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On Topology of epi \ hypo–graphicaloperations in a sense of Mosco 's epi \ hypo graphical

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Abstract: *In this paper, we generalize the topological results of the convergence of convex – sequences in the epigraphical sense to the convex–concave sequences in Mosco – epi \ hypo graphical sense. We actually prove that if two convex – concave sequences are convergent in Mosco – epi \ hypo graphical sense, then the sequence of epi \ hypo graphical – sum of the two sequences is convergent in Mosco – epi \ hypo graphical sense.Also, we useour result to study the convergence of a sequence of Moreau – Yosida functions for convex – concave functions.*

Keywords and Phrases: *convex-concave function, epi-graph, epi\hypo-graph, epi\hypo-sum, epi\hypo-multiplication, parent convex function, parent concave function, Mosco's epi \ hypo graphical convergence.*

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I. Introduction

Epigraphical analysis studies minimization problems by using the epigraph concept:

$$epi\, f = \{(x, r) \in X \times R \;/\; f(x) \le r\}$$

Hypographical analysis studies maximization problems by using the hypograph concept:

$hypo\, f = \{(x, r) \in X \times R \;/\; f(x) \ge r\}$, where X is a vector space .

whereas epi \ hypo graphical analysis studies maximization-maximization problems,sometimes called the saddle points problems. This led to creation of new concepts such as : epi \ hypo graphical convergence - epi \ hypo graphical derivation - epi \ hypo graphical integration - epi \ hypo graphical sum - epi \ hypo graphical multiplication ….. etc.

Many mathematicians haveadopted these concepts in the study ofsaddle points problems. For more details, see [1,11,12,13].

II. Preliminaries

We recall some basic definitions and concepts that will be needed through the paper.

X will be a vector space unless Otherwise is stated.

Definition2.1(the epigraphical operation):

Let $f, g : X \to \overline{R}$. Then the**epi-sum** of f and g is defined by the relation

$$\left(f \underset{e}{+} g\right)(x) = \inf_{u \in X}\{f(u) + g(x - u)\} \qquad \forall x \in X$$

The epi-multiplication of $f : X \to \overline{R}$ by $\lambda > 0$ is defined by the relation

$$\left(\lambda \underset{e}{*} f\right)(x) = \lambda f\left(\lambda^{-1} x\right) \qquad \forall x \in X$$

In [] Attouch andWets proved that

$$epi_s\left(f \underset{e}{+} g\right) = epi_s(f) + epi_s(g) \quad , \quad epi\left(\lambda \underset{e}{*} f\right) = \lambda epi(f)$$

where $epi_s f = \{(x, r) \in X \times R \;/\; f(x) < r\}$

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Definition2.2 (the hypographical operation)

Let $f, g : X \to \overline{R}$. Then the **hypo-sum** of f and g is defined by the relation

$$\left(f \overset{h}{+} g\right)(x) = \sup_{u \in X}\{f(u) + g(x-u)\} \qquad \forall x \in X$$

$$= -\left((-f) \underset{e}{+} (-g)\right)$$

The hypo-multiplication of $f : X \to \overline{R}$ by $\lambda > 0$ is defined by the relation

$$\left(\lambda \overset{h}{*} f\right)(x) = \lambda f\left(\lambda^{-1} x\right) \qquad \forall x \in X$$

Definition2.3 $\left\{ f_n : X \to \overline{R} \quad ; \quad n \in N \right\}$ is equi – coercive function if there exists

$\theta : R^{+} \to [0, +\infty[$ where $\lim_{t \to \infty} \theta(t) = +\infty$, such that:

$$\forall n \in N \quad , \ \forall x \in X \quad ; \ f_n(x) \geq \theta(\|x\|)$$

We alsorecallsome basic definitions and notions from epi \ hypo graphical analysis.
For more information see [2,3,5,7,14].

Let $L : X \times Y \to \overline{R}$. Then, we have

Definition2.4 L is **a convex - concave function** if it is convex with respect to the first variable and concave with respect to the second variable, i.e., $\forall x \in X \quad L(.,y)$ is a convex function, and

$\forall y \in Y \quad L(x,.)$ is a concave function.

Definition2.5 Let L be aconvex - concave function.

The parent convex $F_L : X \times Y^{*} \to \overline{R}$ of L is defined by the relation

$$F_L\left(x, y^{*}\right) = \sup_{y \in Y}\left\{L(x,y) + \left\langle y, y^{*}\right\rangle\right\}$$

The parent concave $G_L : X^{*} \times Y \to \overline{R}$ of L is defined by the relation

$$G_L\left(x^{*}, y\right) = \inf_{x \in X}\left\{L(x,y) - \left\langle x, x^{*}\right\rangle\right\}$$

L is closed if $F_L = -G_L^{*}$, $F_L^{*} = -G_L$ whereas F_L^{*} , G_L^{*} the conjugate functions for F , G respectively.

Definition 2.6 (**epi \ hypo – graphical operators**)

Let $L, K : X \times Y \to \overline{R}$. The **epi\hypo-sum** of L and K is defined by the relation

$$\left(L \underset{e|h}{+} K\right)(x,y) = \inf_{u \in X} \sup_{v \in Y}\left\{L(u,v) + K(x-u, y-v)\right\}$$

$$= \inf_{\substack{u_1, u_2 \in X \\ u_1 + u_2 = x}} \ \sup_{\substack{v_1, v_2 \in Y \\ v_1 + v_2 = y}} \left\{L(u_1, v_1) + K(u_2, v_2)\right\}$$

The **epi\hypo-multiplication** of $L : X \times Y \to \overline{R}$ by $\lambda > 0$ is defined by the relation

$$\left(\lambda \underset{e|h}{*} L\right)(x,y) = \lambda L\left(\lambda^{-1} x, \lambda^{-1} y\right)$$

Theorem 2.1 [14]: Let $L, K : X \times Y \to \overline{R}$ be convex-concave functions and $\lambda > 0$ then $L \underset{e|h}{+} K$ and $\lambda \underset{e|h}{*} L$ are convex-concave functions.

Theorem 2.2 [14]: Let $L, K : X \times Y \to \overline{R}$ be convex-concave functions. Then

$$F_{L \underset{e|h}{+} K}(x, y^*) = \left[F_L(., y^*) \underset{e}{+} F_K(., y^*)\right](x)$$

$$G_{L \underset{e|h}{+} K}(x^*, y) = \left[G_L(x^*, .) \overset{h}{+} G_K(x^*, .)\right](y)$$

where:

$F_{L \underset{e|h}{+} K}, F_L, F_K$ the parent convex functions for $L \underset{e|h}{+} K, L, K$ respectively.

$G_{L \underset{e|h}{+} K}, G_L, G_K$ the parent concave functions for $L \underset{e|h}{+} K, L, K$ respectively.

Definition2.7. (The converges in a sense of Mosco's epi \ hypo graphical)

Let X, Y be Banach reflexive spaces and let the set

$\{K_n, K : X \times Y \to \overline{R}, \ n \in N\}$ be a sequence of closed andconvex – concave functions. The upper limit (Limsup) of the sequence $(K_n)_{n\in N}$ in a sense of Mosco's epi \ hypo graphical is defined by the relation:

$$(e_s / h_w - ls\, K_n)(x, y) = \sup_{y_n \xrightarrow[n]{w} y} \ \inf_{x_n \xrightarrow[n]{s} x} \ \limsup_n K_n(x_n, y_n)$$

and denoted by $e_s / h_w - ls\, K_n$. Also, the liminf of the sequence $(K_n)_{n\in N}$ in a sense of Mosco's epi \ hypo graphical is defined by the relation:

$$(h_s / e_w - li\, K_n)(x, y) = \inf_{x_n \xrightarrow[n]{w} x} \ \sup_{y_n \xrightarrow[n]{s} y} \ \liminf_n K_n(x_n, y_n), \ \forall (x, y) \in X \times Y.$$

Where $(w)\ s$ refers to the weak topology on $X \times Y$.

We say that the sequence $(K_n)_{n\in N}$ converges to K in a sense of Mosco's epi \ hypo graphical, denoted by $K_n \xrightarrow{M-e/h} K$ or $K = M - e/h - \lim_n K_n$, if the following holds true:

$$e_s / h_w - ls\, K_n \le K \le h_s / e_w - li\, K_n$$

Note that when the functions $(K_n)_{n\in N}$ do not depend on the variable Y, the definition of the convergence in a sense of Mosco's epi \ hypo graphical is identical to that of Mosco's epi \ hypo graphical with respect to the first variable. Also, when the functions $(K_n)_{n\in N}$ do not depend on the variable X, the definition of the convergence in a sense of Mosco's epi \ hypo graphical is identical to that of Mosco's epi \ hypo graphical with respect to the second variable. See [5] for more details.

It should be noted that if $(K_n)_{n\in N}$ is a sequence of convex – concave and closed functions, then

$e_s / h_w - lsK^n(., y)$ is convex and semi continuous from below with respect to x and

$h_s / e_w - liK^n(x, .)$ is concave and semi continuous from above with respect to y.

We can give an equivalent definition to the previous one as the following:

Definition2.8[5]. We say that the sequence $(K_n)_{n\in N}$ converges to K in a sense of Mosco's epi \ hypo graphical if the following two conditions hold true:

$i)\ \forall(x,y)\in X\times Y\ ,\forall y_n \xrightarrow[n]{w} y\ ,\exists x_n \xrightarrow[n]{s} x\quad /\quad \limsup_n K_n(x_n,y_n)\le K(x,y)$

$ii)\ \forall(x,y)\in X\times Y\ ,\forall x_n \xrightarrow[n]{w} x\ ,\exists y_n \xrightarrow[n]{s} y\quad /\quad \liminf_n K_n(x_n,y_n)\ge K(x,y)$

Theorem 2.3[5]. Let $X\ ,Y$ be Banach reflexive spaces and let the set

$\{F_n, F : X\times Y^* \to \overline{R}\ ,\ n\in N\}$ be a sequence of parent, convex and closed functions depending on convex - concave and closed functions. Then,

$$i \Leftrightarrow ii,$$

where, $i)\ F_n \xrightarrow{M} F$, $ii)\ K_n \xrightarrow{M-e/h} K$

Definition2.9(Moreau-Yosida function):

Let $L : X\times Y \to \overline{R}$ be a convex – concave function. Moreau-Yosida function with the two indices $\lambda>0, \mu>0$ of the function $L\in \overline{R}^{X\times Y}$, is defined by the relation:

$L_{\lambda,\mu}(x,y) := \inf_x \sup_y \left\{ L(u,v) + \frac{1}{2\lambda}\|x-u\|^2 - \frac{1}{2\mu}\|y-v\|^2 ; u\in X, v\in Y \right\}$. This function is usually denoted by $L_{\lambda,\mu}$. It is well known that $L_{\lambda,\mu}$ is a locally Lipschitz function. In Hilbert spaces, $L_{\lambda,\mu}$ admits a Saddle point denoted by $(x_{\lambda,\mu}, y_{\lambda,\mu})$. For more details, see [5].

It was proved by Autoch and Wets that the convergence of Mosco's epi/hypo graphical of a sequence of convex-concave functions is equivalent to the simple convergence of the sequence of related Moreau-Yosida functions .

III. The Main Result:

In this section, we study the Convergence of epi\ hypo-sum of two sequences of convex - concave functionsby using Mosco's epi \ hypo convergence as the following:

Theorem 3.1: Let $X\ ,Y$ be Banach reflexive spaces and let the set

$\{L_n, K_n, K, L : X\times Y \to \overline{R}\ ,\ n\in N\}$ be a sequence of convex - concave and closed functionsin which each term of the sequence is equi-coercive on X . If the following holds true:

$$L_n \xrightarrow{M-e/h} L$$
$$K_n \xrightarrow{M-e/h} K$$

Then,

$$L_n \underset{e|h}{+} K_n \xrightarrow{M-e/h} L \underset{e|h}{+} K\ .$$

Proof. Using Theorem 1.1, it is enough to prove that $F^n_{e|h} \xrightarrow{M} F_{e|h}$, where $F^n_{e|h}\ ,F_{e|h}$ are the parent and convex functions of the functions $L_n \underset{e|h}{+} K_n\ ,\ L \underset{e|h}{+} K$ respectively for all $n\in N.$ Hence, we have to prove the following two conditions:

1) $\forall(x,y^*)\in X\times Y^*,\ \forall(x_n,y^*_n)\xrightarrow{w}(x,y^*)\ ;\ \liminf_{n\to\infty} F^n_{e|h}(x_n,y^*_n)\ge F_{e|h}(x,y^*)$

2) $\forall(x,y^*)\in X\times Y^*,\ \exists(x_n,y^*_n)\xrightarrow{s}(x,y^*)\ ;\ \limsup_{n\to\infty} F^n_{e|h}(x_n,y^*_n)\le F_{e|h}(x,y^*)$

According to Theorem 2.2, we have:

$$F^{n}_{e|h}\left(x_n,y^*_n\right)=\left[F_{L_n}\left(.,y^*_n\right)\underset{e}{+}F_{K_n}\left(.,y^*_n\right)\right](x_n)$$

$$F_{e|h}\left(x,y^*\right)=\left[F_L\left(.,y^*\right)\underset{e}{+}F_K\left(.,y^*\right)\right](x)$$

Where, F_{L_n} , F_{K_n} , F_L , F_K are the parent convex functions of the functions L_n, K_n, L, K respectively for all $n\in N.$

We prove the first condition:

Let $\left(x_n,y^*_n\right)\xrightarrow[n\to\infty]{w}\left(x,y^*\right)$ and let $\varepsilon_n\xrightarrow[n\to\infty]{}0$. Then, by definition of epi-graphical summation, there exist two sequences $\left(v_n\right)_{n\in N}$, $\left(u_n\right)_{n\in N}$ in X where $u_n+v_n=x_n$ such that

$$\left[F_{L_n}\left(.,y^*_n\right)\underset{e}{+}F_{K_n}\left(.,y^*_n\right)\right](x_n)\geq F_{L_n}\left(u_n,y^*_n\right)+F_{K_n}\left(v_n,y^*_n\right)-\varepsilon_n$$

This implies that

$$\begin{aligned}\liminf_{n\to\infty}F^{n}_{e|h}\left(x_n,y^*_n\right)&=\liminf_{n\to\infty}\left[F_{L_n}\left(.,y^*_n\right)\underset{e}{+}F_{K_n}\left(.,y^*_n\right)\right](x_n)\\&\geq\liminf_{n\to\infty}F_{L_n}\left(u_n,y^*_n\right)+\liminf_{n\to\infty}F_{K_n}\left(v_n,y^*_n\right)\end{aligned}\quad\text{.......(3.1)}$$

Since $\left(L_n\right)_{n\in N}$ is a sequence of equi-coercive functions on X , it follows that F_{L_n} is also equi-coercive functions on X for all $n\in N.$

Using definition 1.14, we find that there exists a function $\theta:R^+\to[0,+\infty[$ satisfying the relation $\lim_{t\to\infty}\theta(t)=+\infty$ such that $F_{L_n}\left(u_n,y^*_n\right)\geq\theta\left(\|u_n\|\right)$ for all y^*_n and for all $n\in N.$ Thus, the sequence $\left(u_n\right)_{n\in N}$ is bounded (otherwise would imply that $\liminf_{n\to\infty}F_{L_n}\left(u_n,y^*_n\right)=+\infty$). The same argument can be applied to show that the sequence $\left(v_n\right)_{n\in N}$ is bounded. So, there exists a subsequence $\left(n_k\right)_{k\in N}$ such that

$$\liminf_{n\to\infty}F_{L_n}\left(u_n,y^*_n\right)=\lim_{k\to\infty}F_{L_{n_k}}\left(u_{n_k},y^*_{n_k}\right)$$

$$\liminf_{n\to\infty}F_{K_n}\left(v_n,y^*_n\right)=\lim_{k\to\infty}F_{K_{n_k}}\left(v_{n_k},y^*_{n_k}\right)$$

On the other hand, since $\left(u_{n_k}\right)_{k\in N}$ and $\left(v_{n_k}\right)_{k\in N}$ are bounded, we can find two subsequences $\left(n_{k'}\right)_{k'\in N}$ and $\left(n_k\right)_{k\in N}$ such that $v_{n_{k'}}\xrightarrow[k'\to\infty]{w}v$, $u_{n_{k'}}\xrightarrow[k'\to\infty]{w}u$. Therefore,

$$\begin{aligned}\lim_{k\to\infty}F_{L_{n_k}}\left(u_{n_k},y^*_{n_k}\right)&=\lim_{k'\to\infty}F_{L_{n_{k'}}}\left(u_{n_{k'}},y^*_{n_{k'}}\right)\\\lim_{k\to\infty}F_{K_{n_k}}\left(v_{n_k},y^*_{n_k}\right)&=\lim_{k'\to\infty}F_{K_{n_{k'}}}\left(v_{n_{k'}},y^*_{n_{k'}}\right)\end{aligned}\quad\text{..................(3.2)}$$

We have $L_n\xrightarrow{M-e|h}L$, $K_n\xrightarrow{M-e|h}K$. According to Theorem 2.3 we find that $F_{L_n}\xrightarrow{M}F_L$, $F_{K_n}\xrightarrow{M}F_K$. So, we have

$$\begin{aligned}\lim_{k'\to\infty}F_{L_{n_{k'}}}\left(u_{n_{k'}},y^*_{n_{k'}}\right)&\geq F_L\left(u,y^*\right)\\\lim_{k'\to\infty}F_{K_{n_{k'}}}\left(v_{n_{k'}},y^*_{n_{k'}}\right)&\geq F_K\left(v,y^*\right)\end{aligned}\quad\text{.........(3.3)}$$

Using (3.2) and (3.3) and substituting in (3.1) we obtain the following:

$$\liminf_{n\to\infty} F_{e|h}^{n}\left(x_n, y_n^*\right) \geq F_L\left(u, y^*\right) + F_K\left(v, y^*\right)$$

$$\geq \inf_{\substack{u,v\in X \\ u+v=x}} \left\{F_L\left(u, y^*\right) + F_K\left(v, y^*\right)\right\}$$

$$\geq \left[F_L\left(., y^*\right) \underset{e}{+} F_K\left(., y^*\right)\right](x) = F_{e|h}\left(x, y^*\right)$$

This proves the first condition. Now, for the second one:
Let $0 < \varepsilon$. Then there exist $\bar{v}$, $\bar{u}$ in X in which $\bar{u} + \bar{v} = x$ such that

$$F_{e|h}\left(x, y^*\right) + \varepsilon \geq F_L\left(\bar{u}, y^*\right) + F_K\left(\bar{v}, y^*\right) \text{........(3.4)}$$

Since $F_{L_n} \xrightarrow{M} F_L$, there exists $\left(\bar{u}_n, y_n^*\right) \xrightarrow[n\to\infty]{s} \left(\bar{u}, y^*\right)$ such that

$$F_L\left(\bar{u}, y^*\right) \geq \limsup_{n\to\infty} F_{L_n}\left(\bar{u}_n, y_n^*\right) \text{.........(3.5)}$$

Also, since $F_{K_n} \xrightarrow{M} F_K$, there exists $\left(\bar{v}_n, y_n^*\right) \xrightarrow[n\to\infty]{s} \left(\bar{v}, y^*\right)$ such that

$$F_K\left(\bar{v}, y^*\right) \geq \limsup_{n\to\infty} F_{K_n}\left(\bar{v}_n, y_n^*\right) \text{.......(3.6)}$$

Substituting (3.5) and (3.6) in (3.4), we obtain:

$$F_{e|h}\left(x, y^*\right) + \varepsilon \geq \limsup_{n\to\infty} F_{L_n}\left(\bar{u}_n, y_n^*\right) + \limsup_{n\to\infty} F_{K_n}\left(\bar{v}_n, y_n^*\right)$$

$$\geq \limsup_{n\to\infty}\left[F_{L_n}\left(\bar{u}_n, y_n^*\right) + F_{K_n}\left(\bar{v}_n, y_n^*\right)\right]$$

$$\geq \limsup_{n\to\infty}\left[F_{L_n}\left(., y_n^*\right) \underset{e}{+} F_{K_n}\left(., y_n^*\right)\right](x_n)$$

$$\geq \limsup_{n\to\infty} F_{e|h}^{n}\left(x_n, y_n^*\right)$$

Since the above inequality holds true for all $0 < \varepsilon$, it follows (by letting $0 < \varepsilon$ tends to zero) that $F_{e|h}\left(x, y^*\right) \geq \limsup_{n\to\infty} F_{e|h}^{n}\left(x_n, y_n^*\right)$. This proves the second condition and completes the proof of the theorem.

Theorem 3.2: Let $L : X \times Y \to \overline{R}$ be a convex- concave function, where X, Y are Banach reflexive spaces. Then, the following conditions are equivalent:

$$i) \quad L_n \xrightarrow{M-e|h} L$$

$$ii) \quad \begin{array}{c} \forall (x, y) \in X \times Y \quad , \quad \forall \lambda > 0 \ , \ \forall \mu > 0 \\ \lim_{n\to\infty} (L_n)_{\lambda,\mu}(x, y) = L_{\lambda,\mu}(x, y) \end{array}$$

It should be noted that the relation (3.4) can be written in the following:

$$L_{\lambda,\mu}(x, y) = \left(L \underset{e|h}{+} \left(\frac{1}{2\lambda}\|.\|^2 - \frac{1}{2\mu}\|.\|^2\right)\right)(x, y).$$

This means that the function $L_{\lambda,\mu}$ is a sum of the functions $K = \left(\frac{1}{2\lambda}\|.\|^2 - \frac{1}{2\mu}\|.\|^2 \right)$ and the epi/hypo graphical of the function L.

By applying Theorem 3.1, we obtain a generalization of the previous theorem as the following:

Theorem 3.3: Let X, Y be Banach reflexive spaces and let the set $\left\{ L_n, L : X \times Y \to \overline{R} \;,\; n \in N \right\}$ be a sequence of convex – concave , closed and equi-coercive functions on X . If

$$L_n \xrightarrow{M-e|h} L \text{ for all } (x, y) \text{ and for all } \lambda > 0,\ \mu > 0, \text{ then } (L_n)_{\lambda,\mu} \xrightarrow{M-e|h} L_{\lambda,\mu}$$

Proof. The proof can be done by putting

$$K_n = K = \left(\frac{1}{2\lambda}\|.\|^2 - \frac{1}{2\mu}\|.\|^2 \right) \text{ in Theorem 3.1.}$$

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