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A Numerical Investigation in Option Valuation

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Abstract:*In modern finance, derivatives like options are actively traded on many exchanges throughout the world. Since pricing option is a challenging task, it attracts the attention of many researchers nowadays. In many cases calculation of large number of prices is required in short time, so fast and accurate calculation of option price is crucial. This paper introduces some fundamental concepts on underlying option valuation theory including implementation of computational tools. To do this, numerical methods such as Binomial Trees, Monte Carlo Simulation are discussed. Both these numerical techniques are used to price the most desirable European options. Hence the results are compared with the standard Black-Scholes-Merton Model with the help of a computer algebra system MATLAB.*

Key Word*:European options, Black-Scholes model, Binomial Tree, Monte Carlo Simulation*.

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I. Introduction

Options are extremely versatile financial contracts and the name itself indicates the implication of choice of the holder. The seller of an option has compulsion of anyhow meeting up the constraints under it while the holder owns free will to decide whether to use it or not [1]. However, to master the art of option trading, one requires proper investment of practice and time, as well as enough money and risk management. Moreover, volatility issues are to be taken under immense attention in order to exert options to the fullest [2].

Option pricing models are generally mathematical models that use certain parameters to calculate theoretical value of an option. The most favored option pricing model is the Black-Scholes-Merton or Black-Scholes model proposed by Fisher Black, Myron Scholes and Robert Merton back in 1973 [3] which led them to win Nobel prize in 1997. It is still considered as the benchmark of all posterior models and is extensively practiced to evaluate the premium of an option as it provides a simple closed form solution in case of a continuous dividend paying stock price. Though most of the option pricing models are mainly modified forms of this model, it some- times gives inaccurate prices while tested against real data due to some rigid assumptions such as geometric Brownian motion with constant drift and volatility which are barely justified in real market [4]. Under these assumptions, the change in the asset price is normally distributed.

On the other hand, the Binomial Tree option pricing model is an alternative option valuation method developed in 1979 by Cox, Ross and Rubinstein [5]. According to them the value of a European option can be obtained by discounting the expected maturity value of the option. This method arises form discrete random walk models of the underlying option and uses an iterative procedure which allows specified nodes or time points, during the time span between the valuation and expiration date of the option. After each time interval, the price can go either up or down by a given percentage [6]. The model provides no analytic solution hence the option price must be evaluated by numerical techniques.

Another substitution can be done by Monte Carlo simulation which is a numerical technique. This technique provides a series of procedures to sample random outcomes for a particular process. In 1987, Hull and White applied this technique to price options with stochastic volatility [7]. As an alternative to Black-Scholesand Binomial Tree methods, Boyle proposed Monte Carlo as an option pricing model in 1977 given the fact that it can be tempered to adapt with different processes under the stock returns [8].Again in 1997, with the association of Glasserman and Broadie, Boyle used Monte Carlo simulation for security pricing [9].

The purpose of this paper is to represent the performance of two different numerical schemes for option pricing. One is a discrete model known as Binomial Tree while the other is Monte Carlo- a simulation process. We consider Black-Scholes generated option price as exact value and hence compare the convergence of the above mentioned models with it. In section II, we discuss the models in details. The results and discussions are added in section III. At last in section IV, we present some concluding remarks.

II. Models

In the next section, we are going to discuss three option pricing methodology, named as, Black-Scholes model, Binomial Tree and Monte Carlo. We also include some numerical results accompanying with MATLAB programming codes.

Black-Scholes Model

Black-Scholes is a pricing model used to determine the fair price or theoretical values for a European call or put option. This model provides a partial differential equation which must be satisfied by price of any derivative dependent on non-dividend asset. The differential equation is based on some assumptions [3], such as, the stock price can be characterized by the following stochastic process with constant expected return (μ) and volatility (σ) :

$$dS = \mu S dt + \sigma S dz \tag{1}$$

where, z is a standard Brownian motion and $z(t) \sim N(0,t)$, S is the stock price, T is time to maturity with $t \in [0,T]$, r is the risk-free interest rate. Solving the partial differential equation given by this model, we get a closed form solution to calculate European call option price (C) as follows:

$$C = S_t N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{2}$$

Similarly, formula for European put option price (P) is,

$$P = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1) \tag{3}$$

Here,

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \; ; \; d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

The Black-Scholes pricing formula leads to a direct calculation of the option value which could make a high computational effort. Also it is not possible to find the option price at any period through this model. Therefore we discuss other two numerical option pricing techniques.

Binomial Model

The Binomial option pricing model reduces the likelihood of price changes and remove the possibility for arbitrage. Under the assumption of a perfectly efficient market, it is able to provide a mathematical value of an option at each point in the defined time-frame. The model takes a risk-neutral approach to valuation [3] and assumes that underlying security prices can only either increase (by the factor u) or decrease (factor d) with time until the option expires worthless as shown in FIGURE 1.

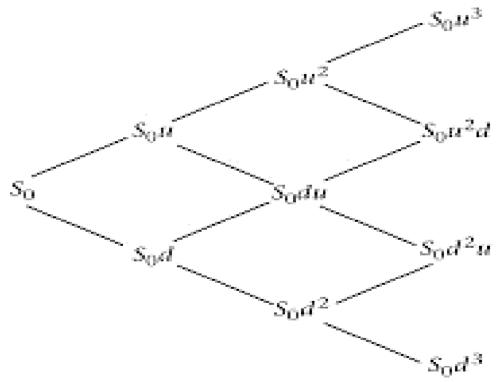

FIGURE 1: Stock price movement in general Binomial Tree.

Risk-neutrality demands: $Se^{rdt} = pSu + (1-p)Sd$. Here, p is the probability of an up movement and $(1-p)$ probability of a down movement. The discounted expected return equals the current price[10].Applying the same argument from time period to time period, it is possible to have Binomial trees with multiple time steps to simulate the movement of the underlying asset more accurately. Adding more steps in the tree leads to a Binomial distribution with more and more possible outcomes which should ultimately approximate to the continuous, lognormal distribution such as continuous time pricing models like Black-Scholes.

The two most popular models for Binomial pricing are Cox, Ross and Rubinstein (1979, CRR for short) whose extra advantage is to set [11],

$$ud = 1; u = e^{\sigma\sqrt{T}}\ ; d = e^{-\sigma\sqrt{T}}\ ;\ p = \frac{e^{rT} - d}{u - d}$$

The other is Rendleman and Bartter (1979) who choose,

$$p = 1\big/2\ ;\ u = e^{(r - 1/2\sigma^2)T + \sigma\sqrt{T}}\ ;\ \ d = e^{(r - 1/2\sigma^2)T - \sigma\sqrt{T}}$$

Let us consider that, S_0 is the current stock price and we have N time steps with time interval $\Delta t = T\big/N$. If we take S_{ij} and C_{ij} as the underlying stock price and European call option price respectively after time step i and upstate j , then we have[12],

$$S_{ij} = S_0 u^i d^{i-j} \tag{4}$$

$$C_{Nj} = \max\left(S_0 u^j d^{N-j} - K, 0\right) \tag{5}$$

$$C_{ij} = e^{-r\Delta t}\left(pC_{i+1,j+1} + (1-p)C_{i+1,j}\right) \tag{6}$$

where $i < N$ and p , u and d are selected according to any preferred model (CRR or alternative).

MONTE CARLO SIMULATION

Monte Carlo is a numerical scheme that uses the probabilistic solution and is very useful for options on more than one underlying asset The main idea behind the Monte Carlo technique is that we simulate paths that could be taken by the underlying asset (under the risk-neutral probability) to estimate an expected option price at expiry,which can be discounted back to today [8]. The convergence of the correct option value will be at a rate of $N^{-\frac{1}{2}}$, where N is the number of sample paths. If we have a sequence of independent, identically distributed random variables Y_n then we have that,

$$\lim_{n\to\infty} \frac{1}{N}\sum_{n=1}^{N} Y_n = \mathrm{E}\left[Y_1\right] \tag{7}$$

which is the law of large numbers[12]. With the Monte Carlo technique what we are trying to do is to evaluate the value of $\mathrm{E}\left[f\left(Y_T\right)\right]$ which is the expectation of a function of a random variable Y_T .Consider a derivative dependent on a single market variable S that provides a payoff at time T . Then we can value the derivative as follows [3]:

1. Sample a random path for S in a risk-neutral world.

2. Calculate the payoff from the derivative.

3. Repeat steps 1 and 2 to get many sample values of the payoff from the derivative in a risk-neutral world.

4. Calculate the mean of the sample payoffs to get an estimate of the expected payoff in a risk-neutral world.

5. Discount this expected payoff at the risk-free rate to get an estimate of the value of the derivative

Since Monte Carlo method is built on the foundation of risk-neutral pricing, the price of underlying asset in a risk-neutral world follows *Ito*process.To simulate the path followed by S, we can divide the life of the derivative into N short intervals of length t and approximate the equation (8)

$$S_{t+\Delta t} - S_t = rS_t\Delta t + \sigma S_t \Phi\sqrt{\Delta t} \tag{8}$$

where, Φ is a random sample from a normal distribution with mean zero and standard deviation of 1.0. So the values of S at time Δt are to be calculated from the initial value of S , the value at time $2\Delta t$ to be calculated from the value at time Δt , and so on. One simulation trial involves constructing a complete path for S using N random samples from a normal distribution. Since in practice it is more accurate to simulate $\ln S$ rather than S . Using *Ito*'s lemma ,we get the process followed by $\ln S$ as follows :

$$d\ln S = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dz \quad (9)$$

which implies,

$$S_{(t+\Delta t)} = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\Phi\sqrt{\Delta t}\right] \quad (10)$$

This equation is used to construct a path for S. For the time interval $[0,T]$ and considering r and σ are constant we get,

$$S_{(t+\Delta t)} = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\Phi\sqrt{T}\right] \quad (11)$$

To estimate the expected option value at time T , then we take random draws from the $N(0,1)$ distribution which enables us to calculate S_T and then calculate $V(S_T)$. To get an approximation of the expectation, we then average $V(S_T)$. Thus if the n th draw from the normal distribution gives $V^n(S_T)$, then by the law of large numbers [12],

$$\frac{1}{N}\sum_{m=1}^{N} V\left(S_T^m\right) \to \mathrm{E}_t^Q\left[V\left(S_T\right)\right]; N \to \infty$$

The following figure shows different paths of the stock price which follows geometric Brownian motion.

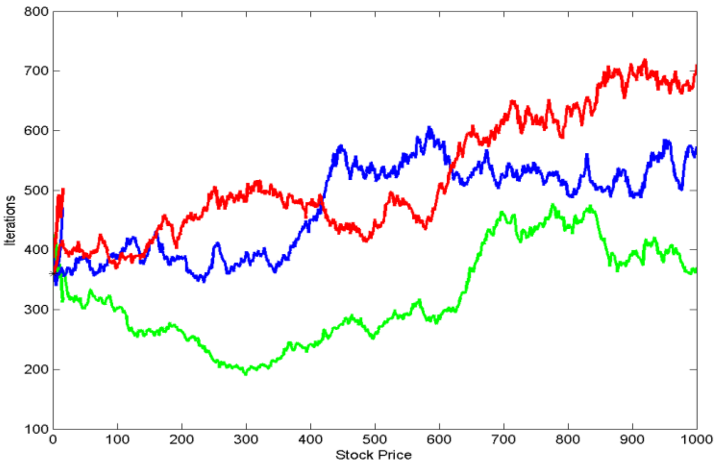

FIGURE 2: Simulating paths for one stock price.

It is very simple to apply Monte Carlo method for different types of options. Here we will profile a general case of European options. We discount back the final payoff by factor $\exp(-rt)$ and derive the payoff function. For a European call option the payoff at maturity $C(S_T)$ is given by,

$$C(S_T) = (S_T - K)^+ \quad (12)$$

and the underlying asset follows,

$$S_T = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\phi_n\sqrt{T-t}\right] \quad (13)$$

So to value the option we need to simulates N possible values or paths for S_T by making N independent draws from $N(0,1)$ then to use these possible values, denoting them ϕ_n for $1 \le n \le N$, we have,

$$S_T^n = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\phi_n\sqrt{T-t}\right] \quad (14)$$

$$C(S_T^n) = (S_T^n - K)^+ \quad (15)$$

$$C(S_t, t) = \exp\big(-r(T-t)\big)\frac{1}{N}\sum_{n=1}^{N} C(S_T^n) \qquad (16)$$

III. Result & Discussion

In this section, we see the calculation of option values by the two methods discussed so far and hence discuss the convergence of these models with the standard Black-Scholes produced option values.

Binomial Model

Let us consider an example to price an European call option using multi step Binomial tree with expiration time $T = 1$ year, stock price S_0 =100, strike price $K = 100$,risk-free interest rate $r = 0.06$ and volatility $\sigma = 0.2$. For three step Binomial tree we get the price 11.552 at time t = 0 produced by MATLAB coding. Black-Scholes model gives 10.9895 for the same data. In FIGURE 3, we see the movement of stock price with different time period. The corresponding call option price is obtained by backward process. The option prices are shown in Figure 4.

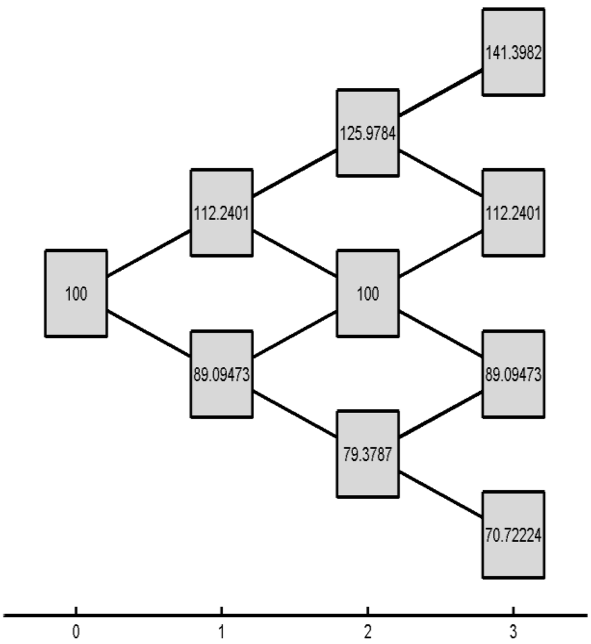

FIGURE 3: Stock price dynamics in different time period.

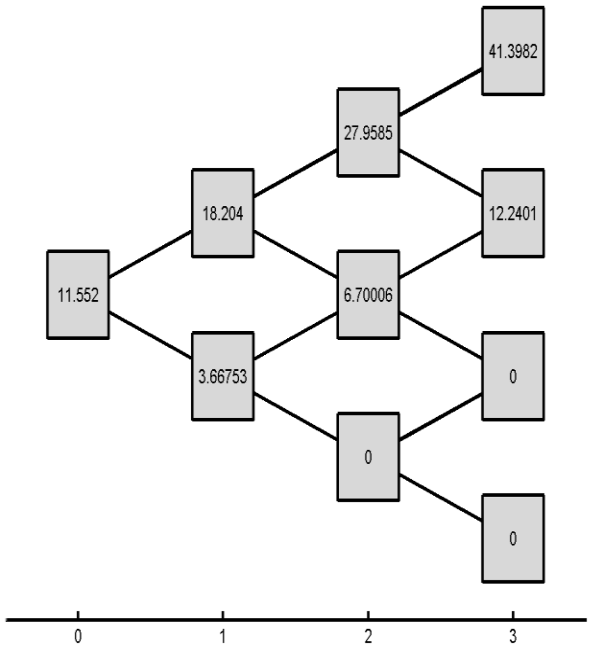

FIGURE 4: Option price dynamics in different ti period.

In Table 1 we observe that, as the number of steps increases Binomial tree prices converge to Black-Scholes price.

Table 1: European call option prices using Binomial tree for different number of steps.

Number of Steps	Binomial Tree Price	Black Scholes Price	Absolute Error	Relative Error
5	11.3272		0.3377	0.0307
10	10.7910		0.1985	0.0181
15	11.1015		0.1120	0.0102
20	10.8896		0.0999	0.0091
25	11.0566		0.0671	0.0061
30	10.9227		0.0668	0.0061
35	11.0374		0.0479	0.0044
40	10.9394		0.0501	0.0046
45	11.0267		0.0372	0.0034
50	10.9494	10.9895	0.0401	0.0037
55	11.0200		0.0305	0.0028
60	10.9561		0.0334	0.0030
65	11.0153		0.0258	0.0023

70	10.9608		0.0287	0.0026
75	11.0118		0.0223	0.0020
80	10.9644		0.0251	0.0023
85	11.0092		0.0197	0.0018
90	10.9672		0.0223	0.0020
95	11.0071		0.0176	0.0016
100	10.9694		0.0201	0.0018

In FIGURE 5, we show the convergence of Binomial tree option price with the Black-Scholes price for different number of time steps. We clearly notice that Binomial tree is not good for small number of steps, but almost converges to the right price as we increase time steps.

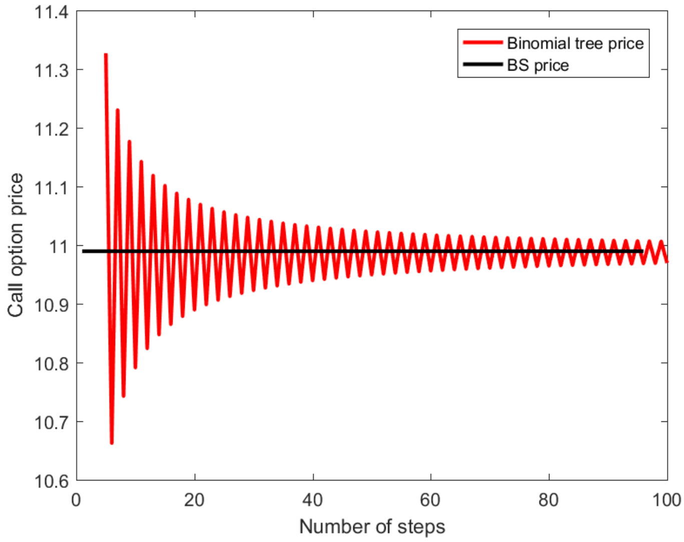

FIGURE 5: Convergence of Binomial tree with increasing number of steps.

Monte Carlo Simulation

Now we consider the previous example to compute the price of European call option using Monte Carlo Simulation as well. We already mentioned that Black-Scholes formula (2) gives 10.9895 for this example. The results can be obtained by a simple MATLAB code. The corresponding figure of the example is showing below. In FIGURE 6, we observe that, simulation of paths for the European call option then finally we obtain the average value (shown in red line).

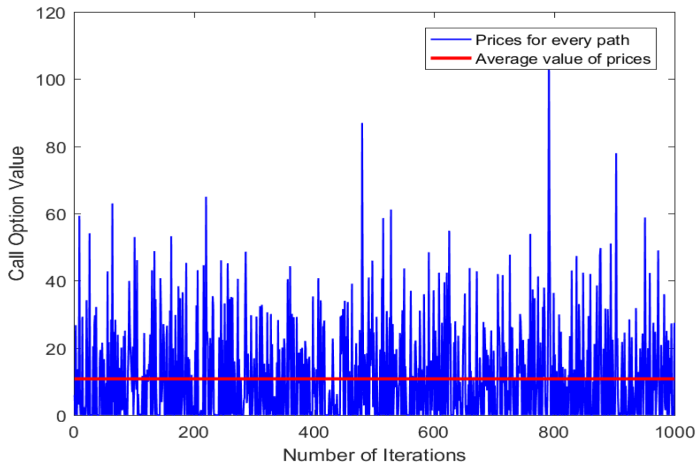

FIGURE 6: Option values for 10000 number of steps.

Increasing the number of iterations in the above Monte Carlo problem gives the following anticipations as shown in Table 2:

Table 2: Monte Carlo method for different number of paths.

Trial	No. of Paths	Option Values	Absolute Error	Relative Error
1	10	8.9900	1.9995	0.1819
2	100	10.6598	0.3306	0.0301
3	1000	10.6247	0.3648	0.0332
4	10000	10.8816	0.1079	0.0098
5	100000	10.9662	0.0233	0.0021
6	1000000	10.9815	0.0080	0.0007
7	10000000	10.9913	0.0018	0.0002

From the Table 2, we conclude that, the option value from Monte Carlo comes closer to the value of Black Scholes with the increasing number of iterations. Now in the following Table 3, we observe the Monte Carlo price for different strike and maturity comparing with the Black-Scholes price.

Table 3: European call option pricing by Monte Carlo for different strike prices and varying maturity with fixed $n = 1000$, $S_0 = 100$, $r = 0.06$ & $\sigma = 0.2$.

K	T	Black-Scholes Price	Monte Carlo Price	Absolute Error	Relative Error
90	0.2	11.4475	11.3103	0.1372	0.0120
	0.4	13.0790	12.8842	0.1948	0.0149
	0.6	14.5966	15.4463	0.8497	0.0582
	0.8	16.0118	15.8425	0.1693	0.0106
	1.0	17.3456	17.6490	0.3033	0.0175
100	0.2	4.1740	4.1847	0.0106	0.0025
	0.4	6.2581	6.1674	0.0907	0.0145
	0.6	7.9957	7.9812	0.0145	0.0018
	0.8	9.5516	10.3542	0.8025	0.0840
	1.0	10.9895	10.6648	0.3248	0.0296
110	0.2	0.8832	0.9024	0.0192	0.0218
	0.4	2.3417	2.2169	0.1248	0.0533
	0.6	3.7672	3.9265	0.1593	0.0423
	0.8	5.1304	4.8066	0.3237	0.0631
	1.0	6.4373	6.2845	0.1528	0.0237

IV. Conclusion

Throughout this paper, we investigated two different numerical techniques for option pricing namely, the Binomial model and Monte Carlo simulation. Then we compared the convergence of these methods to the analytic Black-Scholes price. Binomial model proved to be the fastest converging technique with increasing number of steps. As time interval becomes close to infinity, the binomial pricing formula converges to the Black-Scholes price. On the other hand, Monte Carlo works forward from the beginning to the end of the life of an option. It can cope with a great deal of complexities as far as the payoffs are concerned. As the number of simulations increase, Monte Carlo shows better convergence. But it is quite time consuming as it requires large amounts of random numbers. Finally, we can say that both these numerical techniques discussed throughout this work are satisfactorily flexible to evaluate European options. Though we computed and observed all the outcomes under these models for European call options, they can also be used to achieve similar results for European put options. The basic procedures which are reported here can be used to handle most of the option valuation problems encountered in real world.

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