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Frame operator of K-frame in n-Hilbert space

Prasenjit Ghosh[1],
[1](*Department of Mathematics, University of Calcutta, India*)

Abstract: *In this paper we describe some properties of frame in n-inner product spaces. Characterizations between K-frames and quotient operators in n-inner product space are given.*
Key Word: *Frame, K-frame, quotient operator, n-inner product space, n-normed space.*

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I. Introduction

Reconstruction of functions using a family of elementary functions were first introduced by Gabor [6] in 1946. Later in 1952, Duffin and Schaeffer were introduced frames in Hilbert spaces in their fundamental paper [5], they used frames as a tool in the study of non-harmonic Fourier series. After some decades, frame theory was popularized by Daubechies, Grossman, Meyer [3]. A frame for a separable Hilbert space is a generalization of such an orthonormal basis and this is such a tool that also allows each vector in the space to be written as a linear combination of elements from the frame but, linear independence among the frame elements is not required. Several generalizations of frames namely, K-frame [8], Fusion frame [2], $\boldsymbol{G}$-frame [14], etc. have been introduced in recent times. K-frames for a separable Hilbert space were introduced by Lara Gavruta. K-frame is more generalization than the ordinary frame and many properties of ordinary frame may not hold for such generalization of frame.

The concept of 2-inner product space was introduced by [4]. S. Gahler [7] introduced the notion of 2-normed space. H. Gunawan and Mashadi [9] developed the generalization of 2-norm space for $n \geq 2$.The generalization of 2-inner product space for $n \geq 2$ was developed by A. Misiak [13]. The notion of a frame in a n-inner product space has been presented by P. Ghosh and T. K. Samanta [10] and they also studied frame in tensor product of n-inner product spaces [11]. The author also presented K-frame and some its properties in n-Hilbert space [12].

In this paper, some properties of frame in n-Hilbert space are going to be established. We give a relationship between K-frame and quotient operators in n-Hilbert space.

Throughout this paper, X will denote a separable Hilbert space with the inner product $<.,.>$ and $B(X)$denote the space of all bounded linear operator on X. We also denote $R(T)$ for range set of T, $N(T)$ for null space of T where $T \in B(X)$ and l^2 denote the space of square summable scalar-valued sequences.

II. Preliminaries

Definition 2.1. [1] A sequence $\{ f_i \}$ of elements in X is said to a frame for X if there exist constants A, B > 0 such that

$$A \left\| f \right\|^2 \leq \sum_{i=1}^{\infty} |< f, f_i >|^2 \leq B \left\| f \right\|^2 \text{ for all } f \in X.$$

The constants A, B are called frame bounds. If the collection $\{ f_i \}$ satisfies

$\sum_{i=1}^{\infty} |< f, f_i >|^2 \leq B \left\| f \right\|^2$ for all $f \in X$, then it is called a Bessel sequence.

Definition 2.2. [1] Let $\{ f_i \}$ be a frame for X. Then the operator defined by $T : l^2 \to X, T(\{ c_i \}) = \sum_{i=1}^{\infty} c_i f_i$ is called pre-frame operator and its adjoint operator given by $T^* : X \to l^2, T^*(f) = \{< f, f_i >\}$ is called the analysis operator. The frame operator is given by $S : X \to X, S f = T T^* f = \sum_{i=1}^{\infty} < f, f_i > f_i$.

Definition 2.3. [8] Let $K : X \to X$ be a bounded linear operator. Then a sequence $\{ f_i \}$ in X is said to be a K-frame for X if there exist constants $A, B > 0$ such that

$$A \left\| K^* f \right\|^2 \leq \sum_{i=1}^{\infty} |< f, f_i >|^2 \leq B \left\| f \right\|^2 \text{ for all } f \in X.$$

Definition 2.4. [8] Let $U, V : X \to X$ be two bounded linear operator with $N(U) \subset N(V)$. Then a linear operator $T = [\, {}^{U}/_{V} \,]$ given by

$T = [{}^{U}/_{V}] : R(V) \to R(U), T(Vx) = Ux$ is called quotient operator.

Definition 2.5. [9] A real valued function $\|., \dots, .\| : X^n \to R$ is called a n-norm on X if the following conditions hold:

(I) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,
(II) $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutations of $x_1, x_2, \dots, x_n$,
(III) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\| \ \forall \alpha \in K$,
(IV) $\|x + y, x_2, \dots, x_n\| \le \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$.

The pair $(X, \|., \dots, \|)$ is then called a linear n-normed space.

Definition 2.6. [13] Let $n \in N$ and X be a linear space of dimension greater than or equal to n over the field K, where K is the real or complex numbers field. A function $<. , . | ., \dots, . > : X^{n+1} \to K$ is satisfying the following five properties:

I. $< x_1, x_1 | x_2, \dots, x_n > \ge 0$ and $< x_1, x_1 | x_2, \dots, x_n > = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,
II. $< x, y | x_2, \dots, x_n > = < x, y | x_{i(2)}, \dots, x_{i(n)} >$ for every permutation $(i(2), \dots, i(n))$ of $(2, \dots, n)$,
III. $< x, y | x_2, \dots, x_n > =$ complex conjugate of $< x, y | x_2, \dots, x_n >$,
IV. $< \alpha x, y | x_2, \dots, x_n > = \alpha < x, y | x_2, \dots, x_n >$, for all $\alpha \in K$,
V. $< x + y, z | x_2, \dots, x_n > = < x, z | x_2, \dots, x_n > + < y, z | x_2, \dots, x_n >$,

is called an n-inner product on X and the pair $(X, <. , . | ., \dots, . >)$ is called n-inner product space.

Theorem 2.7. [13] For n-inner product space $(X, <. , . | ., \dots, . >)$,
$|< x, y | x_2, \dots, x_n >| \le \|x, x_2, \dots, x_n\| \|y, x_2, \dots, x_n\|$ hold for all $x, y, x_2, \dots, x_n \in X$.

Theorem 2.8. [13] For every n-inner product space $(X, <. , . | ., \dots, . >)$,
$\|x_1, x_2, \dots, x_n\| = \sqrt{< x_1, x_1 | x_2, \dots, x_n >}$ Defines a n-norm for which
$< x, y | x_2, \dots, x_n > = \frac{1}{4} (\|x + y, x_2, \dots, x_n\|^2 - \|x - y, x_2, \dots, x_n\|^2)$, and
$\|x + y, x_2, \dots, x_n\|^2 + \|x - y, x_2, \dots, x_n\|^2 = 2 (\|x, x_2, \dots, x_n\|^2 - \|y, x_2, \dots, x_n\|^2)$ hold for all $x, y, x_1, x_2, \dots, x_n \in X$.

Definition 2.9. [9] A sequence $\{x_k\}$ in a linear n-normed space X is said to be convergent to some $x \in X$ if for every $x_2, \dots, x_n \in X$ $\lim_{k \to \infty} \|x_k - x, x_2, \dots, x_n\| = 0$ and it is called a Cauchy sequence if $\lim_{l, k \to \infty} \|x_l - x_k, x_2, \dots, x_n\| = 0$ for every $x_2, \dots, x_n \in X$.The space X is said to be complete if every Cauchy sequence in this space is convergent in X. A n-inner product space is called n-Hilbert space if it is complete with respect to its induce norm.

Note 2.10. [10] Let L_F denote the linear subspace of X spanned by the non-empty finite set $F = \{a_2, a_3, \dots, a_n\}$, where $a_2, a_3, \dots, a_n$ are fixed elements in X. Then the quotient space ${}^{X}/_{L_F}$ is a normed linear space with respect to the norm, $\|x + L_F\|_F = \|x, a_2, \dots, a_n\|$ for every $x \in X$. Let M_F be the algebraic complement of L_F, then X can be expressed as the direct sum of L_F and M_F. Define $< x, y >_F = < x, y | a_2, \dots, a_2 >$ on X . Then $<, ., >_F$ is a semi-inner product on X and this semi-inner product induces an inner product on ${}^{X}/_{L_F}$ which is given by
$< x + L_F, y + L_F >_F = < x, y >_F = < x, y | a_2, \dots, a_2 > \forall x, y \in X$. By identifying ${}^{X}/_{L_F}$ with M_F in an obvious way, we obtain an inner product on M_F. Now for every $\in M_F$, we define
$\|x\|_F = \sqrt{< x, x >_F}$ and it can be easily verify that $(M_F, \| \|_F)$ is a norm space. Let X_F be the completion of the inner product space M_F.

For the remaining part of this paper, $(X, <. , . | ., \dots, . >)$ is consider to be a n-Hilbert space and I will denote the identity operator on X_F.

Definition 2.11. [10] A sequence $\{f_i\}$ of elements in X is said to a frame associated to $(a_2, \dots, a_n)$ for X if there exist constants $0 < A \le B < \infty$ such that

$A \left\| f, a_2, \ldots\ldots, a_n \right\|^2 \leq \sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots\ldots, a_n >|^2 \leq B \left\| f, a_2, \ldots\ldots, a_n \right\|^2$ for all $f \in X$. The constants A, B are called frame bounds. If the collection $\{ f_i \}$ satisfies
$\sum_{i=1}^{\infty} |< f, f_i \mid a_2, \ldots\ldots, a_n >|^2 \leq B \left\| f, a_2, \ldots\ldots, a_n \right\|^2$ for all $f \in X$, then it is called a Bessel sequence associated to $(a_2, \ldots\ldots, a_n)$ in X with bound B.

Theorem 2.12. [10] Let $\{ f_i \}$ be a sequence in X. Then $\{ f_i \}$ is a frame associated to $(a_2, \ldots\ldots, a_n)$ with bounds A and B if and only it is a frame for the Hilbert space X_F with bounds A and B.

Definition 2.13. [10] Let $\{ f_i \}$ be a Bessel sequence associated to $(a_2, \ldots\ldots, a_n)$ for X.
Then the bounded linear operator defined by $T_F : l^2 \to X_F$, $T_F(\{ c_i\}) = \sum_{i=1}^{\infty} c_i f_i$ is called pre-frame operator and its adjoint operator given by $T_F^* : X_F \to l^2, T_F^* f = \{< f, f_i \mid a_2, \ldots\ldots, a_n >\}$ is called the analysis operator. The frame operator is given by $S_F : X_F \to X_F$, $S_F f = \sum_{i=1}^{\infty} < f, f_i \mid a_2, \ldots\ldots, a_n > f_i$.

The frame operator S_F is bounded, positive, self-adjoint and invertible.

Definition 2.14. [12] Let K be a bounded linear operator on X_F .Then a sequence $\{ f_i \}$ of elements in X is said to a K- frame associated to $(a_2, \ldots\ldots, a_n)$ for X if there exist constants $0 < A \leq B < \infty$ such that
$A \left\| K^* f, a_2, \ldots\ldots, a_n \right\|^2 \leq \sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots\ldots, a_n >|^2 \leq B \left\| f, a_2, \ldots\ldots, a_n \right\|^2$ for all $f \in X_F$.
This can be written as $A \left\| K^* f \right\|_F^2 \leq \sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots\ldots, a_n >|^2 \leq B \left\| f \right\|_F^2$.

III. Some properties of frame in n-Hilbert space

Theorem 3.1. Let Y be closed subspace of X_F and P_Y be the orthogonal projection on Y. Then for a sequence $\{ f_i \}$ in X_F the following hold:

(i) If $\{ f_i \}$ is a frame associated to $(a_2, \ldots\ldots, a_n)$ for X with frame bounds A,B then $\{P_Y f_i \}$ is a frame for Y with the same bounds.

(ii) If $\{ f_i \}$ is a frame for Y with frame operator S_F, then $P_Y f = \sum_{i=1}^{\infty} < f, S_F^{-1} f_i \mid a_2, \ldots\ldots, a_n > f_i$ for all $f \in X_F$.

Proof: By the definition of orthogonal projection of X_F onto Y, we get

$$P_Y f = \begin{cases} f & if\ f \in Y \\ 0 & if\ f \in Y^{\perp} \end{cases} \tag{1}$$

(i) Suppose $\{ f_i \}_{i=1}^{\infty}$ associated to $(a_2, \ldots\ldots, a_n)$ for X with frame bounds A,B. Then $\{f_i\}_{i=1}^{\infty}$ is a frame for X_F with frame bounds A,B .So,

$$A \|f\|_F^2 \leq \sum_{i=1}^{\infty} |< f, f_i >_F|^2 \leq B \|f\|_F^2 \quad \forall\ f \in X_F$$

So by (1), the above inequality can be write as

$$A \|f\|_F^2 \leq \sum_{i=1}^{\infty} |< f, P_Y f_i >_F|^2 \leq B \|f\|_F^2 \quad \forall\ f \in Y$$

(ii) Let $\{ f_i \}$ be a frame associated to $(a_2, \ldots\ldots, a_n)$ for X with frame operator S_F. Then it is easy to verify that $f = \sum_{i=1}^{\infty} < f, S_F^{-1} f_i \mid a_2, \ldots\ldots, a_n > f_i$ for all $f \in Y$. Therefore by (1), we get $P_Y f = \sum_{i=1}^{\infty} < f, S_F^{-1} f_i \mid a_2, \ldots\ldots, a_n > f_i$ for all $f \in Y$. Now, if f belongs to the orthogonal complement of Y then $< f, S_F^{-1} f_i \mid a_2, \ldots\ldots, a_n > = 0$ and $P_Y f = 0$ if f belongs to the orthogonal complement of Y. Therefore, $P_Y f = \sum_{i=1}^{\infty} < f, S_F^{-1} f_i \mid a_2, \ldots\ldots, a_n > f_i$ for all $f \in X_F$. This completes the proof.

Note 3.2. Let $\{ f_i \}$ be a frame associated to $(a_2, \ldots\ldots, a_n)$ for X and $f \in X_F$. If for some $\{ c_i \} \in l^2$, $f = \sum_{i=1}^{\infty} c_i f_i$, then

$$\sum_{i=1}^{\infty} \left| c_i \right|^2 = \sum_{i=1}^{\infty} \left| < f, S_F^{-1} f_i \mid a_2, \ldots, a_n > \right|^2 + \sum_{i=1}^{\infty} \left| c_i - < f, S_F^{-1} f_i \mid a_2, \ldots . a_n > \right|^2 .$$

Theorem 3.3. Let $\{ f_i \}$ be a frame associated to $(a_2, \ldots\ldots, a_n)$ for X with pre-frame operator T_F.Then the pseudo-inverse of T_F is described by,

$$T_F^1 : X_F \to l^2 , \; T_F^1 f = \{< f, S_F^{-1} f_i \mid a_2, \ldots\ldots, a_n >\},$$

where S_F is the corresponding frame operator.

Proof. By the Theorem (2.12), $\{ f_i \}$ be a frame for X_F . Then for $f \in X_F$ has a representation $f = \sum_{i=1}^{\infty} c_i f_i$, for some $\{ c_i \} \in l^2$ and this can be written as $T_F(\{ c_i\}) = f$. By note (3.2), the frame coefficient $=$ $\{< f, S_F^{-1} f_i \mid a_2, \ldots\ldots, a_n >\}$ have minimal l^2-norm among all the sequences representing f. Hence, the above equation has a unique solution of minimal norm namely, $T_F^1 f = \{< f, S_F^{-1} f_i \mid a_2, \ldots\ldots, a_n >\}$.

Theorem 3.4. Let $\{ f_i \}$ be a frame associated to $(a_2, \ldots \ldots, a_n)$ for X, then the optimal frame bounds A, B are given by $A = \| S_F^{-1} \|^{-1} = \| T_F^{1} \|^{-2}, B = \| S_F \| = \| T_F \|^{2}$, where T_F is the pre-frame operator, T_F^{1} is the pseudo-inverse of T_F and S_F is
is the corresponding frame operator.

Proof. By the definition, the optimal upper frame bound is given by
$B = \sup \{ \sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots \ldots, a_n >|^2 \ | \| f, a_2, \ldots \ldots, a_n \| = 1 \} =$
$\sup \{ < S_F f, f \mid a_2, \ldots \ldots, a_2 > \mid \| f, a_2, \ldots \ldots, a_n \| = 1 \} = \| S_F \|$. Therefore, $B = \| S_F \| = \| T_F \|^2$. We know that the dual frame $\{ S_F^{-1} f_i \}$ has frame operator S_F^{-1} and the optimal upper bound is A^{-1} . So by the above similar process $A^{-1} = \| S_F^{-1} \|$ and this implies that $A = \| S_F^{-1} \|^{-1}$. Now, from the Theorem (3.3), we obtain $\| S_F^{-1} \| = \sup \{ \| T_F^{1} f \|_F^2 \big| \| f \|_F = 1 \} = \| T_F^{1} \|^2$. Thus, $A = \| S_F^{-1} \|^{-1} = \| T_F^{1} \|^{-2}$. This completes the proof.

IV. Frame operator for K- frame

Theorem 4.1. Let $\{ f_i \}$ be a Bessel Sequence associated to $(a_2, \ldots \ldots, a_n)$ for X with frame operator S_F and K be a bounded linear operator on X_F . Then $\{ f_i \}$ is a K- frame associated to $(a_2, \ldots \ldots, a_n)$ for X if and only if the quotient operator $T = \left[K^* / S_F^{\frac{1}{2}} \right]$ is bounded.

Proof. Let $\{ f_i \}$ be a K- frame associated to $(a_2, \ldots \ldots, a_n)$ for X. Then there exist positive constants A, B such that

$$A \| K^* f \|_F^2 \leq \sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots \ldots, a_n >|^2 \leq B \| f \|_F^2 \text{ for all } f \in X_F . \qquad (2)$$

Since S_F is the corresponding frame operator, we can write

$$< S_F f, f \mid a_2, \ldots \ldots, a_n > = \sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots \ldots, a_n >|^2 \text{ for all } f \in X_F \qquad (3).$$

By (3), The inequality (2) can be written as
$A \| K^* f \|_F^2 \leq < S_F f, f \mid a_2, \ldots \ldots, a_n > \leq B \| f \|_F^2$ for all $f \in X_F$. This implies that
$A \| K^* f \|_F^2 \leq < S_F^{\frac{1}{2}} f, S_F^{\frac{1}{2}} f \mid a_2, \ldots \ldots, a_n > \leq B \| f \|_F^2$ for all $f \in X_F$ .This implies that

$$A \| K^* f \|_F^2 \leq \| S_F^{\frac{1}{2}} f \|_F^2 \leq B \| f \|_F^2 \text{ for all } f \in X_F \qquad (4).$$

Let us now define the operator $T = \left[K^* / S_F^{\frac{1}{2}} \right] : R \left(S_F^{\frac{1}{2}} f \right) \to R(K^*)$, by $T \left(S_F^{\frac{1}{2}} f \right) = K^* f \ \forall f \in X_F$.

Now, let $f \in N \left(S_F^{\frac{1}{2}} f \right)$. Then $S_F^{\frac{1}{2}} f = \theta$ implies that $\left\| S_F^{\frac{1}{2}} f \right\|_F^2 = 0$, so by (4), $A \| K^* f \|_F^2 = 0$. This implies that $K^* f = \theta$ implies $f \in N(K^*)$ and this implies that $N \left(S_F^{\frac{1}{2}} f \right) \subset N(K^*)$. This shows that the quotient operator T is well-defined. Also for all $f \in X_F$,
$\left\| T \left(S_F^{\frac{1}{2}} f \right) \right\|_F = \| K^* f \|_F \leq \frac{1}{\sqrt{A}} \| S_F^{\frac{1}{2}} f \|_F$. Hence, T is bounded.

Conversely, suppose that the quotient operator T is bounded. Then there exists $B > 0$ such that
$\left\| T \left(S_F^{\frac{1}{2}} f \right) \right\|_F^2 \leq B \| S_F^{\frac{1}{2}} f \|_F^2 \forall f \in X_F$. This implies that $\| K^* f \|_F^2 \leq B \| S_F^{\frac{1}{2}} f \|_F^2$

$= B < S_F^{\frac{1}{2}} f, S_F^{\frac{1}{2}} f \mid a_2, \ldots \ldots, a_n > = B < S_F f, f \mid a_2, \ldots \ldots, a_n >$ [since $S_F^{\frac{1}{2}}$ is also self-adjoint]

$$= B \sum_{i=1}^{\infty} |< f, f_i | a_2, \ldots \ldots, a_n >|^2 \qquad (5).$$

Also, $\{ f_i \}$ be a Bessel Sequence associated to $(a_2, \dots \dots, a_n)$ in X, so there exists $C > 0$ such that $\sum_{i=1}^{\infty} |< f, f_i | a_2, \dots \dots, a_n >|^2 \le B \, \|f, a_2, \dots \dots, a_n\|^2$ for all $f \in X_F$ (6).
Hence, from (5) and (6), $\{ f_i \}$ is a K- frame associated to $(a_2, \dots \dots, a_n)$ for X.

Theorem 4.2. Let $\{ f_i \}$ is a K- frame associated to $(a_2, \dots \dots, a_n)$ for X with the frame operator S_F and T be a bounded linear operator on X_F . Then the following are equivalent:

(1) Let $\{ Tf_i \}$ is a TK- frame associated to $(a_2, \dots \dots, a_n)$ for X.

(2) $U = \left[(TK)^* \Big/ S_F^{\frac{1}{2}} T^* \right]$ is bounded.

(3) $V = \left[(TK)^* \Big/ (T S_F T^*)^{\frac{1}{2}} \right]$ is bounded.

Proof. (1)⇒ (2) Suppose $\{T f_i\}_{i=1}^{\propto}$ is a TK-frame associated to $(a_2, \dots \dots a_n)$ for X. Then there exists constant A, $B > 0$ such that

$$A \, \|(T K)^* f\|_F^2 \le \sum_{i=1}^{\propto} |< f, T f_i | a_2, \dots \dots, a_n >|^2 \le B \|f\|_F^2 , \forall f \in X_F. \quad (7)$$

Since S_F is the corresponding frame operator, we can write

$$< S_F f , f | a_2, \dots \dots, a_n > = \sum_{i=1}^{\propto} |< f , f_i | a_2, \dots \dots, a_n >|^2 \quad \forall f \in X_F$$

Now,

$$\sum_{i=1}^{\propto} |< f, T f_i | a_2, \dots \dots, a_n >|^2 = \sum_{i=1}^{\propto} |< T^* f, f_i | a_2, \dots \dots, a_n >|^2 = < S_F(T^* f), T^* f | a_2, \dots \dots, a_n >$$

$$= < S_F^{\frac{1}{2}} (T^* f) , S_F^{\frac{1}{2}} (T^* f) \mid a_2, \dots \dots, a_n > = \left\| S_F^{\frac{1}{2}} (T^* f) \right\|_F^2$$

Let us now consider the quotient operator,

$$\left[(T K)^* / S_F^{\frac{1}{2}} \, T^* \right] : \mathrm{R}\left(S_F^{\frac{1}{2}} \, T^* \right) \to \mathrm{R}\left((T K)^* \right) \text{ by } \left(S_F^{\frac{1}{2}} \, T^* \right) f \to (T K)^* f \qquad \forall f \in X_F$$

From (7), we can write

$$A \, \|(T K)^* f\|_F^2 \le \left\| S_F^{\frac{1}{2}} (T^* f) \right\|_F^2 \qquad \forall f \in X_F$$

$$\Rightarrow \|(T K)^* f\|_F^2 \le \frac{1}{A} \left\| S_F^{\frac{1}{2}} (T^* f) \right\|_F^2 \qquad \forall f \in X_F$$

This shows that the quotient operator $\left[(T K)^* / S_F^{\frac{1}{2}} \, T^* \right]$ is bounded.

(2) ⇒ (3) suppose that the quotient operator $\left[(T K)^* / S_F^{\frac{1}{2}} \, T^* \right]$ is bounded.

Then there exists constant B>0 such that

$\|(T\,K)^{*} f\|_F^2 \le B \left\| S_F^{\frac{1}{2}} (T^{*} f) \right\|_F^2 \quad \forall f \in X_F$ (8)

Now for such f $\in X_F$, we have

$\left\| S_F^{\frac{1}{2}} (T^{*} f) \right\|_F^2 = < S_F\, T^{*} f , T^{*} f | \, a_2, \dots \dots, a_n > = < T\, S_F\, T^{*} f , \; f | \, a_2, \dots \dots, a_n >$

$= < (T \;\; S_F\, T^{*})^{\frac{1}{2}} f , (T \;\; S_F\, T^{*})^{\frac{1}{2}} f | \, a_2, \dots \dots, a_n > = \left\| (T \;\; S_F\, T^{*})^{\frac{1}{2}} f) \right\|_F^2$ (9)

From (8) and (9), we get

$\|(T\,K)^{*} f\|_F^2 \le B \left\| (T \;\; S_F\, T^{*})^{\frac{1}{2}} f) \right\|_F^2 \quad \forall f \in X_F.$

Hence, the quotient operator $\left[(T\,K)^{*} / (T \;\; S_F\, T^{*})^{\frac{1}{2}} \right]$ is bounded.

(3)⇒ (1) suppose the quotient operator $\left[(T\,K)^{*} / (T \;\; S_F\, T^{*})^{\frac{1}{2}} \right]$ *is bounded.*

Then there exists constant B>0 such that

$\|(T\,K)^{*} f\|_F^2 \le B \left\| (T \;\; S_F\, T^{*})^{\frac{1}{2}} f) \right\|_F^2 \quad \forall f \in X_F.$ (10)

It is easy to verify that $T \;\; S_F\, T^{*}$ is self-adjoint and positive and hence the square root $T \;\; S_F\, T^{*}$ of exists. Now, for each $f \in X_F$, we have

$\sum_{i=1}^{\propto} |< f , T\, f_i | \, a_2, \dots \dots, a_n >|^2 = \sum_{i=1}^{\propto} |< T^{*} f , f_i | \, a_2, \dots \dots, a_n >|^2$

$= < S_F\, (T^{*} f) , T^{*} f | \, a_2, \dots \dots, a_n > = < S_F^{\frac{1}{2}} T^{*} f , S_F^{\frac{1}{2}} T^{*} f | \, a_2, \dots \dots, a_n >$

$= < \left(S_F^{\frac{1}{2}} \; T^{*} \right)^{*} S_F^{\frac{1}{2}} T^{*} f, \; f | \, a_2, \dots \dots, a_n >$

$= < T\, S_F^{\frac{1}{2}} \; S_F^{\frac{1}{2}} T^{*} f, \; f | \, a_2, \dots \dots, a_n >$

$= < T \;\; S_F\, T^{*} f , f | \, a_2, \dots \dots, a_n >$

$= \left\| (T \;\; S_F\, T^{*})^{\frac{1}{2}} f) \right\|_F^2$ (11)

From (10) and (11),

$\frac{1}{B} \|(T\,K)^{*} f\|_F^2 \le \sum_{i=1}^{\propto} |< f , T\, f_i | \, a_2, \dots \dots, a_n >|^2 \quad \forall f \in X_F$

On the other hand, since $\{f_i\}_{i=0}^{\propto}$is a K- frame associated to $(a_2, \dots \dots, a_n)$,

$\sum_{i=1}^{\propto} |< f , T\, f_i | \, a_2, \dots \dots, a_n >|^2 = \sum_{i=1}^{\propto} |< T^{*} f , f_i | \, a_2, \dots \dots, a_n >|^2 \le C \|T\|^2 \|f\|_F^2$

Hence, $\{T\, f_i\}_{i=1}^{\propto}$ is a TK-frame associated to $(a_2, \dots \dots, a_n)$, for X.

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