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Convergence of new Bernstein type operators.

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ABSTRACT. We concern in this paper new Bernstein type operator with parameter $\lambda \in [-1, 1 - 1/n]$. We discuss the Rate of Convergence of Bernstein operator, prove that Bernstein approximation theorem with that new parameter and also prove that Voronovskaja type theorem using by exponential operators.

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I. Introduction

In 1912 Given by S.N.Bernstein. A function f on $[0, 1]$ they define the polynomial

$$B_n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n}). \tag{1}$$

for each positive integer n. Known as Bernstein polynomials.We will discuss in this paper, if f is continuous on $[0, 1]$, its sequence of Bernstein polynomials converges uniformly to f on $[0, 1]$, thus giving constructive proof of Weierstrass theorem. There are several proof of this fundamental theorem beginning with that given by K.Weierstrass after Bernstein gave its constructive proof in 1912. We introduce by using Exponential operator rate of convergence of Bernstein operator and proof the Bernstein approximation theorem with new parameter $\lambda \in [-1, 1-1/n]$ and also prove that Voronovskaja type theorem by exponential operator with new parameter.Now this new operator known as λ-Bernstein operator such that

$$B_n f(x) = \sum_{k=0}^{n} f(k/n) \tilde{b}_{n,\lambda}(x) \tag{2}$$

where $x \in [0, 1]$, and $n = 1, 2, \ldots$ where the function $b_{n,\lambda}(x)$ are defined as in the paper by Ye.Z. Long [10] new Bezier bases with parameter λ

by

$$\begin{cases} \tilde{b}_{n,0}(\lambda;x) = b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\ \tilde{b}_{n,k}(\lambda;x) = b_{n,k}(x) + \lambda\left(\frac{n-2k+1}{n^2-1} b_{n+1,k}(x) - \frac{n-2k-1}{n^2-1} b_{n+1,k+1}(x)\right), \\ for-all, k \in [1, n-1] \\ \tilde{b}_{n,n}(\lambda;x) = b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x) \end{cases} \tag{3}$$

where $\lambda \in [-1,1]$ and using $\lambda = 0$ then reducing to classical Bernstein polynomial. Above section given by Qing-Bo Cai et al. [8] in 2018 some Bezier type results for λ- Brnstein operators with parameter $\lambda \in [-1,1]$ where

$$B_{n,\lambda}(f;x) = \sum_{k=o}^{n} \tilde{b}_{n,k}(\lambda;x) f(\frac{k}{n}) \tag{4}$$

where $B_{n,\lambda}(f;x)$,and $(k = 0, 1 \ldots n)$ are defined in (3) and $\lambda \in [-1,1]$. Many papers about Bernstein polynomials given by [4, 6, 9]and recently in 2018 Qing-Bo Cai [8] gives Approximation properties of λ-Bernstein operator.In 2015 Ivan Gaverea [7] proves the classical Bernstein -Voronoskaja type theorem for Positive linear approximation operators. In 2019 Adrian holhas [2] presents a Voronovskaya formula for the first derivatives of Positive linear operators.In these paper we discuss the convergence properties of new type Bernstein operator with parameter $\lambda \in [-1, 1-1/n]$ and New Bernstein Approximation theorem also prove that Voronovskaja type theorem for this new parameter with Exponential operator and Bezier bases. This gives new type of work because combination of Exponential operator and Bezier bases formula. In 2018 Cai et al. [8] gave some Lemma and Corollary for λ Bernstein operators and uses Bezier bases.

Lemma 1.0.1. *For λ-Bernstein operators, they have the following equalities.*

$$B_{n,\lambda}(1;x) = 1; \tag{5}$$

$$B_{n,\lambda}(t;x) = x + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda; \tag{6}$$

$$B_{n,\lambda}(t^2;x) = x^2 + \frac{x(x-1)}{n} + \lambda\left[\frac{2x-4x^2+2x^{n+1}}{n(n-1)} + \frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}\right]; \tag{7}$$

Corollary 1.0.2. *For fixed $x \in [0,1]$ and $\lambda \in [-1,1]$, using above lemma and by some easy computation,we have*

$$B_{n,\lambda}(t-x;x) = \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}\lambda \le \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n(n-1)}. \tag{8}$$

$$B_{n,\lambda}((t-x^2);x) = \frac{x(1-x)}{n} + \left[\frac{2x(1-x)^{n+1}+2x^{n+1}-2x^{n+2}}{n(n-1)} + \frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}\right]\lambda$$

$$\le \frac{x(1-x)}{n} + \left[\frac{2x(1-x)^{n+1}+2x^{n+1}-2x^{n+2}}{n(n-1)} + \frac{x^{n+1}+(1-x)^{n+1}-1}{n^2(n-1)}\right] \tag{9}$$

$$lim_{n\to\infty} nB_{n,\lambda}(t-x;x) = 0; \qquad (10)$$
$$lim_{n\to\infty} nB_{n,\lambda}(t-x)^2;x) = x(1-x), x \in (0,1). \qquad (11)$$

Remark 1.0.3. For $\lambda \in [-1, 1-1/n], x \in [0,1]$, λ- Bernstein operators possess the endpoint interpolation property that is

$$B_{n,\lambda}(f;0) = f(0), B_{n,\lambda}(f;1) = f(1).$$

In 2019 Adrian Holhas [2]gave the general exponential operators- Let $I \subset \mathbb{R}$ be an open interval and let $\alpha \geq 0$ be a real number.Consider a continuous function $\theta : (0,\infty) \to \mathbb{R}$ and we denote by $C_{\theta,\alpha}$ the space of continuous functions $f \in C(I)$ with the property that exist $M > 0$ such that $|f(x)| \leq Me^{\alpha\theta(|x|)}$, for every $x \in I$. Because of the symmetry and to simplify the notation we consider in the following that $I \subset (0,\infty)$.

Lemma 1.0.4. *Consider a sequence of positive linear operators (L_n) preserving the constants and having the property that for every $f \in C_{\theta,\alpha}$ there exists an integer $n_\alpha \in \mathbb{N}$ such that $L_n f$ exists for every $n \geq n_\alpha$. Suppose that $L_n(e^{\alpha\theta(t)})$ converges pointwise on I.Then, for every $x \in I$ and for every $\alpha \geq 0$*

$$L_n\left(max(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}); x\right) \leq M_\alpha(x), n \geq n_\alpha,$$

where $M_\alpha(x) > 0$ depends on α and x but not on n.

2. MAIN RESULTS

2.1. New Bernstein Approximation theorem :-.

Theorem 2.1.1. *If f is continuous function in $C[0,1]$ and for any $\epsilon > 0$ then there exists an integer N such that*

$$|B_{n,\lambda}(f;x) - f(x)| < \in, \forall x \in [0,1] \quad and \quad \forall \quad n \geq N.$$

Proof. In 1912 S.N.Bernstein given the famous polynomials known as Bernstein polynomials

$$B_{n,\lambda}(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n}). \qquad (12)$$

Now let

$$B_{n,\lambda}(f;x) - f(x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} - \sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k}$$

Because we know $\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = 1$ and by (12)

$$B_{n,\lambda}(f,x) - f(x) = \sum_{k=0}^{n} [f(\frac{k}{n}) - f(x)] \binom{n}{k} x^k (1-x)^{n-k} \qquad (13)$$

therefor

$$|B_{n,\lambda}(f,x)-f(x)| \leq \sum_{k=0}^{n}[f(\frac{k}{n})-f(x)]\binom{n}{k}x^k(1-x)^{n-k} \qquad (14)$$

We use the term $\sum_{k=0}^{n} = \sum_{k\in S_\delta} + \sum_{k\notin S_\delta}$
Where S_δ denote the set of all values of k. in (14)
$|B_{n,\lambda}(f,x) - f(x)| \leq$

$$\sum_{\substack{k\in S_\delta \\ |k/n-x|\geq\delta}} |f(k/n)_\lambda - f(x)|\binom{n}{k}_\lambda x^k(1-x)^{n-k} + \sum_{\substack{k\notin S_\delta \\ |k/n-x|<\delta}} |f(k/n)_\lambda - f(x)|\binom{n}{k}_\lambda x^k(1-x)^{n-k} \qquad (15)$$

By using Exponetial operator given by in 2019 Holhas [2] as

$$|f(x)| \leq Me^{\alpha\theta(|x|)},$$
$$\forall \quad x \in I, \quad M > 0 and \quad \alpha \geq 0$$

then

$$\sum_{\substack{k\in S_\delta \\ |k/n-x|\geq\delta}} |f(k/n)_\lambda - f(x)| \leq Me^{\alpha\theta(|k/n|)_\lambda} + Me^{\alpha\theta(|x|)} \leq M\big(e^\alpha\theta|(k/n+x)|_\lambda\big),$$

when $\quad n \quad tends \quad to \quad \infty$ then k/n is lowest.

$$\leq Me^{\alpha\theta(|x|)} \leq M_\alpha(x), \quad \forall \quad n \geq N \qquad (16)$$

Now let us

$$\sum_{\substack{k\notin S_\delta \\ |k/n-x|<\delta}} |f(k/n)_\lambda - f(x)|\binom{n}{k}_\lambda x^k(1-x)^{n-k} \qquad (17)$$

is second part of (15) we know that if f is continuous then it is also uniformly continuous on the closed interval $[0, 1]$ and vise versa, then $\forall \quad \epsilon > 0$ there exist $\delta > 0$ depending on $\epsilon \quad f$ and its points. such that

$$|f(x) - f(y)| < \frac{\epsilon}{2}$$

when $|x - y| < \delta$,
$\forall \quad x, y \in [0, 1]$.
Now then in (17) we gets

$$\sum_{\substack{k\notin S_\delta \\ |k/n-x|<\delta}} |f(k/n)_\lambda - f(x)|\binom{n}{k}_\lambda x^k(1-x)^{n-k}$$

$$< \frac{\epsilon}{2}\sum_{k\notin S_\delta} x^k(1-x)^{n-k}\binom{n}{k}_\lambda < \frac{\epsilon}{2} \qquad (18)$$

By (16) and (18) we get by (15)

$$|B_{n,\lambda}(f;x) - f(x)| < M_\alpha(x) + \frac{\epsilon}{2}$$

then if we charge $N > 2M/\epsilon$, for all $n \geq N$. therefore finally we proved that

$$|B_{n,\lambda}(f;x) - f(x)| < \epsilon. for \quad all \quad n \geq N.$$

□

2.2. Rate of convergence of new Bernstein operators :-.

Theorem 2.2.1. *Let $f \in C(I)$ and $x \in [0,1]$ with the new parameter $\lambda \in [-1, 1-1/n]$ then we have*

$$|B_{n,\lambda}(f;x) - f(x)| \leq M_\alpha(x).$$

Now before proving this theorem we give a lemma.

Lemma 2.2.2. *Assuming a sequence of new Bernstein operators $B_{n\lambda}(f;x)$ preserve the constants and having the property that for all $f \in C_{\theta,\alpha}$ then there exists an integers $n_\alpha \in N$ such that $B_{n\lambda}(f;x)$ exist for every $n_\alpha \in N$. Suppose that $B_{n\lambda}(f;x)$ converges pointwise on I then for every $x \in I$ and for every $\alpha \geq 0$*

$$B_{n\lambda}(f;x) \leq M_\alpha(x), \forall \quad n \geq n_\alpha$$

where $M_\alpha(x) > 0$ depends on α and x not on n

Proof. It is easily proved the lemma by using lemma 1.0.4 □

Proof. theorem 2.2.1:- Since $B_{n,\lambda}(f;x)$ is the Bernstein operator and $f \in C(I)$ we have

$$|B_{n,\lambda}(f;x) - f(x)| \leq B_{n,\lambda}(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}; x)$$
$$= B_{n,\lambda}(e^{\alpha\theta(t)+\alpha\theta(x);x}$$
$$= B_{n,\lambda}(f;x) \leq M_\alpha(x).$$

by using the above lemma so proved the theorem. □

Now we give proof of the famous Voronovskaja type theorem for Exponential operator using by new Bernstein polynomials with shape parameter λ which is given by Elizaveta V. Voronovskaya(1898-1972).

Theorem 2.2.3. *If $f(x)$ is a bounded functions on the closed interval $[0,1]$. Then the $\forall \quad x \in [0,1]$ at wchich $f"(x)$ exists s.t.*

$$lim_{n\to\infty} n(B_n, \lambda(f;x) - f(x)) = \frac{1}{2}x(1-x)f"(x).$$

Before proving the famous theorem we use a result which was given by [1].

Theorem 2.2.4. [1] *Let m be a nonnegative integer and let $f \in C_{\theta,\alpha}$ such that f is m times continuously differentiable with $f^{(m)} \in C_{\theta,\alpha}$. Then*

$$|L_n(f,x) - \sum_{k=0}^{m} \frac{f^{(k)}(x)}{k!}\dot{\mu}_{n,k}(x)| \leq \frac{1}{m}\left(A_{n,m}(x) + \frac{B_{n,m}(x)}{\delta_n}\right)\omega_{\varphi,\theta,\alpha}\left(f^{(m)}, \delta_n\right)$$

where

$$A_{n,m}(x) = L_n\Big(max(e^{\alpha\theta(t)}, e^{\alpha\theta(x)})\Big)|t-x|^m, x\Big),$$

$$B_{n,m}(x) = L_n\Big(max(e^{\alpha\theta(t)}, e^{\alpha\theta(x)})\Big)|t-x|^m|\varphi(t) - \varphi(x)|, x\Big),$$

$$\omega_{\varphi,\theta,\alpha}(f,\delta) = \sup_{\substack{x,t\in I\\ |\varphi(t)-\varphi(x)|\leq\delta}} \frac{|f(t)-f(x)|}{max(e^{\alpha\theta(t)}, e^{\alpha\theta(x)})}$$

and φ is a continuous and strictly increasing function on I such that $\theta o\varphi^{-1}$ is uniformly continuous on $\varphi(I)$.

Proof. Theorem(2.2.3) Let $x \in [0,1]$ and using the Taylors expansion for proving that

$$f(a) = f(x)+f'(x)(a-x)+\frac{1}{2}f"(x)(a-x)^2+h_m(x,a)(a-x)^m \qquad (19)$$

where $m=2$ Now using the properties [1]lemma4

$$|h_m(x,a)| \leq \frac{1}{m!}.max(e^\alpha\theta(a), e^\alpha\theta(x))(1+\frac{|\phi(a)-\phi(x)|}{\delta}\omega\varphi\theta\alpha(f^m.\delta) \qquad (20).$$

now using the above theorem(2.2.4) we get by (20)

$$|h_2(x,a)| \leq \frac{1}{2!}.max(e^\alpha\theta(a), e^\alpha\theta(x))\Big(1+\frac{\delta}{\delta}\sup(\frac{|f(a)-f(x)|}{max(e^\alpha\theta(a), e^\alpha\theta(x))})\Big)$$

$\leq \frac{1}{2}(1+1).\epsilon \leq \epsilon$ where ϵ is an arbitrary and very small therefore the term $|h_m(x,a)|$ in (19)will be left. Then now (19) given by

$$f(a) = f(x) + f'(x)(a-x) + \frac{1}{2}f"(x)(a-x)^2 \qquad (21)$$

$$B_{n,\lambda}(f;x) - f(x) = f'(x)B_{n,\lambda}(a-x;x) + \frac{f"(x)}{2}B_{n,\lambda}((a-x)^2;x)$$

We take limits on both sides

$$\lim_{n\to\infty} n[B_{n,\lambda}(f;x) - f(x)] =$$

$$f'(x)\lim_{n\to\infty} nB_{n,\lambda}(a-x;x)+\frac{f"(x)}{2}\lim_{n\to\infty} nB_{n,\lambda}((a-x)^2;x) \qquad (22)$$

We using Corrollary (1.0.2) equation(10) and (11) gives that

$$\lim_{n\to\infty} nB_{n,\lambda}(a-x;x) = 0$$

and

$$\lim_{n\to\infty} n^2B_{n,\lambda}((a-x)^2;x) = x(1-x).$$

after using these results we get by (22)

$$lim_{n\to\infty}n(B_n,\lambda(f;x) - f(x)) = \frac{1}{2}x(1-x)f"(x).$$

So proved the famous theorem using property Exponential operators and Bezier form. □

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