

Weierstrass Transform on Howell's Space-A New Theory

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Abstract: In this paper we have explained a new theory of Weierstrass transform defined on Howell's space. The definition of the Weierstrass transform on this space of test functions is discussed. Some Fundamental theorems are proved and some results are developed. Weierstrass transform defined on Howell's space is linear, continuous and one-one mapping from G to G^c is also discussed.

Key Word: Weierstrass transform, Howell's Space, Fundamental theorem.

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I. Introduction

A new theory of Fourier analysis developed by Howell, presented in an elegant series of papers [2-6] supporting the Fourier transformation of the functions on R^n . This theory is similar to the theory for tempered distributions and to some extent, can be viewed as an extension of the Fourier theory. Bhosale [1] had extended Fractional Fourier transform on Howell's space. Mahalle *et. al.* [7, 8] described a new theory of Mellin and Laplace transform defined on Howell's space. Motivated by the above work, we are extending this theory to Weierstrass transform.

In this paper, we have introduced the concept of Weierstrass transform defined on Howell's space. In section II new notation and basic spaces are discussed. We have proved one lemma and and fundamental theorem described in section III. Definition of Weierstrass transform on Howell's space is given in Section IV. Some results are proved in section V. The Weierstrass transform defined on this space is linear, continuous and one-one, is proved in the form of theorems in section VI. Lastly conclusion is given.

II. Notations and Conventions

The notations, conventions and the terminologies in this work follow those of [2] and Zemanian [9]. Throughout this paper n will denote some arbitrary fixed positive integer and Λ will denote the Lebesgue measure on R^n . Unless otherwise indicated, all functions are to be assumed to have domain in C^n and the range in C^1 . The notations $y = (y_1, y_2, \dots, y_n)$ and $\sigma + i\zeta$ will be considered interchangeable and all denote the n -dimensional complex variable with real part $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ and imaginary part $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$.

2.1 Here we list some notation form Howell [2], which will be used prominently in this work.

(1) The translation: Let $f : C^n \rightarrow C$. For each $\xi \in C^n$, $\theta_\xi f$ will denote the translation of f by ξ , given by

$$\theta_\xi f |_y = f(y - \xi).$$

(2) Scaling operator: For each $\alpha \in C$, $\sum_\alpha f$ will denote the function f with variable scaled by α , that is

$$\sum_\alpha f |_y = f(\alpha y)$$

(3) Exponential function: Let $\omega \in C^n$. Where convenient, e_ω will denote the function given by

$$e_\omega(y) = e^{\omega y}$$

(4) Normal Gaussian: Let $\alpha > 0$. Where convenient η_α will denote the function given by

$$\eta_\alpha(y) = \left(\frac{\alpha}{\sqrt{2\pi}} \right)^n \exp \left[-\frac{(\alpha y)^2}{2} \right]$$

(5) Rho seminorm: The symbol ρ will be reserved for the seminorm on C^n given by $\rho(y) = \sum_{k=1}^n |\operatorname{Re}[y_k]|$

(6) Strips in C^n : For each $\alpha \geq 0$, B_α will denote the subset of C^n given by,

$$B_\alpha = \{y = (y_1, y_2, \dots, y_n) \in C^n : |\operatorname{Im}[y_k]| \leq \alpha, k = 1, 2, \dots, n\}$$

Note that $B_0 = R^n$ and that if $\alpha < \beta$, then $B_\alpha \subset B_\beta$.

2.2 Basic Spaces G and G^c : Two spaces of functions will be of especial importance. They will be denoted by G and G^c and are defined below:

2.2.1 Definition (G): A function $\phi : C^n \rightarrow C$, is an element of G if and only if both of the following holds:

- (1) ϕ is an analytic function of each complex variable.
- (2) For every $\alpha > 0$, $\operatorname{Sup} \{e^{\alpha \rho(y)} |\phi(y)| : y \in B_\alpha\} < \infty$

2.2.2 Definition (G^c): A function $f : C^n \rightarrow C$, is an element of G^c if and only if both of the following holds:

- (1) f is an analytic function of each complex variable
- (2) For every $\alpha > 0$ there is a Corresponding $\gamma \geq 0$ such that $\operatorname{Sup} \{e^{-\gamma \rho(y)} |f(y)| : y \in B_\alpha\} < \infty$

G with suitable topology, will serve as the space of test functions. The following facts will be of importance and should all be obvious from the above definitions

- (1) G and G^c both are linear spaces closed under multiplication.
- (2) $G \subset G^c$
- (3) G contains all functions of the form $\phi = \theta_\xi \eta_\alpha$, where $\alpha > 0$ and $\xi \in C^n$
- (4) If $\phi \in G$ and $\xi, \zeta \in C^n$ then $e_{i\xi} \theta_\zeta \phi$ is also in G and its restriction to R^n is in $L^1(R^n)$
- (5) G^c contains all polynomial and exponential functions on C^n
- (6) Any exponential function e_ξ with $\xi \in C^n$ is an element of G^c .

The topology on G is defined by the set of norms $\{\|\cdot\|_\alpha : \alpha \geq 0\}$, where

$$\|\phi\|_\alpha = \operatorname{Sup} \{e^{\alpha \rho(y)} |\phi(y)| : y \in B_\alpha\}.$$

Any function ϕ , which is analytic in each variable is clearly in G if and only if $\|\phi\|_\alpha$ is finite for every $\alpha > 0$. It is also obvious that, if $\alpha \leq \beta$, then $\|\phi\|_\alpha \leq \|\phi\|_\beta$.

Similarly the topology on G^c is defined by the set of norms $\{\|\cdot\|_\alpha : \alpha \geq 0\}$ where

$$\|\phi\|_\alpha = \operatorname{Sup} \{e^{-\gamma \rho(y)} |\phi(y)| : y \in B_\alpha\}, \text{ for every } \alpha > 0, \text{ there is a corresponding } \gamma \geq 0.$$

G^c with suitable topology will serve as the space of test functions. It is also obvious that, if $\alpha \leq \beta$, $\|\phi\|_\alpha \leq \|\phi\|_\beta$. A slightly more general relation is expressed in the following lemma.

2.3 Lemma: Let $\alpha > 0$, $k \geq 0$ and $\phi \in G$. Then for any $y \in B_\alpha$ and $\lambda \in C$ with $|\lambda| \leq \alpha$, $|\phi(y)e^{\lambda \rho(y)}| \leq \|\phi\|_{\alpha+k} e^{-k \rho(y)}$

Proof: Let y be any arbitrarily fixed elements of B_α and $\lambda \in C$ with $|\lambda| \leq \alpha$. Then

$$\begin{aligned} \|\phi\|_{\alpha+k} &= \text{Sup} \{ e^{(\alpha+k)\rho(y)} |\phi(y)| : y \in B_{\alpha+k} \} \\ &\geq \text{Sup} \{ e^{\lambda \rho(y)} e^{k \rho(y)} |\phi(y)| : y \in B_{\alpha+k} \} \\ \|\phi\|_{\alpha+k} e^{-k \rho(y)} &\geq |\phi(y) e^{\lambda \rho(y)}| \end{aligned}$$

Therefore,

$$|\phi(y) e^{\lambda \rho(y)}| \leq \|\phi\|_{\alpha+k} e^{-k \rho(y)}.$$

III. Fundamental Theorem

Let $\phi \in G$ and $f \in G^c$ then mapping $\phi \rightarrow f\phi$ is continuous linear mapping from G to G .

Proof: Let ϕ and f are fixed elements of G and G^c respectively. The analyticity of $f\phi$ is obvious and it suffices to produce a suitable bound for $\|f\phi\|_\alpha$, where α is arbitrary. Since $f \in G^c$, therefore for all $\alpha \geq 0$, there exists $k > 0$ such that,

$$e^{-k \rho(y)} |f(y)| \leq N \quad (\text{Say}) \quad (3.1)$$

Now, to show that $f\phi \in G$.

$$\|f\phi\|_\alpha = \text{Sup} \{ e^{\alpha \rho(y)} |f(y)\phi(y)| : y \in B_\alpha \},$$

where B_α denote the subset of C^n given in section II.

But $\phi \in G$ then by lemma 2.3 with $\lambda = \alpha$

$$|\phi(y) e^{\alpha \rho(y)}| \leq \|\phi\|_{\alpha+k} e^{-k \rho(y)}$$

Now,

$$e^{\alpha \rho(y)} |\phi(y)| |f(y)| \leq \|\phi\|_{\alpha+k} e^{-k \rho(y)} |f(y)| \quad (3.2)$$

From equation (3.1), we get

$$e^{\alpha \rho(y)} |\phi(y)| |f(y)| \leq N \|\phi\|_{\alpha+k}$$

Thus, for any $\phi \in G$, $\|f\phi\|_\alpha \leq N \|\phi\|_{\alpha+k} < \infty$

From this it follows that $f\phi$ is an element of G . Therefore the mapping $\phi \rightarrow f\phi$ is continuous and linear from G to G .

IV. Definition: Weierstrass transform on G

For each f in G , $W(f)$ be the function on C^n given by,

$$[W(f)](x) = \frac{1}{\sqrt{4\pi}} \int_{R^n} f(y) e^{-\frac{(x-y)^2}{4}} d\Lambda_y,$$

where Λ is explained in section II.

This transform is well defined on G , since G^c contains all polynomial and exponential functions on C^n that is $e^{-\frac{(x-y)^2}{4}} \in G^c$. Hence for $f(y) \in G$, $f(y) e^{-\frac{(x-y)^2}{4}} \in G$ by fundamental theorem. Now by property (4)

p.no.344 of [2], restriction of $f(y) e^{-\frac{(x-y)^2}{4}} \in G$ to R^n belongs to $L^1(R^n)$. Hence right hand side exists.

V. Elementary Weierstrass Analysis on G

Here we have proved some results which are all analogous to well known results in the classical theory of Weierstrass transform for smooth integrable function on R^n .

5.1 Result: (Scaling and Weierstrass transform on G):

Let β be any non zero real number and let f be any element of G . Then

$$\left[W \left(\sum_{\beta} f \right) \right] (x) = |\beta|^{-n} \sum_{\beta^{-1}} [W (f)] (x)$$

5.2 Result: Let $f \in G$. Then for any ξ and β in C^n , $\left[W \left(e_{\xi} \sum_{\beta} f \right) \right] (x) = |\beta|^{-n} \sum_{\beta^{-1}} \left[W \left(e_{\xi \beta^{-1}} f \right) \right] (x)$

5.3 Result: (Differentiation and Weierstrass transform on G):

If $f \in G$, then $\psi = W(f)$ is an analytic function of each variables and for $k = 1, 2, 3, \dots, n$

$$\frac{\partial}{\partial \sigma_k} \psi (\sigma + i\zeta) = W \left\{ -\frac{1}{2} [(\sigma_k + i\zeta_k) - y_k] f \right\} (\sigma + i\zeta)$$

$$\frac{\partial}{\partial \zeta_k} \psi (\sigma + i\zeta) = W \left\{ -\frac{i}{2} [(\sigma_k + i\zeta_k) - y_k] f \right\} (\sigma + i\zeta)$$

$$\frac{d}{dx} \psi (x) = W \left[-\frac{(x-y)}{2} f \right] (x)$$

5.4 Result: If $f, g \in G$, then the restriction to R^n of the products $gW[f]$ and $fW[g]$ are in $L^1(R^n)$ and

$$\int_{R^n} g W [f] d\Lambda = \int_{R^n} f W [g] d\Lambda$$

5.5 Result: (Transform of Normal Gaussian):

The Weierstrass transform of Normal Gaussian is $W [\eta_{\alpha} (y)] (x) = \left(\frac{\alpha \mu}{2\pi} \right)^n W \left[\eta_{\mu}^{-1} e^{\frac{y^2}{4}} \right] (x)$,

where $\mu = \left[\frac{-(2\alpha^2 + 1)}{2} \right]^{\frac{1}{2}}$.

5.6 Result: For $f \in G$ and any $\omega \in C^n$

$$\left[W (e_{\omega} f) \right] (x) = e^{-\frac{\omega^2}{4}} \theta_{\omega} \left\{ W \left[e^{\frac{\omega}{2}(3y-x)} f(y) \right] (x) \right\}$$

VI. Continuity of $W(f)$ and other operators:

We prove the following theorem for continuity of $W(f)$ in G^c

6.1 Theorem: For each $f \in G$, the mapping $f \rightarrow W(f)$ is continuous, linear and one-one mapping from G to G^c .

Proof: Let $f \in G$ and $\psi = W(f)$. Let x be any element of β_{α} .

$$\|\psi\|_{\alpha} = \text{Sup} \left\{ e^{-\gamma \rho(x)} |\psi(x)| : x \in B_{\alpha} \right\}$$

Now,

$$\left| e^{-\gamma \rho(x)} \left| \psi(x) \right| \right| \leq \frac{1}{\sqrt{4\pi}} \left| e^{-\gamma \rho(x)} \right| \int_{R^n} \left| f(y) e^{-\frac{(x-y)^2}{4}} \right| d\Lambda_y \tag{6.1}$$

Now, $f \in G$ and $e^{-\frac{(x-y)^2}{4}} \in G^c$ as G^c contains all polynomial and exponential function on C^n . Therefore,

$$f(y) e^{-\frac{(x-y)^2}{4}} \in G.$$

Therefore,
$$\left\| f(y) e^{-\frac{(x-y)^2}{4}} \right\|_\alpha < \infty$$

i.e.
$$\text{Sup} \left\{ e^{\alpha \rho(y)} \left| f(y) e^{-\frac{(x-y)^2}{4}} \right| \right\} < \infty, \quad \text{for all } \alpha > 0 \text{ and } y \in B_\alpha$$

$< K \quad (\text{Say})$

Therefore,

$$\left| f(y) e^{-\frac{(x-y)^2}{4}} \right| < K e^{-\alpha n y}$$

From equation (6.1), we get

$$\left| e^{-\gamma \rho(x)} \left| \psi(x) \right| \right| < \frac{-k}{\alpha n \sqrt{4\pi}} e^{-\gamma \rho(x) - \alpha n y}$$

$< \infty \text{ for } \gamma \geq 0$

Therefore $\psi : f \rightarrow W(f)$ is continuous, linear and one-one mapping from G to G^c .

6.2 Theorem: For each non-zero real value β , \sum_β is a continuous, linear and one-one mapping from G to

G^c with inverse $\sum_{\beta^{-1}}$. Moreover for each $f \in G$ and $\psi(y) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(x-y)^2}{4}} \in G^c$,

$$\int_{R^n} \left[\sum_\beta f \right] \psi d\Lambda = |\beta|^{-n} \int_{R^n} f \left[\sum_{\beta^{-1}} \psi \right] d\Lambda$$

Proof: The first part proof is simple so not given here.

Now,

$$\begin{aligned} \int_{R^n} \left[\sum_\beta f \right] \psi d\Lambda &= \frac{1}{\sqrt{4\pi}} \int_{R^n} f(\beta y) e^{-\frac{(x-y)^2}{4}} d\Lambda_y \\ &= \frac{1}{\sqrt{4\pi}} \int_{R^n} f(v) e^{-\frac{\left(\frac{x-v}{\beta}\right)^2}{4}} \frac{1}{\beta} d\Lambda_v \\ &= |\beta|^{-n} \int_{R^n} f(v) \left[\sum_{\beta^{-1}} \psi(v) \right] d\Lambda_v \end{aligned}$$

Therefore,

$$\int_{R^n} \left[\sum_\beta f \right] \psi d\Lambda = |\beta|^{-n} \int_{R^n} f \left[\sum_{\beta^{-1}} \psi \right] d\Lambda$$

6.3 Theorem: For each $\omega \in C^n$, θ_ω is continuous, linear and one-one mapping from G to G^c with inverse

$\theta_{-\omega}$. Moreover for each $f \in G$ and $\psi = \frac{1}{\sqrt{4\pi}} e^{-\frac{(x-y)^2}{4}} \in G^c$, θ_ω satisfies the following equality

$$\int_{R^n} [\theta_\omega f] \psi \, d\Lambda = \int_{R^n} f [\theta_{-\omega} \psi] \, d\Lambda$$

Proof: First part is so simple, so proof is not given here.

Now consider,

$$\begin{aligned} \int_{R^n} [\theta_\omega f] \psi \, d\Lambda &= \frac{1}{\sqrt{4\pi}} \int_{R^n} [f(y - \omega)] e^{-\frac{(x-y)^2}{4}} \, d\Lambda_y \\ &= \frac{1}{\sqrt{4\pi}} \int_{R^n} f(v) e^{-\frac{[x-(v+\omega)]^2}{4}} \, d\Lambda_v \\ &= \int_{R^n} f(v) [\theta_{-\omega} \psi(v)] \, d\Lambda_v \end{aligned}$$

Therefore,

$$\int_{R^n} [\theta_\omega f] \psi \, d\Lambda = \int_{R^n} f [\theta_{-\omega} \psi] \, d\Lambda$$

VII. Conclusions

In the present paper we have introduced a new theory of Weierstrass transform defined on Howell's space. By using notations and terminology we found some results. Proved Fundamental theorem and also proved that Weierstrass transform defined on Howell's space is linear, continuous and one-one mapping from G to G^c .

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