)$	$\pi(1)$	...	$\pi(k-1)$	$\pi(k)$
$\pi(2k+1)$	$\pi(1)$	...	$\pi(k-1)$	$\pi(k)$	$\pi(k+1)$	...	$\pi(2k-1)$	$\pi(2k)$
$\pi(k)$	$\pi(k+1)$	...	$\pi(2k-1)$	$\pi(2k)$	$\pi(2k+1)$	...	$\pi(k-2)$	$\pi(k-1)$
$\pi(2k)$	$\pi(2k+1)$	...	$\pi(k-2)$	$\pi(k-1)$	$\pi(k)$	...	$\pi(2k-2)$	$\pi(2k-1)$
$\pi(k-1)$	$\pi(k)$	...	$\pi(2k-2)$	$\pi(2k-1)$	$\pi(2k)$	...	$\pi(k-3)$	$\pi(k-2)$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\pi(3)$	$\pi(4)$	...	$\pi(k+2)$	$\pi(k+3)$	$\pi(k+4)$	...	$\pi(1)$	$\pi(2)$
$\pi(k+3)$	$\pi(k+4)$	...	$\pi(1)$	$\pi(2)$	$\pi(3)$	...	$\pi(k+1)$	$\pi(k+2)$
$\pi(2)$	$\pi(3)$	...	$\pi(k+1)$	$\pi(k+2)$	$\pi(k+3)$	...	$\pi(2k+1)$	$\pi(1)$
$\pi(k+2)$	$\pi(k+3)$	...	$\pi(2k+1)$	$\pi(1)$	$\pi(2)$	...	$\pi(k)$	$\pi(k+1)$

$K[2k+1][2k+1]$

Second, let us assume that $(\pi(0), \pi(2k+1), \ldots, \pi(2k(2k+1)))$ is a fixed permutation of the numbers $0, 2k+1, \ldots, 2k(2k+1)$. Let $\bar{k} = 2k+1$ and let

$\pi(0)$	$\pi(\bar{k})$	...	$\pi((k-1)\bar{k})$	$\pi(k\bar{k})$	$\pi((k+1)\bar{k})$	...	$\pi((2k-1)\bar{k})$	$\pi(2k\bar{k})$
$\pi((k+1)\bar{k})$	$\pi((k+2)\bar{k})$	...	$\pi(2k\bar{k})$	$\pi(0)$	$\pi(\bar{k})$	...	$\pi((k-1)\bar{k})$	$\pi(k\bar{k})$
$\pi(\bar{k})$	$\pi(2\bar{k})$	...	$\pi(k\bar{k})$	$\pi((k+1)\bar{k})$	$\pi((k+2)\bar{k})$	...	$\pi(2k\bar{k})$	$\pi(0)$
$\pi((k+2)\bar{k})$	$\pi((k+3)\bar{k})$	...	$\pi(0)$	$\pi(\bar{k})$	$\pi(2\bar{k})$	...	$\pi(k\bar{k})$	$\pi((k+1)\bar{k})$
$\pi(2\bar{k})$	$\pi(3\bar{k})$	...	$\pi((k+1)\bar{k})$	$\pi((k+2)\bar{k})$	$\pi((k+3)\bar{k})$	...	$\pi(0)$	$\pi(\bar{k})$
$\pi((k+3)\bar{k})$	$\pi((k+4)\bar{k})$	...	$\pi(\bar{k})$	$\pi(2\bar{k})$	$\pi(3\bar{k})$	...	$\pi((k+1)\bar{k})$	$\pi((k+2)\bar{k})$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\pi((2k-1)\bar{k})$	$\pi(2k\bar{k})$	...	$\pi((k-3)\bar{k})$	$\pi((k-2)\bar{k})$	$\pi((k-1)\bar{k})$	...	$\pi((2k-3)\bar{k})$	$\pi((2k-2)\bar{k})$
$\pi((k-1)\bar{k})$	$\pi(k\bar{k})$	...	$\pi((2k-2)\bar{k})$	$\pi((2k-1)\bar{k})$	$\pi(2k\bar{k})$	...	$\pi((k-3)\bar{k})$	$\pi((k-2)\bar{k})$
$\pi(2k\bar{k})$	$\pi(0)$	...	$\pi((k-2)\bar{k})$	$\pi((k-1)\bar{k})$	$\pi(k\bar{k})$	...	$\pi((2k-2)\bar{k})$	$\pi((2k-1)\bar{k})$
$\pi(k\bar{k})$	$\pi((k+1)\bar{k})$	...	$\pi((2k-1)\bar{k})$	$\pi(2k\bar{k})$	$\pi(0)$	...	$\pi((k-2)\bar{k})$	$\pi((k-1)\bar{k})$

$L[2k+1][2k+1]$

understanding that $0 = 0 \cdot (2k+1)$ and $2k+1 = 1 \cdot (2k+1)$. Then we can see that the matrix $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[\ell_{ij}]$ is a semi-magic square of order $2k+1$, where $K_{2k+1}[k_{ij}] = K[2k+1][2k+1]$ and $L_{2k+1}[\ell_{ij}] = L[2k+1][2k+1]$. Indeed, since

$(k_{11}, k_{12}, \ldots, k_{1,2k+1}) = (\pi(1), \pi(2), \ldots, \pi(2k+1))$ and since the i-th row of the matrix K_{2k+1} is a cyclic permutation of its first row, we get

$$\sum_{j=1}^{2k+1} k_{ij} = \sum_{j=1}^{2k+1} k_{1j} = \sum_{j=1}^{2k+1} \pi(j) = \sum_{j=1}^{2k+1} j = (k+1)(2k+1)$$

for $i = 1, 2, \ldots, 2k+1$. According to $K[2k+1][2k+1]$, we have that

$$k_{i1} = \begin{cases} \pi(k+2-t), & \text{if } i = 2t \\ \pi(2k+2-t), & \text{if } i = 2t+1 \end{cases}$$

for $t = 1, 2, \ldots, k$. Since $k+2-t \le k+1 < k+2 \le 2k+2-t$ it follows that

$\pi(k+2-t) = \pi(2k+2-t)$ for $t = 1, 2, \ldots, k$. So we find that $(k_{11}, k_{21}, \ldots, k_{2k+1,1})^{>}$ is a permutation $(\pi_c(1), \pi_c(2), \ldots, \pi_c(2k+1))$ of the numbers $1, 2, \ldots, 2k+1$. Next, the j-th column of the matrix $K[2k+1][2k+1]$ is a cyclic permutation of its first column. In the view of this, we get

$$\sum_{i=1}^{2k+1} k_{ij} = \sum_{i=1}^{2k+1} k_{i1} = \sum_{i=1}^{2k+1} \pi_c(i) = \sum_{i=1}^{2k+1} i = (k+1)(2k+1)$$

for $j = 1, 2, \ldots, 2k+1$. We note (i) since the i-th row of the matrix K_{2k+1} is a cyclic permutation of its first row then[5] any fixed number $p \in \{1, 2, \ldots, 2k+1\}$ is presented in the i-th row of the matrix K_{2k+1} only one time and (ii) since the j-th column of the matrix K_{2k+1} is a cyclic permutation of its first column then[6] any fixed number $p \in \{1, 2, \ldots, 2k+1\}$ is presented in the j-th column of the matrix K_{2k+1} only one time.

Further, since $(\ell_{11}, \ell_{12}, \ldots, \ell_{1,2k+1}) = (\pi(0), \pi(2k+1), \ldots, \pi(2k(2k+1)))$ and since the i-th row of the matrix L_{2k+1} is a cyclic permutation of its first row, we get

$$\sum_{j=1}^{2k+1} \ell_{ij} = \sum_{j=1}^{2k+1} \ell_{1j} = \sum_{j=1}^{2k+1} \pi((j-1)(2k+1)) = (2k+1)\sum_{j=1}^{2k+1} (j-1) = k(2k+1)^2$$

for $i = 1, 2, \ldots, 2k+1$. According to $L[2k+1][2k+1]$, we have that

$$\ell_{i1} = \begin{cases} \pi((k+t)(2k+1)), & \text{if } i = 2t \\ \pi(t(2k+1)), & \text{if } i = 2t+1 \end{cases}$$

for $t = 1, 2, \ldots, k$. Next, since $t \le k < k+1 \le k+t$ it follows that $\pi(t(2k+1)) = \pi((k+t)(2k+1))$ for $t = 1, 2, \ldots, k$. So we find that $(\ell_{11}, \ell_{21}, \ldots, \ell_{2k+1,1})^{>}$ is a permutation $(\pi_c(0), \pi_c(2k+1), \ldots, \pi_c(2k(2k+1)))$ of the numbers $0, 2k+1, \ldots, 2k(2k+1)$. Since the j-th column of the matrix L[2k + 1][2k + 1] is a cyclic permutation of its first column, we obtain

$$\sum_{i=1}^{2k+1} `_{ij} = \sum_{i=1}^{2k+1} `_{i1} = \sum_{i=1}^{2k+1} \pi_c((i-1)(2k+1)) = (2k+1)\sum_{i=1}^{2k+1}(i-1) = k(2k+1)^2$$

for $j = 1, 2, \ldots, 2k+1$. We note (i) since the i-th row of the matrix L_{2k+1} is a cyclic permutation of its first row then[7] any fixed number $q \in \{0, 2k+1, \ldots, 2k(2k+1)\}$ is presented in the i-th row of the matrix L_{2k+1} only one time and (ii) since the j-th column of the matrix L_{2k+1} is a cyclic permutation of its first column then[8] any fixed number $q \in \{0, 2k+1, \ldots, 2k(2k+1)\}$ is presented in the j-th column of the matrix L_{2k+1} only one time. Since $m_{ij} = k_{ij} + `_{ij}$ we get

$$\sum_{i=1}^{2k+1} m_{ij} = \sum_{j=1}^{2k+1} m_{ij} = (k+1)(2k+1) + k(2k+1)^2 = (2k+1)\,\frac{\cdots}{2}$$

for $i, j = 1, 2, \ldots, 2k+1$. It remains to show that $m_{ij} \in \{1, 2, \ldots, (2k+1)^2\}$ and that m_{ij} are mutually different for $i, j = 1, 2, \ldots, 2k+1$. Indeed, since $k_{ij} \in \{1, 2, \ldots, 2k+1\}$ and $`_{ij} \in \{0, 2k+1, \ldots, 2k(2k+1)\}$ we have $m_{ij} \in \{1, 2, \ldots, (2k+1)^2\}$ for $i, j = 1, 2, \ldots, 2k+1$. Next, according to K[2k + 1][2k + 1] we have that

$$(3)\qquad k_{ij} = \begin{cases} \pi(k+1-t+j), & \text{if } i = 2t \wedge k+1-t+j \le 2k+1 \\ \pi(k+1-t+j-(2k+1)), & \text{if } i = 2t \wedge k+1-t+j > 2k+1 \\ \pi(2k+1-t+j), & \text{if } i = 2t+1 \wedge 2k+1-t+j \le 2k+1 \\ \pi(2k+1-t+j-(2k+1)), & \text{if } i = 2t+1 \wedge 2k+1-t+j > 2k+1 \end{cases}$$

for $t = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, 2k+1$. Next, according to L[2k + 1][2k + 1] we have that

$$(4)\qquad `_{ij} = \begin{cases} \pi((k+t+j-1)(2k+1)), & \text{if } i = 2t \wedge k+t+j-1 \le 2k \\ \pi((k+t+j-1-(2k+1))(2k+1)), & \text{if } i = 2t \wedge k+t+j-1 > 2k \\ \pi((t+j-1)(2k+1)), & \text{if } i = 2t+1 \wedge t+j-1 \le \\ \pi((t+j-1-(2k+1))(2k+1)), & \text{if } i = 2t+1 \wedge t+j-1 > \end{cases}$$

for $t = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, 2k+1$. Since $m_{ij} = k_{ij} + `_{ij} = \pi(p) + \pi(q(2k+1))$ and since the numbers $\pi(p) \in \{1, 2, \ldots, 2k+1\}$ and $\pi(q(2k+1)) \in \{0, 2k+1, \ldots, 2k(2k+1)\}$, it follows that $k_{ij}$ and $`_{ij}$ are uniquely determined. In other words, if $m_{ij} = k_{ij} + `_{ij}$, $m_{st} = k_{st} + `_{st}$ and $m_{ij} = m_{st}$ then $k_{ij} = k_{st}$ and $`_{ij} = `_{st}$. We now proceed to show that m_{ij} are mutually different for $i, j = 1, 2, \ldots, 2k+1$. On the contrary, assume that $m_{ij} = m_{\mu\nu}$ for some $(i, j) = (\mu, \nu)$. Then $m_{ij} = \pi(p_0) + \pi(q_0(2k+1)) = m_{\mu\nu}$ for some

$\pi(p_0) \in \{1, 2, \ldots, 2k+1\}$ and $\pi(q_0(2k+1)) \in \{0, 2k+1, \ldots, 2k(2k+1)\}$, which provides that $k_{ij} = k_{\mu\nu}$ and $`_{ij} = `_{\mu\nu}$. Without loss of generality we may assume that $i = \mu$. Since $\pi(q_0(2k+1))$ is presented in the j-th column of the matrix L_{2k+1} only one time, we find that $j = \nu$. Since[9] the i-th row and the μ-th row of the matrix K_{2k+1} is a cyclic permutation of its first row and since the i-th row and the μ-th row of the matrix L_{2k+1} is a cyclic permutation of its first row, we can easily see that any m_{is} in the i-th row is also presented in the μ-th row. Indeed, we have

$$m_{i,j+1} = \pi(p_0 + 1) + \pi((q_0 + 1)(2k + 1)) = m_{\mu,\nu+1} ,$$

(i) understanding that $\pi(p_0 + 1) = \pi(1)$ if $p_0 + 1 = 2k + 2$ and $\pi((q_0 + 1)(2k + 1)) = \pi(0)$ if $q_0 + 1 = 2k + 1$ and (ii) understanding that $m_{i,j+1} = m_{i1}$ if $j + 1 = 2k + 2$ and $m_{\mu,\nu+1} = m_{\mu 1}$ if $\nu + 1 = 2k + 2$. In the view of this, we can assume that $j = 1$. Since $j = \nu$ we have
that $\nu \in \{2, 3, . . . , 2k + 1\}$. Finally, in order to prove that M_{2k+1} is a semi-magic square
we shall consider the following four cases:

Case 1. ($i = 2t$ and $\mu = 2s$). Consider the case when $k + 1 - s + \nu \leq 2k + 1$ and $k + s + \nu - 1 \leq 2k$. Using (3) and (4) we obtain that $\pi(k + 2 - t) = \pi(k + 1 - s + \nu)$ and $\pi((k + t)(2k + 1)) = \pi((k + s + \nu - 1)(2k + 1))$, which provides that (i) $k + 2 - t = k + 1 - s + \nu$ and (ii) $k + t = k + s + \nu - 1$. Using (i) and (ii) we obtain $\nu = 1$, a contradiction because $\nu > 1$. Consider the case when $k + 1 - s + \nu \leq 2k + 1$ and $k + s + \nu - 1 > 2k$. Using (3) and (4) we obtain that $\pi(k + 2 - t) = \pi(k + 1 - s + \nu)$ and $\pi((k + t)(2k + 1)) = \pi((k + s + \nu - 1 - (2k + 1))(2k + 1))$, which provides that (iii) $k + 2 - t = k + 1 - s + \nu$ and (iv) $k + t = k + s + \nu - 1 - (2k + 1)$. Using (iii) and (iv) we obtain $2\nu = 2k + 3$, a contradiction because 2 - 2k + 3. Consider the case when $k + 1 - s + \nu > 2k + 1$ and $k + s + \nu - 1 \leq 2k$. Using (3) and (4) we obtain that $\pi(k + 2 - t) = \pi(k + 1 - s + \nu - (2k + 1))$ and $\pi((k + t)(2k + 1)) = \pi((k + s + \nu - 1)(2k + 1))$, which provides that (v) $k + 2 - t = k + 1 - s + \nu - (2k + 1)$ and (vi) $k + t = k + s + \nu - 1$. Using (v) and (vi) we obtain $2\nu = 2k + 3$, a contradiction because 2 - 2k + 3. Consider the case when $k + 1 - s + \nu > 2k + 1$ and $k + s + \nu - 1 > 2k$. Using (3) and (4) we obtain that $\pi(k + 2 - t) = \pi(k + 1 - s + \nu - (2k + 1))$ and $\pi((k + t)(2k + 1)) = \pi((k + s + \nu - 1 - (2k + 1))(2k + 1))$, which provides that (vii) $k + 2 - t = k + 1 - s + \nu - (2k + 1)$ and (viii) $k + t = k + s + \nu - 1 - (2k + 1)$. Using (vii) and (viii) we obtain $\nu = 2k + 2$, a contradiction because $\nu \in \{2, 3, . . . , 2k + 1\}$.

Case 2. ($i = 2t$ and $\mu = 2s + 1$). Consider the case when $2k + 1 - s + \nu \leq 2k + 1$ and
$s + \nu - 1 \leq 2k$. Using (3) and (4) we obtain that $\pi(k + 2 - t) = \pi(2k + 1 - s + \nu)$ and

$\pi((k + t)(2k + 1)) = \pi((s + \nu - 1)(2k + 1))$, which provides that (i) $k + 2 - t = 2k + 1 - s + \nu$ and (ii) $k + t = s + \nu - 1$. Using (i) and (ii) we obtain $\nu = 1$, a contradiction because $\nu >$

1.
Consider the case when $2k+1-s+\nu \leq 2k+1$ and $s+\nu-1 > 2k$. Using (3) and (4) we obtain
that $\pi(k+2-t) = \pi(2k+1-s+\nu)$ and $\pi((k+t)(2k+1)) = \pi((s+\nu-1-(2k+1))(2k+1))$, which provides that (iii) $k+2-t = 2k+1-s+\nu$ and (iv) $k+t = s+\nu-1-(2k+1)$.
Using (iii) and (iv) we obtain $2\nu = 2k+3$, a contradiction because 2 - 2k + 3. Consider the case when $2k+1-s+\nu > 2k+1$ and $s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(2k+1-s+\nu-(2k+1))$ and $\pi((k+t)(2k+1)) = \pi((s+\nu-1)(2k+1))$, which provides that (v) $k+2-t = 2k+1-s+\nu-(2k+1)$ and (vi) $k+t = s+\nu-1$. Using (v)
and (vi) we obtain $2\nu = 2k+3$, a contradiction because 2 - 2k + 3. Consider the case when
$2k+1-s+\nu > 2k+1$ and $s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(k+2-t) = \pi(2k+1-s+\nu-(2k+1))$ and $\pi((k+t)(2k+1)) = \pi((s+\nu-1-(2k+1))(2k+1))$, which provides that (vii) $k+2-t = 2k+1-s+\nu-(2k+1)$ and (viii) $k+t = s+\nu-1-(2k+1)$. Using (vii) and (viii) we obtain $\nu = 2k+2$, a contradiction because $\nu \in \{2, 3, \ldots, 2k+1\}$.

Case 3. ($i = 2t+1$ and $\mu = 2s$). Consider the case when $k+1-s+\nu \leq 2k+1$ and $k+s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(k+1-s+\nu)$ and $\pi(t(2k+1)) = \pi((k+s+\nu-1)(2k+1))$, which provides that (i) $2k+2-t = k+1-s+\nu$ and (ii) $t = k+s+\nu-1$. Using (i) and (ii) we obtain $\nu = 1$, a contradiction because $\nu > 1$. Consider the case when $k+1-s+\nu \leq 2k+1$ and $k+s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(k+1-s+\nu)$ and $\pi(t(2k+1)) = \pi((k+s+\nu-1-(2k+1))(2k+1))$, which provides that (iii) $2k+2-t = k+1-s+\nu$ and (iv) $t = k+s+\nu-1-(2k+1)$.
Using (iii) and (iv) we obtain $2\nu = 2k+3$, a contradiction because 2 - 2k + 3. Consider the case when $k+1-s+\nu > 2k+1$ and $k+s+\nu-1 \leq 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(k+1-s+\nu-(2k+1))$ and $\pi(t(2k+1)) = \pi((k+s+\nu-1)(2k+1))$, which provides that (v) $2k+2-t = k+1-s+\nu-(2k+1)$ and (vi) $t = k+s+\nu-1$.
Using (v) and (vi) we obtain $2\nu = 2k+3$, a contradiction because 2 - 2k + 3. Consider the case when $k+1-s+\nu > 2k+1$ and $k+s+\nu-1 > 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(k+1-s+\nu-(2k+1))$ and $\pi(t(2k+1)) = \pi((k+s+\nu-1-(2k+1))(2k+1))$, which provides that (vii) $2k+2-t = k+1-s+\nu-(2k+1)$ and (viii) $t = k+s+\nu-1-(2k+1)$. Using (vii) and (viii) we obtain $\nu = 2k+2$, a contradiction because $\nu \in \{2, 3, \ldots, 2k+1\}$.

Case 4. ($i = 2t + 1$ and $\mu = 2s + 1$). Consider the case when $2k + 1 - s + \nu \leq 2k + 1$ and $s + \nu - 1 \leq 2k$. Using (3) and (4) we obtain that $\pi(2k + 2 - t) = \pi(2k + 1 - s + \nu)$ and $\pi(t(2k+1)) = \pi((s+\nu-1)(2k+1))$, which provides that (i) $2k+2-t = 2k+1-s+\nu$ and (ii) $t = s + \nu - 1$. Using (i) and (ii) we obtain $\nu = 1$, a contradiction because $\nu > 1$. Consider the case when $2k + 1 - s + \nu \leq 2k + 1$ and $s + \nu - 1 > 2k$. Using (3) and (4) we obtain

that $\pi(2k+2-t) = \pi(2k+1-s+\nu)$ and $\pi(t(2k+1)) = \pi((s+\nu-1-(2k+1))(2k+1))$, which provides that (iii) $2k+2-t = 2k+1-s+\nu$ and (iv) $t = s+\nu-1-(2k+1)$. Using (iii) and (iv) we obtain $2\nu = 2k + 3$, a contradiction because 2 - $2k + 3$. Consider the case when $2k + 1 - s + \nu > 2k + 1$ and $s + \nu - 1 \leq 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(2k+1-s+\nu-(2k+1))$ and $\pi(t(2k+1)) = \pi((s+\nu-1)(2k+1))$, which provides that (v) $2k+2-t = 2k+1-s+\nu-(2k+1)$ and (vi) $t = s+\nu-1$. Using

(v) and (vi) we obtain $2\nu = 2k + 3$, a contradiction because 2 - $2k + 3$. Consider the case when $2k + 1 - s + \nu > 2k + 1$ and $s + \nu - 1 > 2k$. Using (3) and (4) we obtain that $\pi(2k+2-t) = \pi(2k+1-s+\nu-(2k+1))$ and $\pi(t(2k+1)) = \pi((s+\nu-1-(2k+1))(2k+1))$, which provides that (vii) $2k+2-t = 2k+1-s+\nu-(2k+1)$ and (viii) $t = s+\nu-1-(2k+1)$. Using (vii) and (viii) we obtain $\nu = 2k + 2$, a contradiction because $\nu \in \{2, 3, \ldots, 2k+1\}$.

Theorem 1. Let $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[`_{ij}]$ where $K_{2k+1}[k_{ij}] = K[2k+1][2k+1]$ and $L_{2k+1}[`_{ij}] = L[2k+1][2k+1]$. Then $M_{2k+1}[m_{ij}]$ is a semi-magic square of order $2k+1$ for $k \geq 2$.

Theorem 2. Let $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[`_{ij}]$ where $K_{2k+1}[k_{ij}] = K[2k+1][2k+1]$ and $L_{2k+1}[`_{ij}] = L[2k+1][2k+1]$. Then[10] $M_{2k+1}[m_{ij}]$ is a magic square of order $2k+1$ if 3 - $2k+1$.

Proof. In order to prove that M_{2k+1} is a magic square it is sufficient to show that the all elements in both diagonals of the matrix K_{2k+1} and the matrix L_{2k+1} are mutually different. First, according to $K[2k+1][2k+1]$ we have that

$$k_{ii} = \begin{cases} \pi(k+t+1) & \text{if } i = 2t \\ \pi(t+1), & \text{if } i = 2t+1 \end{cases}$$

for $t = 1, 2, \ldots, k$. Since $t + 1 < k + t + 1$ it follows that k_{ii} are mutually different for $i = 1, 2, \ldots, 2k+1$. Next, according to $K[2k+1][2k+1]$ we have that

$$k_{i,2k+2-i} = \begin{cases} \pi(k+2-3t), & \text{if } i = 2t \wedge k+2-3t \geq 0 \\ \pi(k+2-3t+2k+1), & \text{if } i = 2t \wedge k+2-3t < 0 \\ \pi(2k+1-3t), & \text{if } i = 2t+1 \wedge 2k+1-3t \geq \\ \pi(2(2k+1)-3t), & \text{if } i = 2t+1 \wedge 2k+1-3t < \end{cases}$$

for $t = 1, 2, \ldots, k$. Since 3 - $2k + 1$ and $k + 2 = 3(k + 1) - (2k + 1)$ it follows that $k + 2 - 3t = 0$ and $2k + 1 - 3t = 0$. Let $2k + 1 \equiv \varepsilon \mod 3$ where $\varepsilon \in \{-1, 1\}$. Then we

have (i) $k + 2 - 3t \equiv -\varepsilon \mod 3$ and (ii) $k + 2 - 3t + 2k + 1 \equiv 0 \mod 3$, which that $k_{2t,2k+2-2t}$ are mutually different for $t = 1, 2, \ldots, k$. Since (iii) $2k+1 - 3t \equiv \varepsilon \bmod 3$ and (iv) $2(2k+1) - 3t \equiv -\varepsilon \bmod 3$, we find that $k_{2t+1,2k+2-(2t+1)}$ are mutually different for $t = 1, 2, \ldots, k$. Of course, since $3 \nmid 2k+1$ we have $k_{1,2k+1} = \pi(2k+1) = k_{2t+1,2k+2-(2t+1)}$ for $t = 1, 2, \ldots, k$. On the contrary, assume that $k_{i,2k+2-i} = k_{j,2k+2-j}$ for some $i \neq j$. Then according to (i), (ii), (iii) and (iv) it must be $\pi(k+2-3t) = \pi(2(2k+1) - 3s)$, which provides that $k+2-3t = 2(2k+1) - 3s$ for some $t = 1, 2, \ldots, k$ and $s = 1, 2, \ldots, k$. Then $k+2-3t \leq k-1 < k+1 < 2(2k+1) - 3s$, a contradiction.

Next, we shall now demonstrate that the all elements in both diagonals of the matrix $L[2k+1][2k+1]$ are mutually different. Indeed, according to $L[2k+1][2k+1]$ we have that

$$\ell_{ii} = \begin{cases} \pi((k-1+3t)(2k+1)), & \text{if } i = 2t \wedge k-1+3t \leq 2k+1 \\ \pi((k-1+3t-(2k+1))(2k+1)), & \text{if } \\ \pi(3t(2k+1)), & \text{if } i = 2t+1 \wedge 3t \leq 2k+1 \\ \pi(3t-(2k+1))(2k+1)), & \text{if } i = 2t+1 \wedge 3t > 2k+1 \end{cases}$$

for $t = 1, 2, \ldots, k$. Since $3 \nmid 2k+1$ and $k-1 = 3k-(2k+1)$ it follows that $k-1+3t \neq 2k+1$ and $3t \neq 2k+1$. Let $2k+1 \equiv \varepsilon \bmod 3$ where $\varepsilon \in \{-1, 1\}$. Then we have (i) $k-1+3t \equiv -\varepsilon \bmod 3$ and (ii) $k-1+3t-(2k+1) \equiv \varepsilon \bmod 3$, which provides that $\ell_{2t,2t}$ are mutually different for $t = 1, 2, \ldots, k$. Since (iii) $3t \equiv 0 \bmod 3$ and (iv) $3t-(2k+1) \equiv -\varepsilon \bmod 3$, we find that $\ell_{2t+1,2t+1}$ are mutually different for $t = 1, 2, \ldots, k$. Of course, since $3 \nmid 2k+1$ we have $\ell_{11} = \pi(0) = \ell_{2t+1,2t+1}$ for $t = 1, 2, \ldots, k$. On the contrary, assume that $\ell_{ii} = \ell_{jj}$ for some $i \neq j$. Then according to (i), (ii), (iii) and (iv) it must be $\pi((k-1+3t)(2k+1)) = \pi((3s-(2k+1))(2k+1))$, which provides that $k-1+3t = 3s-(2k+1)$ for some $t = 1, 2, \ldots, k$ and $s = 1, 2, \ldots, k$. Then $k-1+3t \geq k+2 > k > 3s-(2k+1)$, a contradiction. Next, according to $L[2k+1][2k+1]$ we have that

$$\ell_{i,2k+2-i} = \begin{cases} \pi((k-t)(2k+1)) & \text{if } i = 2t \\ \pi((2k-t)(2k+1)), & \text{if } i = 2t+1 \end{cases}$$

for $t = 1, 2, \ldots, k$. Since $k-t < 2k-t$ it follows that ℓ_{ii} are mutually different for $i = 1, 2, \ldots, 2k+1$.

Corollary 1. Let $M_n[m_{ij}] = K_n[k_{ij}] + L_n[\ell_{ij}]$ for $n \in 2N+1$, where $K_n[k_{ij}] = K[n][n]$ and $L_n[\ell_{ij}] = L[n][n]$. Then

$$M_n[m_{ij}] = \begin{cases} \text{the magic square,} & \text{if } n = 6k-1 \\ \text{the magic square,} & \text{if } n = 6k+1 \\ \text{the semi-magic square,} & \text{if } n = 6k+3 \end{cases}$$

for $k \in N$.

Remark 5. In case that $k = 2$ the applied method of cyclic permutations for creating the magic squares is reduced to the method of cyclic permutations for creating the magic squares of order 5 established by French mathematician Philippe de La Hire.

III. Two infinite classes of strongly regular graphs

Let $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[\ell_{ij}]$ be a semi-magic square of order $2k + 1$ for $k \geq 2$. Let $G[M_{2k+1}]$ be a graph obtained from the matrix $M_{2k+1}[m_{ij}]$ in the following way: (i) the vertex set of the graph $G[M_{2k+1}]$ is $V(G[M_{2k+1}]) = \{m_{ij} \; i, j = 1, 2, \ldots, 2k + 1\}$ and (ii) the neighborhood of the vertex $m_{ij} = k_{ij} + \ell_{ij}$ is $S_{m_{ij}} = S_{m_{i,-j}} \cup S_{m_{-i,j}} \cup K_{ij} \cup L_{ij}$ where

(5) $$K_{ij} = \{m_{st} \mid k_{st} = k_{ij} \text{ and } (s, t) = (i, j)\},$$

(6) $$L_{ij} = \{m_{st} \mid \ell_{st} = \ell_{ij} \text{ and } (s, t) = (i, j)\},$$

for $s, t = 1, 2, \ldots, 2k + 1$. We note that $K_{ij} \cap L_{ij} = \emptyset$ for $i, j = 1, 2, \ldots, 2k + 1$. Indeed, on the contrary, assume that $m_{st} \in K_{ij} \cap L_{ij}$. Then $m_{st} = k_{st} + \ell_{st} = k_{ij} + \ell_{ij} = m_{ij}$, a contradiction. Namely, it is easy to see that $S_{m_{i,-j}}, S_{m_{-i,j}}, K_{ij}, L_{ij}$ are mutually disjoint. For the sake of an example, let us show that $S_{m_{i,-j}} \cap K_{ij} = \emptyset$. On the contrary, assume that $m_{st} \in S_{m_{i,-j}} \cap K_{ij}$. Using (1) it follows that $s = i$ and $t = j$. Since $m_{it} \in K_{ij}$ and
$k_{it} = k_{ij}$ we find that k_{ij} is presented in the i-th row of the matrix $K_{2k+1}[k_{ij}]$ two times, a contradiction. Since $k_{ij} \in K_{2k+1} = K_{2k+1}[k_{ij}]$ is presented in the i-th row and the j-th column only one time and $m_{ij} \notin K_{ij}$, we obtain $|K_{ij}| = (2k + 1) - 1$. Similarly, since
$\ell_{ij} \in L_{2k+1} = L_{2k+1}[\ell_{ij}]$ is presented in the i-th row and the j-th column only one time and $m_{ij} \notin L_{ij}$, we obtain $|L_{ij}| = (2k + 1) - 1$. Therefore, we have

$$|S_{m_{ij}}| = |S_{m_{i,-j}}| + |S_{m_{-i,j}}| + |K_{ij}| + |L_{ij}| = 2k + 2k + 2k + 2k,$$

which provides that $G[M_{2k+1}]$ is a regular graph of order $n = (2k + 1)^2$ and degree $r = 8k$.

Theorem 3. Let $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[\ell_{ij}]$ be a semi-magic square of order $2k + 1$ for $k \geq 2$. Then $G[M_{2k+1}]$ is a strongly regular graph of order $n = (2k + 1)^2$ and degree $r = 8k$ with $\tau = 2k + 5$ and $\theta = 12$.

Proof. First, assume that m_{ij} and $m_{\mu\nu}$ are two distinct non-adjacent vertices of the graph $G[M_{2k+1}]$. In this case we have $\mu = i$ and $\nu = j$. On the contrary, assume that $\mu = i$ or $\nu = j$. Without loss of generality we can assume that $\mu = i$ and $\nu = j$. Then $m_{i\nu} \in S_{m_{i,-j}}$, which means that $m_{i\nu}$ and m_{ij} are adjacent, a contradiction. Since $m_{\mu\nu} = k_{\mu\nu} + \ell_{\mu\nu}$ it is easy to see $k_{\mu\nu} = k_{ij}$ and $\ell_{\mu\nu} = \ell_{ij}$. Indeed, if we assume $k_{\mu\nu} = k_{ij}$ then $m_{\mu\nu} \in K_{ij}$, which means that $m_{\mu\nu}$ and m_{ij} are adjacent, a contradiction. We shall now (1^0) prove that $|S_{m_{ij}} \cap S_{m_{\mu,-\nu}}| = 3$. Since k_{ij} is presented in the μ-th row of the matrix K_{2k+1} it follows that there exist $s = \nu$ so that $k_{\mu s} = k_{ij}$, which provides that $m_{\mu s} \in S_{m_{\mu,-\nu}}$ and $m_{\mu s} \in K_{ij} \subseteq S_{m_{ij}}$. Similarly, since ℓ_{ij} is presented in the μ-th row of the matrix L_{2k+1} it follows that there exist $t = \nu$ so that $\ell_{\mu t} = \ell_{ij}$, which provides that $m_{\mu t} \in S_{m_{\mu,-\nu}}$ and $m_{\mu t} \in L_{ij} \subseteq S_{m_{ij}}$. Since $S_{m_{\mu,-\nu}} \cap S_{m_{i,-j}} = \emptyset$ and since $S_{m_{\mu,-\nu}} \cap S_{m_{-i,j}} = \{m_{\mu j}\} \subseteq S_{m_{ij}}$, we

obtain[11] that $|S_{m_{ij}} \cap S_{m_{\mu,-\nu}}| \geq 3$. Next, let $m_{\mu x} \in S_{m_{\mu,-\nu}}$ and let $m_{\mu x} \notin \{m_{\mu j}, m_{\mu s}, m_{\mu t}\}$,

which provides that $x \notin \{j, s, t\}$. It remains to demonstrate that $m_{\mu x} \notin S_{m_{ij}}$. On the

contrary, assume that $m_{\mu x} = k_{\mu x} + \ell_{\mu x} \in S_{m_{ij}}$. Then according to (1), (2), (5) and (6) we find that $m_{\mu x} \in K_{ij}$ or $m_{\mu x} \in L_{ij}$. Without loss of generality we may assume $m_{\mu x} \in K_{ij}$. In this case we have $k_{\mu x} = k_{ij}$. Since $k_{\mu s} = k_{ij}$ we find that k_{ij} is presented in the μ-th row of the matrix K_{2k+1} two times, a contradiction. This completes the assertion (1^0). Using the same arguments as in the proof of (1^0), we can (2^0) prove that $|S_{m_{ij}} \cap S_{m_{-\mu,\nu}}| = 3$. We shall now (3^0) prove that $|S_{m_{ij}} \cap K_{\mu\nu}| = 3$. Since $k_{\mu\nu}$ is presented in the i-th row of the matrix K_{2k+1} it follows that there exist $t = j$ so that $k_{it} = k_{\mu\nu}$, which provides that $m_{it} \in K_{\mu\nu}$ and $m_{it} \in S_{m_{i,-j}} \subseteq S_{m_{ij}}$. Since $k_{\mu\nu}$ is presented in the j-th column of

the matrix K_{2k+1} it follows that there exist $s = i$ so that $k_{sj} = k_{\mu\nu}$, which provides that $m_{sj} \in K_{\mu\nu}$ and $m_{sj} \in S_{m_{-i,j}} \subseteq S_{m_{ij}}$. We shall now demonstrate that $K_{ij} \cap K_{\mu\nu} = \emptyset$. On the contrary, assume that $m_{xy} \in K_{ij} \cap K_{\mu\nu}$. Then $k_{xy} = k_{ij}$ and $k_{xy} = k_{\mu\nu}$, which provides that $k_{\mu\nu} = k_{ij}$, a contradiction. Further, let $P_{ij} = \{p + \ell_{ij} \mid p \in \{1, 2, \ldots, 2k+1\} \setminus \{k_{ij}\}\}$ and let $Q_{ij} = \{k_{ij} + q \mid q \in \{0, 2k+1, \ldots, 2k(2k+1)\} \setminus \{\ell_{ij}\}\}$ for $i, j = 1, 2, \ldots, 2k+1$.

Due to the fact that k_{ij} is presented in the i-th row and the j-th column of the matrix K_{2k+1} only one time, we easily see $P_{ij} = L_{ij}$ for $i, j = 1, 2, \ldots, 2k+1$. Due to the fact that ℓ_{ij} is presented in the i-th row and the j-th column of the matrix L_{2k+1} only one time, we easily see $Q_{ij} = K_{ij}$ for $i, j = 1, 2, \ldots, 2k+1$. Let $p_0 \in \{1, 2, \ldots, 2k+1\} \setminus \{k_{ij}\}$ such that $p_0 = k_{\mu\nu}$ and let $q_0 \in \{0, 2k+1, \ldots, 2k(2k+1)\} \setminus \{\ell_{\mu\nu}\}$ such that $q_0 = \ell_{ij}$. Then $p_0 + \ell_{ij} \in L_{ij} \subseteq S_{m_{ij}}$ and $k_{\mu\nu} + q_0 \in K_{\mu\nu}$. So we obtain $p_0 + \ell_{ij} = p_0 + q_0 = k_{\mu\nu} + q_0$, which

provides[12] that $|L_{ij} \cap K_{\mu\nu}| \geq 1$ and $|S_{m_{ij}} \cap K_{\mu\nu}| \geq 3$. Since $p_0 \in \{1, 2, \ldots, 2k+1\} \setminus \{k_{ij}\}$ and $q_0 \in \{0, 2k+1, \ldots, 2k(2k+1)\} \setminus \{\ell_{\mu\nu}\}$ are uniquely determined we obtain $|L_{ij} \cap K_{\mu\nu}| = 1$, which completes the assertion (3^0). Using the same arguments as in the proof of (3^0), we can (4^0) prove that $|S_{m_{ij}} \cap L_{\mu\nu}| = 3$. Finally, using (1^0), (2^0), (3^0) and (4^0) we obtain that

$$|S_{m_{ij}} \cap S_{m_{\mu\nu}}| = |S_{m_{ij}} \cap S_{m_{\mu,-\nu}}| + |S_{m_{ij}} \cap S_{m_{-\mu,\nu}}| + |S_{m_{ij}} \cap K_{\mu\nu}| + |S_{m_{ij}} \cap L_{\mu\nu}|,$$

from which we obtain $|S_{m_{ij}} \cap S_{m_{\mu\nu}}| = 12$ for any two distinct non-adjacent vertices m_{ij} and $m_{\mu\nu}$. Next, let m_{ij} and $m_{\mu\nu}$ be two adjacent vertices of the graph $G[M_{2k+1}]$. We shall now consider the following two cases:

Case 1. ($m_{\mu\nu} \in S_{m_{i,-j}}$ or $m_{\mu\nu} \in S_{m_{-i,j}}$). Without loss of generality we can assume that $m_{\mu\nu} \in S_{m_{i,-j}}$. In this case we have $\mu = i$ and $\nu = j$. We shall now (1^0) prove that $|S_{m_{ij}} \cap S_{m_{i,-\nu}}| = 2k - 1$. Since $m_{ij} \notin S_{m_{i,-j}}$ and $m_{i\nu} \notin S_{m_{i,-\nu}}$ we have $|S_{m_{i,-j}} \cap S_{m_{i,-\nu}}| = (2k + 1) - 2$, which provides that $|S_{m_{ij}} \cap S_{m_{i,-\nu}}| \geq 2k - 1$. Since $m_{ij} \notin S_{m_{ii}}$ and $S_{m_{i,-j}}, S_{m_{-i,j}}, K_{ij}$ and L_{ij} are mutually disjoint it follows that $S_{m_{i,-\nu}}, S_{m_{-i,j}}, K_{ij}$ and L_{ij} are also mutually disjoint, which completes the assertion (1^0). We shall now (2^0) prove that $|S_{m_{ij}} \cap S_{m_{-i,\nu}}| = 2$. Since $m_{i\nu} \notin S_{m_{-i,\nu}}$ we have that $S_{m_{i,-j}} \cap S_{m_{-i,\nu}} = \emptyset$ and $S_{m_{-i,j}} \cap S_{m_{-i,\nu}} = \emptyset$. Since k_{ij} is presented in the ν-th column of the matrix K_{2k+1} it follows that there exist $s = \mu$ so that $k_{s\nu} = k_{ij}$, which provides that $m_{s\nu} \in S_{m_{-i,\nu}}$ and $m_{s\nu} \in K_{ij} \subseteq S_{m_{ij}}$. Similarly, since ℓ_{ij} is presented in the ν-th column of the matrix L_{2k+1} it follows that there exist $t = \mu$ so that $\ell_{t\nu} = \ell_{ij}$, which provides that $m_{t\nu} \in S_{m_{-i,\nu}}$ and $m_{t\nu} \in L_{ij} \subseteq S_{m_{ij}}$. This completes the assertion (2^0). We shall now (3^0) prove that $|S_{m_{ij}} \cap K_{i\nu}| = 2$. Since $m_{i\nu} \notin K_{i\nu}$ and $S_{m_{i,-\nu}} \cap K_{i\nu} = \emptyset$ it follows that $S_{m_{i,-j}} \cap K_{i\nu} = \emptyset$.

Since $k_{i\nu}$ is presented in the j-th column of the matrix K_{2k+1} it follows that there exist $s = i$ so that $k_{sj} = k_{i\nu}$, which provides that $m_{sj} \in K_{i\nu}$ and $m_{sj} \in S_{m_{-i,j}} \subseteq S_{m_{ij}}$. We shall now demonstrate that $K_{ij} \cap K_{i\nu} = \emptyset$. On the contrary, assume that $m_{st} \in K_{ij} \cap K_{i\nu}$. Then $k_{st} = k_{ij}$ and $k_{st} = k_{i\nu}$ which yields $k_{ij} = k_{i\nu}$, a contradiction. Next, since $K_{i\nu} = Q_{i\nu}$ and $Q_{i\nu} = \{k_{i\nu} + q \mid q \in \{0, 2k+1, \ldots, 2k(2k+1)\} \setminus \{\ell_{i\nu}\}\}$ there exist $q_0 \in \{0, 2k+1, \ldots, 2k(2k+1)\} \setminus \{\ell_{i\nu}\}$ such that $q_0 = \ell_{ij}$. In the view of this, we have $k_{i\nu} + q_0 \in K_{i\nu}$ and $k_{i\nu} + q_0 \in L_{ij} \subseteq S_{m_{ii}}$, which completes the assertion (3^0). We shall now (4^0) prove that $|S_{m_{ij}} \cap L_{i\nu}| = 2$. Since $m_{i\nu} \notin L_{i\nu}$ and $S_{m_{i,-\nu}} \cap L_{i\nu} = \emptyset$ it follows that $S_{m_{i,-j}} \cap L_{i\nu} = \emptyset$. Since $\ell_{i\nu}$ is presented in the j-th column of the matrix L_{2k+1} it follows that there exist $s = i$ so that $\ell_{sj} = \ell_{i\nu}$, which provides that $m_{sj} \in L_{i\nu}$ and $m_{sj} \in S_{m_{-i,j}} \subseteq S_{m_{ij}}$. We shall now demonstrate that $L_{ij} \cap L_{i\nu} = \emptyset$. On the contrary, assume that $m_{st} \in L_{ij} \cap L_{i\nu}$. Then $\ell_{st} = \ell_{ij}$ and $\ell_{st} = \ell_{i\nu}$ which yields $\ell_{ij} = \ell_{i\nu}$, a contradiction. Next, since $L_{i\nu} = P_{i\nu}$ and $P_{i\nu} = \{p + \ell_{i\nu} \mid p \in \{1, 2, \ldots, 2k+1\} \setminus \{k_{i\nu}\}\}$ there exist $p_0 \in \{1, 2, \ldots, 2k+1\} \setminus \{k_{i\nu}\}$ such that $p_0 = k_{ij}$. In the view of this, we have $p_0 + \ell_{i\nu} \in L_{i\nu}$ and $p_0 + \ell_{i\nu} \in K_{ij} \subseteq S_{m_{ij}}$, which completes the assertion (4^0). Finally, using (1^0), (2^0), (3^0) and (4^0) we obtain that

$$|S_{m_{ij}} \cap S_{m_{i\nu}}| = |S_{m_{ij}} \cap S_{m_{i,-\nu}}| + |S_{m_{ij}} \cap S_{m_{-i,\nu}}| + |S_{m_{ij}} \cap K_{i\nu}| + |S_{m_{ij}} \cap L_{i\nu}|,$$

from which we obtain $|S_{m_{ij}} \cap S_{m_{i\nu}}| = (2k - 1) + 2 + 2 + 2$ for any two adjacent vertices m_{ij} and $m_{i\nu}$.

Case 2. ($m_{\mu\nu} \in K_{ij}$ or $m_{\mu\nu} \in L_{i,j}$). Without loss of generality we can assume that $m_{\mu\nu} \in K_{ij}$. Since $S_{m_{i,-j}}, S_{m_{-i,j}}$ and K_{ij} are mutually disjoint it follows that $\mu = i$ and $\nu = j$. Since $m_{\mu\nu} = k_{\mu\nu} + \ell_{\mu\nu}$ and $m_{\mu\nu} \in K_{ij}$ we obtain $m_{\mu\nu} = k_{ij} + \ell_{\mu\nu}$, from which we obtain $k_{\mu\nu} = k_{ij}$ and $\ell_{\mu\nu} = \ell_{ij}$. We shall now ($1^0$) prove that $|S_{m_{ij}} \cap S_{m_{\mu,-\nu}}| = 2$. Since $\mu = i$ and $\nu = j$ we have $S_{m_{i,-j}} \cap S_{m_{\mu,-\nu}} = \emptyset$ and $S_{m_{-i,j}} \cap S_{m_{\mu,-\nu}} = \{m_{\mu j}\} \subseteq S_{m_{ij}}$. We shall now demonstrate that $K_{ij} \cap S_{m_{\mu,-\nu}} = \emptyset$. On the contrary, assume that $m_{\mu t} \in K_{ij} \cap S_{m_{\mu,-\nu}}$. Then $m_{\mu t} = k_{\mu t} + \ell_{\mu t} \in S_{m_{\mu,-\nu}}$ and $m_{\mu t} = k_{ij} + \ell_{\mu t} \in K_{ij}$ which yields $k_{\mu t} = k_{ij}$. Since $k_{\mu\nu} = k_{ij}$ and $k_{\mu t} = k_{ij}$ we have $k_{\mu\nu} = k_{\mu t}$. Finally, since $k_{\mu\nu}$ is presented in the μ-th row of the matrix K_{2k+1} only one time we obtain $t = \nu$. In the view of this, we find that $m_{\mu\nu} \in S_{m_{\mu,-\nu}}$, a contradiction. Next, since ℓ_{ij} is presented in the μ-th row of the matrix L_{2k+1} it follows that there exist $t = \nu$ so that $\ell_{\mu t} = \ell_{ij}$, which provides that $m_{\mu t} \in S_{m_{\mu,-\nu}}$ and $m_{\mu t} \in L_{ij} \subseteq S_{m_{ij}}$. This completes the assertion ($1^{\cdot}$). We shall now ($2^{\cdot}$) prove that $|S_{m_{ij}} \cap S_{m_{-\mu,\nu}}| = 2$. Since $\mu = i$ and $\nu = j$ we have $S_{m_{i,-j}} \cap S_{m_{-\mu,\nu}} = \{m_{i\nu}\} \subseteq S_{m_{ij}}$ and $S_{m_{-i,j}} \cap S_{m_{-\mu,\nu}} = \emptyset$. We shall now demonstrate that $K_{ij} \cap S_{m_{-\mu,\nu}} = \emptyset$. On the contrary, assume that $m_{s\nu} \in K_{ij} \cap S_{m_{-\mu,\nu}}$. Then $m_{s\nu} = k_{s\nu} + \ell_{s\nu} \in S_{m_{-\mu,\nu}}$ and $m_{s\nu} = k_{ij} + \ell_{s\nu} \in K_{ij}$ which yields $k_{s\nu} = k_{ij}$. Since $k_{\mu\nu} = k_{ij}$ and $k_{s\nu} = k_{ij}$ we have $k_{\mu\nu} = k_{s\nu}$. Finally, since $k_{\mu\nu}$ is presented in the ν-th column of the matrix K_{2k+1} only one time we obtain $s = \mu$. In the view of this, we find that $m_{\mu\nu} \in S_{m_{-\mu,\nu}}$, a contradiction. Next, since ℓ_{ij} is presented in the ν-th column of the matrix L_{2k+1} it follows that there exist $t = \mu$ so that $\ell_{t\nu} = \ell_{ij}$, which provides that $m_{t\nu} \in S_{m_{-\mu,\nu}}$ and $m_{t\nu} \in L_{ij} \subseteq S_{m_{ij}}$. This completes the assertion (2^0). We shall now (3^0) prove that $|S_{m_{ij}} \cap K_{\mu\nu}| = 2k - 1$. Since $k_{\mu\nu} = k_{ij}$ we have that $K_{ij} = \{m_{st} \mid k_{st} = k_{ij}$ and $(s,t) = (i,j)\} \subseteq S_{m_{ij}}$ and $K_{\mu\nu} = \{m_{st} \mid k_{st} = k_{ij}$ and $(s,t) = (\mu,\nu)\}$. Since $m_{ij} \notin K_{ij}$ and $m_{\mu\nu} \notin K_{\mu\nu}$ we find that $|K_{ij} \cap K_{\mu\nu}| = (2k+1) - 2$.

Since $m_{ij} \notin S_{m_{ij}}$ and $S_{m_{i,-j}}, S_{m_{-i,j}}, K_{ij}, L_{ij} \subseteq S_{m_{ij}}$ are mutually disjoint it follows that $S_{m_{i,-j}}, S_{m_{-i,j}}, L_{ij}$ and $K_{\mu\nu}$ are also mutually disjoint, which completes the assertion ($3^{\cdot}$). We shall now (4^0) prove that $|S_{m_{ij}} \cap L_{\mu\nu}| = 2$. Since $\ell_{\mu\nu}$ is presented in the i-th row of the matrix L_{2k+1} it follows that there exist $t = j$ so that $\ell_{it} = \ell_{\mu\nu}$, which provides that $m_{it} = k_{it} + \ell_{\mu\nu} \in L_{\mu\nu}$ and $m_{it} = k_{it} + \ell_{it} \in S_{m_{i,-j}} \subseteq S_{m_{ij}}$. Since $\ell_{\mu\nu}$ is presented in the j-th column of the matrix L_{2k+1} it follows that there exist $s = i$ so that $\ell_{sj} = \ell_{\mu\nu}$, which provides that $m_{sj} = k_{sj} + \ell_{\mu\nu} \in L_{\mu\nu}$ and $m_{sj} = k_{sj} + \ell_{sj} \in S_{m_{-i,j}} \subseteq S_{m_{ij}}$. We shall now demonstrate that $K_{ij} \cap L_{\mu\nu} = \emptyset$. On the contrary, assume that $m_{st} = k_{st} + \ell_{st} \in K_{ij} \cap L_{\mu\nu}$.

Then $k_{st} = k_{ij}$ and $\ell_{st} = \ell_{\mu\nu}$. Since $k_{\mu\nu} = k_{ij}$ we obtain $k_{st} = k_{\mu\nu}$, which provides that $m_{\mu\nu} = k_{\mu\nu} + \ell_{\mu\nu} \in L_{\mu\nu}$, a contradiction. We shall now demonstrate that $L_{ij} \cap L_{\mu\nu} = \emptyset$. On the contrary, assume that $m_{st} = k_{st} + \ell_{st} \in L_{ij} \cap L_{\mu\nu}$. Then $\ell_{st} = \ell_{ij}$ and $\ell_{st} = \ell_{\mu\nu}$, which provides that $\ell_{\mu\nu} = \ell_{ij}$, a contradiction. This completes the assertion (4^0). Finally, using (1^0), (2^0), (3^0) and (4^0) we obtain that

$$|S_{m_{ij}} \cap S_{m_{\mu\nu}}| = |S_{m_{ij}} \cap S_{m_{\mu,-\nu}}| + |S_{m_{ij}} \cap S_{m_{-\mu,\nu}}| + |S_{m_{ij}} \cap K_{\mu\nu}| + |S_{m_{ij}} \cap L_{\mu\nu}|$$

from which we obtain $|S_{m_{ij}} \cap S_{m_{\mu\nu}}| = 2 + 2 + (2k - 1) + 2$ for any two adjacent vertices m_{ij} and $m_{\mu\nu}$, which[13] completes the[14] proof.

Let $G^-[M_{2k+1}]$ be a graph obtained from the matrix $M_{2k+1}[m_{ij}]$ in the following way: (i) the vertex set of the graph $G^-[M_{2k+1}]$ is $V(G^-[M_{2k+1}]) = \{m_{ij} \; i, j = 1, 2, \ldots, 2k+1\}$ and (ii) the neighborhood[15] of the vertex $m_{ij} = k_{ij} + `_{ij}$ is $S_{m_{ij}} = S_{m_{i,-j}} \cup S_{m_{-i,j}} \cup K_{ij}$. Using the same[16] arguments as in the proof of Theorem 3, we can prove the following result.

Theorem 4. Let $M_{2k+1}[m_{ij}] = K_{2k+1}[k_{ij}] + L_{2k+1}[`_{ij}]$ be a semi-magic square of order $2k+1$ for $k \geq 2$. Then $G^-[M_{2k+1}]$ is a strongly regular graph of order $n = (2k+1)^2$ and degree $r = 6k$ with $\tau = 2k+1$ and $\theta = 6$.

Acknowledgments. The author is very grateful to Marko Lepović for his valuable comments and suggestions.

References

[1] Filippe de La Hire (18 March 1640 – 21 April 1718). He was a French mathematician, astronomer, architect and painter. Acording to Bernard le Bovier de Fontenelle he was an "academy unto himself".

[2] D. Cvetković, M. Dob, H. Sachs, Spectra of graphs – Theory and applications, 3rd revised and enlarged edition, J.A. Barth Vela, Heidelberg – Leipzig, 1995.

4. Appendix

Using the applied method of cyclic permutations for creating the magic and semi-magic squares, in this section with a minor modification of "the first permutation" we create the magic squares of order $6k+3$ for $k \geq 1$. First, let us assume that $(\pi(1), \pi(2), \ldots, \pi(6k+3))$ is a fixed permutation of the numbers $1, 2, \ldots, 6k+3$. Let

π(1)	π(2)	...	π(3k+1)	π(3k+2)	π(3k+3)	...	π(6k+2)	π(6k+3)
π(3k+2)	π(3k+3)	...	π(6k+2)	π(6k+3)	π(1)	...	π(3k)	π(3k+1)
π(6k+3)	π(1)	...	π(3k)	π(3k+1)	π(3k+2)	...	π(6k+1)	π(6k+2)
π(3k+1)	π(3k+2)	...	π(6k+1)	π(6k+2)	π(6k+3)	...	π(3k−1)	π(3k)
π(6k+2)	π(6k+3)	...	π(3k−1)	π(3k)	π(3k+1)	...	π(6k)	π(6k+1)
π(3k)	π(3k+1)	...	π(6k)	π(6k+1)	π(6k+2)	...	π(3k−2)	π(3k−1)
:	:	:	:	:	:	:	:	:
π(3)	π(4)	...	π(3k+3)	π(3k+4)	π(3k+5)	...	π(1)	π(2)
π(3k+4)	π(3k+5)	...	π(1)	π(2)	π(3)	...	π(3k+2)	π(3k+3)
π(2)	π(3)	...	π(3k+2)	π(3k+3)	π(3k+4)	...	π(6k+3)	π(1)
π(3k+3)	π(3k+4)	...	π(6k+3)	π(1)	π(2)	...	π(3k+1)	π(3k+2)

$K[6k+3][6k+3]$

Second, let us assume that $(\pi(0), \pi(6k+1), \ldots, \pi((6k+2)(6k+3)))$ is a fixed permutation of the numbers $0, 6k+3, \ldots, (6k+2)(6k+3)$. Let $\pi_+(p) = \pi(p\,(6k+3))$ for $p = 0, 1, \ldots, 6k+2$ and let

$\pi_+(0)$	$\pi_+(1)$	...	$\pi_+(3k)$	$\pi_+(3k+1)$	$\pi_+(3k+2)$	...	$\pi_+(6k+1)$	$\pi_+(6k+2)$
$\pi_+(3k+2)$	$\pi_+(3k+3)$	...	$\pi_+(6k+2)$	$\pi_+(0)$	$\pi_+(1)$	...	$\pi_+(3k)$	$\pi_+(3k+1)$
$\pi_+(1)$	$\pi_+(2)$	...	$\pi_+(3k+1)$	$\pi_+(3k+2)$	$\pi_+(3k+3)$	...	$\pi_+(6k+2)$	$\pi_+(0)$
$\pi_+(3k+3)$	$\pi_+(3k+4)$	...	$\pi_+(0)$	$\pi_+(1)$	$\pi_+(2)$	...	$\pi_+(3k+1)$	$\pi_+(3k+2))$
$\pi_+(2)$	$\pi_+(3)$	...	$\pi_+(3k+2)$	$\pi_+(3k+3)$	$\pi_+(3k+4)$	...	$\pi_+(0)$	$\pi_+(1)$
$\pi_+(3k+4)$	$\pi_+(3k+5)$	...	$\pi_+(1)$	$\pi_+(2)$	$\pi(3)$	...	$\pi_+(3k+2)$	$\pi_+(3k+3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi_+(6k+1))$	$\pi_+(6k+2)$	...	$\pi_+(3k-2)$	$\pi_+(3k-1)$	$\pi_+(3k)$	...	$\pi_+(6k-1)$	$\pi_+(6k)$
$\pi_+(3k)$	$\pi_+(3k+1)$	...	$\pi_+(6k)$	$\pi_+(6k+1)$	$\pi_+(6k+2)$	...	$\pi_+(3k-2)$	$\pi_+(3k-1)$
$\pi_+(6k+2)$	$\pi_+(0)$	...	$\pi_+(3k-1)$	$\pi_+(3k)$	$\pi_+(3k+1)$	...	$\pi_+(6k)$	$\pi_+(6k+1)$
$\pi_+(3k+1)$	$\pi_+(3k+2)$	...	$\pi_+(6k+1)$	$\pi_+(6k+2)$	$\pi_+(0)$	...	$\pi_+(3k-1)$	$\pi_+(3k)$

$L[6k+3][6k+3]$

understanding that $0 = 0 \cdot (6k+3)$ and $6k+3 = 1 \cdot (6k+3)$. Let us define $X = \{k+2, k+4, \ldots, k+2(2k+1)\} \subseteq \{1, 2, \ldots, 6k+3\}$ and let $Y = \{1, 2, \ldots, 6k+3\} \setminus X$. Let us define $X_+ = \{(k+1)k, (k+3)k, \ldots, (k+4k+1)k\} \subseteq \{0, k, \ldots, (6k+2)k\}$ and let

$Y_+ = \{0, k, \ldots, (6k+2)k\} \setminus X_+$, where $k = 6k+3$. Let $\pi(X)$ be the set of all permutations of the set X and let $\pi(Y)$ be the set of all permutations of the set Y. Of course, since $|X| = 2k+1$ and $|Y| = 4k+2$ we have $|\pi(X)| = (2k+1)!$ and $|\pi(Y)| = (4k+2)!$. Similarly, let $\pi(X_+)$ be the set of all permutations of the set X_+ and let $\pi(Y_+)$ be the set of all permutations of the set Y_+. Of course, since $|X_+| = 2k+1$ and $|Y_+| = 4k+2$ we have $|\pi(X_+)| = (2k+1)!$ and $|\pi(Y_+)| = (4k+2)!$. Let $\text{sum}\,\pi(x)$ be the sum of all elements in a fixed permutation $\pi(x) \in \pi(X)$. Then we have

(7)
$$\text{sum}\,\pi(x) = \sum_{t=1}^{2k+1} (k+2t) = (2k+1)(3k+2).$$

Let $\text{sum}\,\pi_+(x)$ be the sum of all elements in a fixed permutation $\pi_+(x) \in \pi(X_+)$. Then we have

(8)
$$\text{sum}\,\pi_+(x) = (6k+3) \sum_{t=1}^{2k+1} (k + (2t-1)) = (2k+1)(3k+1)(6k+3).$$

The first row of the matrix K_{6k+3} contains the numbers of a fixed permutation $\pi(x) \in \pi(X)$ and the numbers of a fixed permutation $\pi(y) \in \pi(Y)$ obtained in the following way: (i) on the position $6k+3, 6k, \ldots, 3$ set up the numbers of $\pi(x)$ and (ii) on the position $t \not\in \{6k+3, 6k, \ldots, 3\}$ set up the numbers of $\pi(y)$. According to $K[6k+3][6k+3]$ we
note that the numbers of the permutation $\pi(x)$ are presented 3 times in the non-main diagonal of the matrix K_{6k+3}, understanding that $K_{6k+3} = K[6k+3][6k+3]$.

The first row of the matrix L_{6k+3} contains the numbers of a fixed permutation $\pi_+(x) \in \pi(X_+)$ and the numbers of a fixed permutation $\pi_+(y) \in \pi(Y_+)$ obtained in the following way: (i) on the position $1, 4, \ldots, 6k+1$ set up the numbers of $\pi_+(x)$ and (ii) on the position $t \not\in \{1, 4, \ldots, 6k+1\}$ set up the numbers of $\pi_+(y)$. According to $L[6k+3][6k+3]$
we note that the numbers of the permutation $\pi_+(x)$ are presented 3 times in the main diagonal of the matrix L_{6k+3}, understanding that $L_{6k+3} = L[6k+3][6k+3]$. Using (7) and (8) we obtain[17]

$$3 \text{ sum } \pi(x) + 3 \text{ sum } \pi_+(x) = (6k+3) \frac{(6k+3)^2+1}{2},$$

which provides that $M_{6k+3}[m_{ij}] = K_{6k+3}[k_{ij}] + L_{6k+3}[\ell_{ij}]$ is a magic square of order $6k+3$ for $k \geq 1$.

Remark 6. In this section we present a source program `magic.cpp` which has been written by the author in the programming language `Borland C++ Builder 5.5` for creating the magic squares[18] of order $3, 5, \ldots, 999$. The algorithm described in this section is also valid for $k = 0$, a case that is related to the magic square of order 3.

```
//---------------------------------------------------------------------------
#include <stdlib.h>
#include <string.h>
#include <stdio.h>
#include <math.h>
#include <time.h>

#define CR 13
#define LF 10

char *_String (int n, int Size);
void CreateMagicSquare (int Menu);
void CreateRandomPermutation (int *FirstRow, int Menu);

void main (void)
{
    randomize ();

    CreateMagicSquare  (5);
    CreateMagicSquare  (7);
    CreateMagicSquare (9);
```

```
    CreateMagicSquare  (501);
    CreateMagicSquare  (503);
    CreateMagicSquare (505);
}
//------------------------------------------------------------------------
void CreateMagicSquare (int Menu)
{
    int i, j, k, m, n, One, Two, Size, _Size, _Menu, *Diagonal, *_Diagonal;
    int *x, *y, *p, *q, *_p, *_q, *Flag, *a[999];
    char *s, *t;
    FILE *FP;

    static char *MagicFile = "Magic$$$.Lap";


  x = new int [Menu];
  y = new int [Menu];
  p = new int [Menu];
  q = new int [Menu];

  for (i = 0; i < Menu; i++) a[i] = new int [Menu];
  for (i = 0; i < Menu; i++) {
     p[i] = i + 1;
     q[i] = i * Menu;
  }

  if (Menu % 3) _Menu = 1; else _Menu = 2;
  switch (_Menu) {
     case 1:
        CreateRandomPermutation  (p,Menu);
        CreateRandomPermutation  (q,Menu);
        break;
     case 2:
       _p = new int [Menu];
       _q = new int [Menu];

        Diagonal = new int [Menu];
       _Diagonal = new int [Menu];

        Flag = new int [Menu];
        for (i = 0; i < Menu; i++) {
           Diagonal[i] = 0;
          _Diagonal[i] = 0;
           Flag[i] = 0;
        }
```

```
j = Menu / 3;
k = Menu / 6;
m = k + 1;
n = 2 * j;
for (i = n; i < Menu; i++) {
   _p[i] = p[m];
   _q[i] = q[m];
    Flag[m] = 1;
    m = m + 2;
}
n = 0;
for (i = 0; i < Menu; i++) {
   if (Flag[i]) continue;
  _p[n] = p[i];
  _q[n] = q[i];
   n++;
}
```

```
   CreateRandomPermutation (_p,n);
   CreateRandomPermutation (_q,n);

   CreateRandomPermutation (_p + n,j);
   CreateRandomPermutation (_q + n,j);

   n = Menu - 1;
   m = 0;
   for (i = 0; i < j; i++) {
      Diagonal[m] = 1;
     _Diagonal[n] = 1;
      n = n - 3;
      m = m + 3;
   }
   n = 2 * j;
   for (i = 0; i < Menu; i++) {
      if (!_Diagonal[i]) continue;
      p[i] = _p[n];
      n++;
   }
   n = 0;
   for (i = 0; i < Menu; i++) {
      if (_Diagonal[i]) continue;
      p[i] = _p[n];
      n++;
   }
   n = 2 * j;
   for (i = 0; i < Menu; i++) {
      if (!Diagonal[i]) continue;
      q[i] = _q[n];
      n++;
       }
       n = 0;
       for (i = 0; i < Menu; i++) {
          if (Diagonal[i]) continue;
          q[i] = _q[n];
          n++;
       }
       delete [] _p;
       delete [] _q;
       delete []  Flag;

       delete  []   Diagonal;
       delete  [] _Diagonal;
       break;
 };
 One = Menu / 2;
```

```
  Two = Menu - 1;

  x[0] = 0;
  k = 1;
  while (k < Menu) {
     x[k] = One;
     k++;
     x[k] = Two;
     k++;
     One--;
     Two--;
  }
  One = 1 + Menu / 2;
  Two = 1;

  y[0] = 0;
  k = 1;
  while (k < Menu) {
     y[k] = One;
     k++;
     y[k] = Two;
     k++;
     One++;
     Two++;
  }
  for (i = 0; i < Menu; i++) {
     n = x[i];
     m = y[i];
     for (j = 0; j < Menu; j++) {
        a[i][j] = p[n] + q[m];
        n++;
        m++;
       if (n == Menu) n = 0;
       if (m == Menu) m = 0;
    }
 }
_Menu = Menu * Menu;

 t = _String ( Menu,3);
 s = _String (_Menu,6);

 movmem (t,MagicFile + 5,3);
 delete [] t;
 t = s;

 FP = fopen (MagicFile, "wb");
```

```
  Size = 6;
  while (*s++ == '0' ) Size--;
  delete [] t;

 _Size = Menu * (Size + 1) + 1;
  s = new char [_Size];

  for (i = 0; i < _Size; i++) s[i] = ' ';
  s[_Size - 2] = CR;
  s[_Size - 1] = LF;

  for (i = 0; i < Menu; i++) {
     for (j = 0; j < Menu; j++) {
        t = _String (a[i][j], Size);
        k = (Size + 1) * j;
        movmem (t, s + k, Size);
        delete [] t;
     }
     fwrite (s, 1, _Size, FP);
  }
  fclose (FP);

  delete  [] x;
  delete  [] y;
  delete  [] p;
  delete  [] q;
  delete [] s;

    for (i = 0; i < Menu; i++) delete [] a[i];
}
//----------------------------------------------------------------------
void CreateRandomPermutation  (int *FirstRow, int Menu)
{
     int i, j, *p, *Flag;

     p = new int [Menu];
     Flag = new int [Menu];

     for (i = 0; i < Menu; i++) Flag[i] = 0;
     for (i = 0; i < Menu;) {
        j = random (Menu);
        if (Flag[j]) continue;
        p[i] = FirstRow[j];
        Flag[j] = 1;
        i++;
     }
     for (i = 0; i < Menu; i++) FirstRow[i] = p[i];
```

```
      delete [] p;
      delete [] Flag;
}
//------------------------------------------------------------------------
char *_String (int n, int Size)
{
      char *p = new char [Size + 1];
      int i, j;

      p[Size]  = 0;
      j = Size - 1;

      for (i = 0; i < Size; i++) {
         p[j] = n % 10 + '0';
         n = n / 10;
         j--;
      }
      return p;
}
//------------------------------------------------------------------------
```

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