

A Possible New Upper and Lower Bound From The Infinite Set Of Nonlinear Equations For The Li-Keiper Coefficients

Danilo Merlini^{1, 2, 3}, Massimo Sala^{2, 3}, Nicoletta Sala^{2, 3}

Cerfim, Switzerland

ISSI, Switzerland

IRFIA, Switzerland

(Research Institute in Arithmetic Physics)

Abstract

Starting with an infinite set of nonlinear equations for the Li-Keiper coefficients, we first specify a lower bound emerging from the infinite set and give a characterization of it. Then, we propose a possible new upper and lower bound for the coefficients in few of the partitions occurring in the cluster functions furnishing in a nonlinear way the coefficients.

A numerical experiment up to $n=15$ confirms the proposed bounds and an experiment, i.e. the counting of the zeros in the binary representation of an integer for a constant related to the Glaisher-Kinkelin constant is also given up to $n=32$.

Key Words: Li-Keiper coefficients and an Infinite set of nonlinear Equations, partition function, partitions, Bell Numbers, upper and lower bounds, Glaisher-Kinkelin constant, Riemann Wave Background, Riemann Hypothesis.

Date of Submission: 08-03-2021

Date of Acceptance: 22-03-2021

I. Introduction

In this work we analyze an alternative approach to our recent work concerning a possible proof of the Riemann Hypothesis using Theorems of Statistical Mechanics for a model in dimension 1, with long range interaction related to a polynomial truncation of the Xi function in the variable $z = 1-1/s$ (the zeros in z are on the unit circle).

The alternative approach consists in the analysis of an infinite set of Equations (called equilibrium equation) for the partial partition functions furnishing the Li-Keiper coefficient.

II. An infinite set of equations and a numerical check

Recently, we have obtained an infinite set of nonlinear Equations for the Li-Keiper coefficients, by means of their definition. They are different from other known Equations and involve here the partial partition function φ_n at the left hand side and the successive derivatives of the function at the right border of the critical strip: these could be called the “equilibrium equations” for the Li-Keiper coefficients ($\lambda(n)$). They read [1]

$$\sum_{k=0}^{n-1} (-1)^k \cdot \binom{n-1}{k} \cdot \varphi_{n-k} = \left(\frac{\xi^n(1)}{\xi(1)} \cdot \frac{1}{\Gamma(n+1)} \right) \quad (1)$$

where φ_n is the coefficient of z^n in the expansion of the partition function, i.e. the Xi function given by

$$\begin{aligned} 2 \cdot \xi(z) &= \exp\left(\sum (\lambda_n/n) \cdot z^n\right) = \sum \varphi_n \cdot z^n = \\ &= \varphi_0 + \varphi_1 \cdot z + \varphi_2 \cdot z^2 + \varphi_3 \cdot z^3 + \dots = \\ &= 1 + \lambda_1 \cdot z + (1/2) \cdot (\lambda_1^2 + \lambda_2) \cdot z^2 + (1/3) \cdot (\lambda_3 + (3/2) \cdot \lambda_1 \cdot \lambda_2 + (1/2) \cdot \lambda_1^3) \cdot z^3 + \dots \end{aligned}$$

and

$$\xi^n(1) = \left(\frac{d^n}{ds^n} (\xi(s)) \right) \Big|_{s=1}.$$

The first few six Equations of the set are:

$$\begin{aligned}
 1 \cdot \varphi_1 &= (\xi^1/\xi)_{s=1}/1! \\
 1 \cdot \varphi_2 - 1 \cdot \varphi_1 &= (\xi^2/\xi)_{s=1}/2! \\
 1 \cdot \varphi_3 - 2 \cdot \varphi_2 + 1 \cdot \varphi_1 &= (\xi^3/\xi)_{s=1}/3! \\
 1 \cdot \varphi_4 - 3 \cdot \varphi_3 + 3 \cdot \varphi_2 - 1 \cdot \varphi_1 &= (\xi^4/\xi)_{s=1}/4! \\
 1 \cdot \varphi_5 - 4 \cdot \varphi_4 + 6 \cdot \varphi_3 - 4 \cdot \varphi_2 + 1 \cdot \varphi_1 &= (\xi^5/\xi)_{s=1}/5! \\
 1 \cdot \varphi_6 - 5 \cdot \varphi_5 + 10 \cdot \varphi_4 - 10 \cdot \varphi_3 + 5 \cdot \varphi_2 - 1 \cdot \varphi_1 &= (\xi^6/\xi)_{s=1}/6!
 \end{aligned}
 \tag{2}$$

As a check, we compute the left hand side of the Equations in calculating the φ 's using the values (up to some decimals) of the lambda's given in [2] and for the right hand side using the Riemann series at $s=1/2$ or simply the expansion with Maple of the Taylor series of $\Xi(s)$ at $s=1$ (Table 1).

Left	Right
0.0230957089	0.0230957089
0.0233438645	0.0233438645
0.0004979838	0.0004979846
0.0002531817	0.0002531898
0.0000050504	0.0000050493
0.0000017212	0.0000172087

Table 1

III. Upper and lower bound for the Li-Keiper coefficients and special solutions: Riemann Wave Background

In the above Equations we set $\varphi_k = k \cdot \varphi_1 = k \cdot \lambda_1$ for every positive integer k ; thus, for the left hand side of Eq.(1) we have:

$$\sum_{k=0}^{n-1} (-1)^k \cdot \binom{n-1}{k} \cdot (n-k) \cdot \lambda_1 = 0
 \tag{3}$$

for every n .

Thus:

$$0 = (\xi^n(1)/\xi(1)) / n! > 0
 \tag{4}$$

In fact relation (4) is to understand as follows:

From $\varphi_2 - \varphi_1 = (\xi^2/\xi)_{s=1}/2!$, we have that $\varphi_2 = 2 \cdot \varphi_1 + \delta_2$
 with $\delta_2 = (\xi^2/\xi)_{s=1}/2! - \varphi_1$, with $\delta_2 > 0$ [1, 3].

Thus

$$\varphi_2 = 2 \cdot \varphi_1 + \delta_2 \text{ with } \delta_2 > 0.$$

Next from:

$$\varphi_3 - 2 \cdot \varphi_2 + \varphi_1 = (\xi^3/\xi)_{s=1}/3! = \delta_3 > 0$$

we have:

$$\varphi_3 = 2 \cdot \varphi_2 - \varphi_1 + \delta_3 = 2 \cdot (2 \cdot \varphi_1 + \delta_2) - \varphi_1 + \delta_3 = 3 \cdot \varphi_1 + 2 \cdot \delta_2 + \delta_3$$

Thus: $\varphi_3 = 3 \cdot \varphi_1 + 2 \cdot \delta_2 + \delta_3$ with $2 \cdot \delta_2 + \delta_3 > 0$.

From

$$\varphi_4 = 3 \cdot \varphi_3 - 3 \cdot \varphi_2 + \varphi_1 + (\xi^4/\xi)_{s=1}/4! = 3 \cdot \varphi_3 - 3 \cdot \varphi_2 + \varphi_1 + \delta_4$$

with $\delta_4 = (\xi^4/\xi)_{s=1}/4! > 0$,

$$\varphi_4 = 3 \cdot (3 \cdot \varphi_1 + 2 \cdot \delta_2 + \delta_3) - 3 \cdot (2 \cdot \varphi_1 + \delta_2) + \varphi_1 + \delta_4.$$

Thus:

$$\varphi_4 = 4 \cdot \varphi_1 + 3 \cdot \delta_2 + 3 \cdot \delta_3 + \delta_4 \text{ with } 3 \cdot \delta_2 + 3 \cdot \delta_3 + \delta_4 > 0.$$

From $\varphi_5 = 4 \cdot \varphi_4 - 6 \cdot \varphi_3 + 4 \cdot \varphi_2 - \varphi_1 + (\xi^5/\xi)_{s=1}/5!$

In the same way, with $\delta_5 = (\xi^5/\xi)_{s=1}/5! > 0$, we obtain

$$\varphi_5 = 5 \cdot \varphi_1 + 4 \cdot \delta_2 + 6 \cdot \delta_3 + 4 \cdot \delta_4 + \delta_5 = 5 \cdot \varphi_1 + \varepsilon_5$$

with $\varepsilon_5 > 0$.

Finally, from $\varphi_6 - 5 \cdot \varphi_5 + 10 \cdot \varphi_4 - 10 \cdot \varphi_3 + 5 \cdot \varphi_2 - 1 \cdot \varphi_1 = (\xi^6/\xi)_{s=1} / 6!$

We obtain:

$$\varphi_6 = 6 \cdot \varphi_1 + 5 \cdot \delta_2 + 10 \cdot \delta_3 + 10 \cdot \delta_4 + 5 \cdot \delta_5 + \delta_6 = 6 \cdot \varphi_1 + \varepsilon_6$$

with $\varepsilon_6 = 5 \cdot \delta_2 + 10 \cdot \delta_3 + 10 \cdot \delta_4 + 5 \cdot \delta_5 + \delta_6 > 0$.

Then:

$$\varphi_n = n \cdot \varphi_1 + \varepsilon_n = n \cdot \lambda_1 + \varepsilon_n, \text{ with } \varepsilon_n > 0 \tag{5}$$

(thanks to the Riemann Series) and ε_n is increasing with n , and

$$\lambda_1 = 1 + \gamma/2 - (1/2) \cdot \log(4\pi) = 0.0230957\dots$$

where γ is the Euler-Mascheroni constant, Wolf [4].

Moreover we obtain:

$$\varphi_n = n \cdot \varphi_1 + \sum_{k=1}^{n-1} \binom{n-1}{k} \cdot \delta_{k+1} \tag{6}$$

Our rigorous positive lower bound reads:

$$\varphi_n > n \cdot \lambda_1 \text{ for all } n > 1. \tag{7}$$

Concerning the fluctuation of the φ_s of Eq.(5) there is a binomial structure given by:

$$\varepsilon_n = \sum_{k=1}^{n-1} \binom{n-1}{k} \cdot \delta_{k+1} \tag{8}$$

$$\varphi_n = n \cdot \varphi_1 + \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \delta_{k+1} \tag{9}$$

Explicitly:

$$\varphi_n = n \cdot \varphi_1 + (n-1) \cdot \delta_2 + \sum_{k=2}^{n-1} \binom{n-1}{k} \cdot \frac{\xi^{k+1}}{\xi} \Big|_{s=1} \cdot \frac{1}{(k+1)!} \tag{10}$$

or, since from above we defined $\delta_2 = (\xi^2/\xi)_{s=1}/2! - \varphi_1$, where $\varphi_1 = (\xi^1/\xi)_{s=1} = \lambda_1$, we have:

$$\begin{aligned} \varphi_n &= \frac{\xi^1}{\xi} \Big|_{s=1} + (n-1) \cdot \frac{\xi^2}{\xi} \Big|_{s=1} \cdot \frac{1}{2!} + \sum_{k=2}^{n-1} \binom{n-1}{k} \cdot \frac{\xi^{k+1}}{\xi} \Big|_{s=1} \cdot \frac{1}{(k+1)!} \\ \varphi_n &= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \frac{\xi^{k+1}}{\xi} \Big|_{s=1} \cdot \frac{1}{(k+1)!} \end{aligned} \tag{11}$$

Remark

We notice here that the φ_n are related to the complete Bell Polynomials or to the Faà di Bruno's formula [5] in the form of determinants. We recall here the formula for φ_3 .

The known general formula is:

$$\exp\left(\sum_{j=1}^{\infty} a_j \cdot \frac{z^j}{j!}\right) = \sum_{k=0}^{\infty} B_k(a_1, a_2, \dots, a_k) \cdot \frac{z^k}{k!}$$

where B_k is the complete k -Bell Polynomial and where the Li-Keiper coefficients are given by $\lambda_j = a_j / (j-1)!$, the corresponding determinant for $n=3$ reads:

$$B_3(a_1, a_2, a_3) = \text{Det} \begin{pmatrix} a_1 & 2 \cdot a_2 & a_3 \\ -1 & a_1 & a_2 \\ 0 & -1 & a_1 \end{pmatrix} = a_1^3 + a_1 \cdot a_2 + 2 \cdot a_1 \cdot a_2 + a_3 = \lambda_1^3 + 3 \cdot \lambda_1 \cdot \lambda_2 + 2 \cdot \lambda_3$$

since $a_3 = 2 \cdot \lambda_3$, then :

$$(1/3!) \cdot (\lambda_1^3 + 3 \cdot \lambda_1 \cdot \lambda_2 + 2 \cdot \lambda_3) = (1/3) \cdot (\lambda_3 + (3/2) \cdot \lambda_1 \cdot \lambda_2 + \lambda_1^3/2) = \varphi_3.$$

We now continue and first check the Formula above for $n=5$ for which we know, inserting the first 5 lambda 's values taken from Ref [2], that $\varphi_5 = 0.12047684\dots$

From Eq.(9) or Eq.10 with $n=5$, we obtain $\varphi_5 = 0.12047684\dots$ in full agreement with the above number.

For an analysis of subsequent Expressions we analyze numerically the behavior of φ_n as a function of n , calculated with Eq.(11) at $s=1$ that is a Formula with the absence of non trivial zeros; the results up to $n= 10$ are given in the Table 2 and in the plot (notice: $\xi(s) = \sum_{n=0} a_n \cdot (s - 1)^n$).

$$\varphi_n = \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \left. \frac{\xi^{k+1}}{\xi} \right|_{s=1} = \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot 2 \cdot a_{k+1} \tag{12}$$

n	φ_n
1	0.0230957088
2	0.0464395736
3	0.0702814232
4	0.0948744395
5	0.1204768540
6	0.1473536683
7	0.1757784078
8	0.2060349171
9	0.2384192110
10	0.2732413926

Table 2

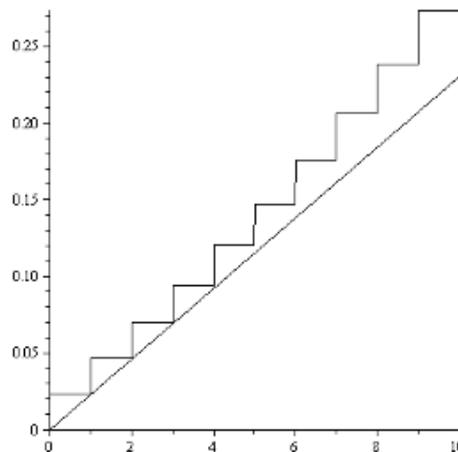


Figure 1. Plot of $\varphi_n(n)$ and the lower bound $n \cdot \varphi_1 = n \cdot \lambda_1$

A characterization of the above special “solution” $\varphi_n = n \cdot \varphi_1$, furnishes, after summation over n in Eq.(2), the function [1]

$$\exp(\log(1 + \varphi_1 \cdot z / (1-z)^2)) = \exp(\log(1 + \varphi_1 \cdot K(z))) \tag{13}$$

where $K(z)$ is the Koebe function of argument z . Moreover the expansion of the above log function i.e. the coefficient of z^n in the expansion of $\log(1 + \lambda_1 \cdot z / (1-z)^2)$ (up to the factor $1/n$) is our lower bounds for each n on the Li-Keiper coefficients and such a sequence constitutes our Riemann Wave Background [1]. The series is given explicitly by:

$$\exp(\log(1 + \lambda_1 \cdot z / (1-z)^2)) = \lambda_1 \cdot z + (1/2) \cdot (4 \cdot \lambda_1 - \lambda_1^2) \cdot z^2 + (1/3) \cdot (9 \cdot \lambda_1 - 6 \cdot \lambda_1^2 + (1/2) \cdot \lambda_1^3) \cdot z^3 + (1/4) \cdot (16 \cdot \lambda_1 - 20 \cdot \lambda_1^2 + 8 \cdot \lambda_1^3 - \lambda_1^4) \cdot z^4 + \dots + \dots \tag{14}$$

We may also consider the Formula given by Eq.(11) to obtain the general formula for the Li-Keiper coefficients. We have:

$$\varphi_1 = (\xi^{-1}/\xi)|_{s=1} = \lambda_1 = 1 \cdot \varphi_1$$

From $\varphi_2 = (1/2) \cdot (\lambda_2 + \lambda_1^2) \rightarrow \lambda_2 = 2 \cdot \varphi_2 - 1 \cdot \varphi_1 \cdot \varphi_1$

From $\varphi_3 = (1/3) \cdot (\lambda_3 + (3/2) \cdot \lambda_1 \cdot \lambda_2 + \lambda_1^3/2) \rightarrow$
 $\lambda_3 = 3 \cdot \varphi_3 - (3/2) \cdot \varphi_1 \cdot (2 \cdot \varphi_2 - 1 \cdot \varphi_1 \cdot \varphi_1) - \varphi_1^3/2 = 3 \cdot \varphi_3 - 3 \cdot \varphi_1 \cdot \varphi_2 + 1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1.$

From $\varphi_4 = (1/4) \cdot (\lambda_4 + (4/3) \cdot \lambda_1 \cdot \lambda_3 + (1/2) \cdot \lambda_2^2 + \lambda_1^2 \cdot \lambda_2 + \lambda_1^4/6) \rightarrow$
 $\lambda_4 = 4 \cdot \varphi_4 - (4 \cdot \varphi_1 \cdot \varphi_3 + 2 \cdot \varphi_2 \cdot \varphi_2) + 4 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_2 - 1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1.$

From $\varphi_5 = (1/5) \cdot (\lambda_5 + (5/4) \cdot \lambda_1 \cdot \lambda_4 + (5/6) \cdot \lambda_2 \cdot \lambda_3 + (5/6) \cdot \lambda_1^2 \cdot \lambda_3 +$
 $+ (5/8) \cdot \lambda_1 \cdot \lambda_2^2 + (5/12) \cdot \lambda_1^3 \cdot \lambda_2 + (5/120) \cdot \lambda_1^5$

we have:

$$\lambda_5 = 5 \cdot \varphi_5 - (5 \cdot \varphi_1 \cdot \varphi_4 + 5 \cdot \varphi_2 \cdot \varphi_3) + (5 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_4 + 5 \cdot \varphi_1 \cdot \varphi_2 \cdot \varphi_2) - 5 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_2 +$$

$$+ 1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1$$

i.e. here (5, -10, +10, -5, +1).

As above from the expression of φ_6 we obtain:

$$\lambda_6 = 6 \cdot \varphi_6 - (6 \cdot \varphi_1 \cdot \varphi_5 + 6 \cdot \varphi_2 \cdot \varphi_4 + 3 \cdot \varphi_3 \cdot \varphi_3) + (6 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_4 + 12 \cdot \varphi_1 \cdot \varphi_2 \cdot \varphi_3 +$$

$$+ 2 \cdot \varphi_2 \cdot \varphi_2 \cdot \varphi_2) - (9 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_2 \cdot \varphi_2 + 6 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_3) + (6 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_2) -$$

$$- 1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1.$$

i.e. here (6, -15, +20, -15, +6, -1).

Then, we have for each n:

$$\lambda_n = \sum_{k=1}^n (-1)^{(k-1)} \cdot \left(\sum_{\sum_{i=1}^k p_i = n} a_k(p_1, p_2, \dots, p_k) \cdot \prod_{i=1}^k \varphi_{p_i} \right) \tag{15}$$

Where

$$\sum_{\sum_{i=1}^k p_i = n} a_k = \binom{n}{k}$$

Above for n=5, we have p(5)= 7 i.e. the partitions of 5: (5), (1, 4), (2, 3), (1, 1, 3), (1, 2, 2), (1, 1, 1, 2), and (1, 1, 1, 1) with the weights a_k given by (5), (5, 5), (5, 5), (5), (1) and the φ 's given by Eq.(11). For n=6 we also have that

$$\lambda_6 = 6 \cdot \varphi_6 - (6 \cdot \varphi_1 \cdot \varphi_5 + 6 \cdot \varphi_2 \cdot \varphi_4 + 3 \cdot \varphi_3 \cdot \varphi_3) + (12 \cdot \varphi_1 \cdot \varphi_2 \cdot \varphi_3 + 6 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_4 + 2 \cdot \varphi_2 \cdot \varphi_2 \cdot \varphi_2) +$$

$$- (6 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_3 + 9 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_2 \cdot \varphi_2) + 6 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_2 - 1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_1.$$

with the partition: (6), (1, 5), (2, 4), (3, 3), (1, 2, 3), (1, 1, 4), (2, 2, 2), (1,1, 1, 3), (1,1,2,2), (1,1,1,1,2), (1, 1, 1, 1, 1) (p(6)=11).

For the asymptotic and more about the partition function p(n) we refer to one of the fundamental work of Ramanujan [6, p.116].

In particular as

$$p(n) \sim \frac{e^{2\pi \cdot \sqrt{\frac{n}{6}}}}{(4 \cdot n \cdot \sqrt{3})} = \frac{e^{k \cdot \sqrt{n}}}{(4 \cdot n \cdot \sqrt{3})} \tag{16}$$

Thus, we have obtained an alternating binomial sequence -with “weighted cluster partitions” - whose values decrease rapidly to zero (the first two terms - the first as an upper bound - the first two as a lower bound - are expected to ensure the positivity of the λ 's at low values of n).

As a numerical experiment, we first give on the Table below, the numerical values of the sequences furnishing the first 6 exact values of the λ 's.

$$\begin{aligned} \lambda_1 &= \mathbf{0.023095708966} \dots = 1+\gamma/2 - (1/2)\cdot\log(4\pi) \\ \lambda_2 &= 0.0928791468860 - 0.000533411769594 = \\ &= \mathbf{0.09234573511} \dots \\ \lambda_3 &= 0.210844265482 - 0.00321766460904 + 0.000012319522 \dots = \\ &= \mathbf{0.207638920396} \dots \\ \lambda_4 &= 0.3794977079944 - 0.0108060650201 + 0.000099077079 - 2.84528115941\cdot 10^{-7} = \\ &= \mathbf{0.368790470678} \dots \\ \lambda_5 &= 0.602359196025 - 0.0272751580080 + 0.000249373225556 - 0.00000286056695512 + \\ &\quad + 6.57137853964\cdot 10^{-9} = \\ &= \mathbf{0.575330557247} \dots \\ \lambda_6 &= 0.8841219582034230 - 0.057948994032944 + 0.0014085173503118162574 + \\ &\quad - 0.00001554836720655352 + 7.928018702647952\cdot 10^{-8} - 1.517706484293658\cdot 10^{-10} = \\ &= \mathbf{0.8275659331} \dots \end{aligned}$$

(exact to some digits, the last one instead of 0.827566012.....[2]).

We notice that the first term in each of the 6 sequences is an upper bound to the true value of the corresponding Li-Keiper coefficients and the first two terms is a lower bound to it as stated above.

As an example, for λ_4 , where the true value is $\lambda_4 = 0.368790479\dots$ [2], the upper bound is 0.37949.. and the lower one is 0.36869..

Thus: $0.36869\dots < \lambda_4 = 0.368790479\dots < 0.37949$.

In Appendix 1, we illustrate the calculation for $n=7, 8, 9, 10$ where we find the inequality (the value of λ_7 is still taken from Ref [2]).

$$1.12078 < \lambda_7 = 1.12446011757 < 1.23044$$

and give also the expressions for the lower and the upper bound for $n=8, n=9$ and $n=10$.

The general relationship for the Li-Keiper coefficients in this approximation is formulated as:

$$\lambda_n \sim n \cdot \varphi_n - \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} a_p \cdot \varphi_p \cdot \varphi_{n-p} \tag{17}$$

Where $\lfloor \cdot \rfloor$ is the floor function.

Since we are in the presence of an alternating decreasing sequence in absolute values, the successive contributions (+, -) are positive confirming our inequality above for the upper and for - to begin with - the lower bound along our lines.

In a numerical contest, an interesting numerical advanced experiment consists thus in using the first two “terms” above (the contribution of the “cluster” φ_n (one block) and that of the product of only two “cluster contributions”(two blocks) as given in Eq. (17), where a_p are the positive constants. The Table up to $n=10$ is given below (lower bound, true value, upper bound) with the plot.

The true values are taken from Maslanka [2].

$\lambda_1 = \mathbf{0.023095708966} \dots = 1+\gamma/2 - (1/2)\cdot\log(4\pi)$
$\lambda_2 = 0.0928791468860 - 0.000533411769594 =$ $= \mathbf{0.09234573522} \dots$
$0.2076276009 < \lambda_3 = \mathbf{0.20763892055} < 0.21084426548$
$0.3686916430 < \lambda_4 = \mathbf{0.36879047949} < 0.379497707994$
$0.5750840380 < \lambda_5 = \mathbf{0.57542714461} < 0.602359196025$
$0.8261729642 < \lambda_6 = \mathbf{0.8275660122} < 0.884121958203$
$1.1207865182 < \lambda_7 = \mathbf{1.12446011757} < 1.230448804543$
$1.4573142832 < \lambda_8 = \mathbf{1.46575567715} < 1.648279333807$
$1.8334010001 < \lambda_9 = \mathbf{1.85091604838} < 2.145772946851$
$2.245753497 < \lambda_{10} = \mathbf{2.27933936319} < 2.732413708$

Table 3

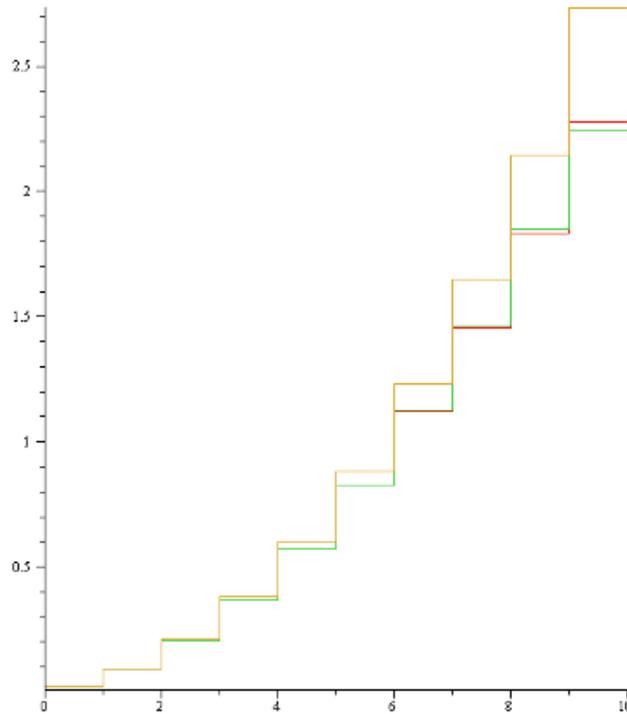


Fig.2. Plots of the lower, true values and upper bounds up to n=10 (From Table 3).

We have also pursued our numerical experiment and we have calculated our upper and lower bounds up to n=15.

In the next Fig. 3 we present the plot of our upper and lower bound from n=1 to n=15 together with a straight line i.e. $0.3 \cdot x + \text{constant}$ and the convex curve $0.1 \cdot x \cdot \log(x) + \text{constant}$ which illustrate the positivity property of our bounds (at low values of n) given by Eq.(17). (If R.H. is true then the function will be given by $0.5 \cdot (x \cdot \log(x)) - c \cdot x$, for large integers x and $c \sim 1.13$ [2].

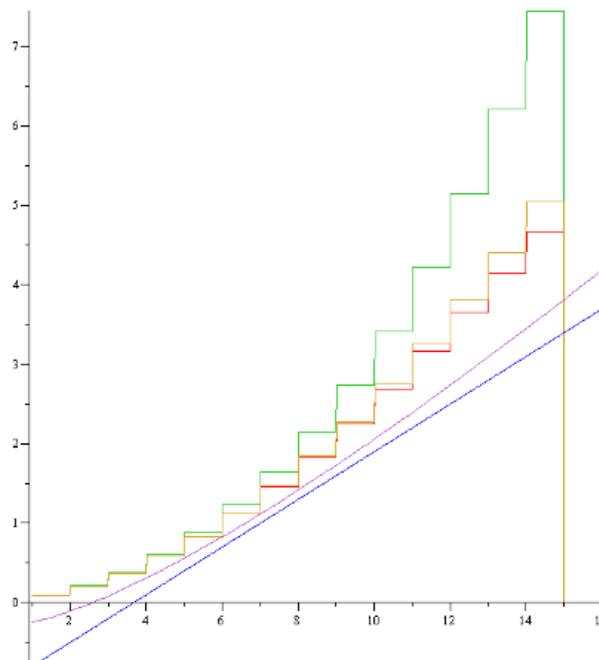


Fig.3. Plot of the lower bound, the true value, the upper bound and the function $0.1 \cdot x \cdot \log(x) + \text{constant}$, the function $0.3 \cdot x + \text{constant}$ in the range (1..15) of x, i.e. for low values of n.

IV. Shift of one unity in Xi(s) and the Glaisher-Kinkelin constant

The infinite set, given by Eq.(1), is in terms of the values of the successive derivatives of the Xi function at the right border of the critical strip, i.e. at s=1 (or at z=0, where z=1-1/s).

Summation of the right hand sides of the set (which results in a strong cancellation at the left hand side) leads to a shift of one unity (from s=1 to s=2) in the argument of the Xi function and we obtain :

$$\begin{aligned}
 c &= \sum_{n=1}^{\infty} \left(\frac{1}{n!}\right) \cdot \left(\frac{\xi^n(1)}{\xi(1)}\right) = \sum_{n=1}^{\infty} \left(\frac{\xi^n(1)}{\xi(1)}\right) \cdot \frac{(2-1)^n}{(n!)} = \\
 &= \left(\frac{\xi'(2)}{\xi(1)}\right) = \left(\frac{\pi}{3}\right) \cdot \left(\frac{\xi'(2)}{\xi(2)}\right)
 \end{aligned}
 \tag{18}$$

Since $\xi(2) = (1/\pi) \cdot (\pi^2/6) = (\pi/3) \cdot (1/2) = (\pi/3) \cdot \xi(1)$, then:

$$\begin{aligned}
 c &= \sum_{n=1}^{\infty} \left(\frac{1}{n!}\right) \cdot \left(\frac{\xi^n(1)}{\xi(1)}\right) = \left(\frac{\pi}{3}\right) \cdot \left(\frac{\xi'(2)}{\xi(2)}\right) = \\
 &= (\pi/3) \cdot ((1/2) \cdot (\gamma + \log(4 \cdot \pi) + 3) - 12 \cdot \log(A)) = 0.072325988...
 \end{aligned}
 \tag{19}$$

where A is the Glaisher-Kinkelin constant [7], given by $\zeta'(s=-1) = 1/12 - \log(A) = -0.1654211937..$

(20)

Remark

Eq.(1) holds also for ξ replaced by $(s/2) \cdot \pi^{-s/2} \cdot \Gamma(s/2)$ and the ϕ 's related to the λ_{trend} 's as well as for the function $(s-1) \cdot \zeta(s)$ and the ϕ 's related to the λ_{tiny} 's ($\lambda_{\text{trend}}(n) + \lambda_{\text{tiny}}(n) = \lambda(n) = \lambda_n$), where $\lambda_{\text{tiny}}(n)$ is the tiny part of the coefficient [2]. Then, our above constant splits into the two contributions respectively given by:

$$(1/6) \cdot (-\gamma \cdot \pi - \pi \cdot \log(\pi) + \pi) = -0.3780098064 \text{ (trend)}
 \tag{21}$$

and

$$(1/6) \cdot ((12/\pi) \cdot \zeta'(s=2)) + 2 \cdot \pi = 0.4503357950 \text{ (tiny)}
 \tag{22}$$

so that $0.4503357950 - 0.3780098064 = 0.0723259886 = c$, our above constant.

Moreover, using the formula involving the number of zeros $N_0(n)$ [8], appearing in the binary representation of an integer n, we have - using the fundamental Formulas for γ and $\log(\pi)$ - [8], i.e.

$$\log\left(\frac{2}{\pi}\right) = -\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{N_1(n) - N_0(n)}{(2n) \cdot (2n+1) \cdot (2n+2)}\right)
 \tag{23}$$

and

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{N_1(n) + N_0(n)}{(2n) \cdot (2n+1) \cdot (2n+2)}\right)
 \tag{24}$$

that

$$c = \left(\frac{\pi}{3}\right) \cdot \left(1 + \frac{3}{2} \log(2) + 12 \cdot \zeta'(-1) + \sum_{n=1}^{\infty} \left(\frac{N_0(n)}{(2n) \cdot (2n+1) \cdot (2n+2)}\right)\right)
 \tag{25}$$

Numerically, using $N_0(n)$ up to n=32, we obtain:

$$c = \left(\frac{\pi}{3}\right) \cdot \left(1 + \frac{3}{2} \log(2) + 12 \cdot \zeta'(-1) + \frac{1}{4 \cdot 5 \cdot 6} \dots\right)
 \tag{26}$$

(see Appendix 3). As noticed above, we note the strong cancellation in the left hand side and equal, now, to the linear function in the Li-Keiper coefficients which reads:

$$c = \left(\frac{\pi}{3}\right) \cdot \left(\frac{d}{dz} (\log(\xi(z))) \cdot \frac{dz}{ds}\right) \Big|_{z=\frac{1}{2}} = \left(\frac{\pi}{3}\right) \cdot \sum_{n=1}^{\infty} \lambda_n \cdot \left(\frac{1}{2}\right)^{n+1} \quad (27)$$

Another interesting formula is given in appendix 4. Here, using the first Li-Keiper coefficients [2] up to n=15 we obtain $c = 0.072222733376\dots$ a value, as above, which may be compared with the exact value 0.072325...

An upper bound for c is also given by $\pi \cdot \lambda_1 = 0.07255730\dots$ [1] a value greater than the true value of only the amount $2 \cdot 10^{-4}$.

V. Concluding Remark and Riemann Hypothesis

In this work, we presented an analysis of the infinite set of Equations for the Li-Keiper coefficients, already obtained from their definition Eq.(1) [1]. An alternative to the approach of a possible proof of the R.H. using the Lee-Yang theorem on the zeros of some partitions function for ferromagnetic Ising spin models, with long range 2-body interactions in the presence of a magnetic field [1].

The set of Equations given by (1) is of course related to the Faà di Bruno's formula as well as to the complete Bell Polynomials for the ϕ 's as well as to the partitions present in the structure of the algebraic expression for the Li-Keiper coefficients as functions of the ϕ 's (the complete Bell polynomials) [5].

We are in presence of an alternating strong decaying sequence of contributions; in this context we have found a new formula for an upper and a lower positive bound (above positive at least for the low values of n) to the Li-Keiper coefficients which involves only a single and a two cluster contributions.

The new formula has been checked up to n=15 and deserves future analysis both in a theoretical as well as in a numerical context. The constant c, related to the Glaisher-Kinkelin constant already obtained [1, 3], has been computed here using the formula for γ and for $\log(\pi)$ of Matiyasevich [8] and of Vacca and Sondow [9,10], in counting the zeros occurring in the binary representation of an integer n up to n=32, and also with the series with the λ 's values up to λ_{15} taken from Maslanka [2].

Finally, concerning the Riemann Hypothesis along our lines we may subsume the following:

a. Dealing with Statistical Mechanics of spin-systems with two-body long range ferromagnetic interactions in the high temperature region - in connection with a truncation of the Xi function - we have found a possible proof of the R.H. by means of a lower "periodic" bound for the Li-Keiper coefficients given by the Riemann-Wave Background [1]. In this approach, we do not considered directly the behavior of nontrivial zeros and the main ingredient is the Lee-Yang theorem on the zeros of some of the partition functions of Statistical Mechanics. The lower bound is here "only" in the form of a "periodic" function of maximum value, which is equal to 4 [1, 3].

b. In the alternative approach along the above lines, we have used the definition of the Li-Keiper coefficients to obtain an infinite set of equilibrium equations for the Li-Keiper coefficients in connections with important functions like Bell polynomials, Faà di Bruno's formulas and partitions, i.e. a structure involving an alternating decaying sequence with the emergence of a new lower and upper bound for the Li-Keiper coefficients. To the best of our knowledge the bounds have not been considered before or along our lines (the lower bounds are positive at least for low values of n).

These new bounds have been checked up to n=15 and they deserve of course further research, i.e. including the terms with more blocks.

The strong decaying alternating sequence related to the series expansion of the Xi function will be difficult to disprove. Moreover, the constant c, related to the Glaisher-Kinkelin [7] constant has been computed here in the binary system using important contributions of Vacca, Sondow and Matiyasevich.

References

- [1].a. D. Merlini, M. Sala, N. Sala: "A possible Non Negative lower bound on the Li-Keiper coefficients", *IOSR Journal of Mathematics*, Vol.15 Issue 6 (Ser. IV). pp. 01-16 (2019).
- b. D. Merlini, M. Sala, N. Sala: "The binomials coefficients in the Riemann Wave Background: a possible proof of the Riemann Hypothesis", *IOSR Journal of Mathematics*, Vol.16, Ser. III (2020), pp. 22-36.
- c. D. Merlini, M. Sala, N. Sala: "Primitive Riemann Wave at $\text{Re}(s)=0.9$ and Application of Gauss-Lucas Theorem", *Chaos and Complexity Letters*, Vol. 14, Issue 1 (2020), NOVA SCIENCE, New-York, pp. 3-32.K.
- [2]. Maslanka: "Effective method for computing Li's coefficients and their properties" Arxiv: math/0402168v5 (2004).
- [3]. D. Merlini, M. Sala, N. Sala: "Spin system on a hexagon and Riemann hypothesis", *IOSR Journal of Mathematics*, Vol. 16, Issue 5 (Ser. II) (2020), pp.14-24.
- [4]. M. Wolf: "6+ ∞ new expressions for the Euler- Mascheroni constant", arXiv:1904.0985v1 [math NT](2020).
- [5].a. G.C. Rota, (1964). "The number of partitions of a set". *The American Mathematical Monthly*, 71(5), 498-504.
- b. Bell polynomial. *Encyclopedia of Mathematics*. Retrieved 10 November, 2020.
URL: http://encyclopediaofmath.org/index.php?title=Bell_polynomial&oldid=46007

- c. E.W. Weisstein, "Faà di Bruno's Formula." From MathWorld--A Wolfram Web Resource. Retrieved 10 November, 2020. URL: <https://mathworld.wolfram.com/FaadiBrunosFormula.html>
- d. E.W. Weisstein, "Cumulant." From MathWorld--A Wolfram Web Resource. Retrieved 10 November, 2020. URL: <https://mathworld.wolfram.com/Cumulant.html>
- e. E.W. Weisstein, "Partition." From MathWorld--A Wolfram Web Resource. Retrieved 10 November, 2020. URL: <https://mathworld.wolfram.com/Partition.html>
- [6]. G. H. Hardy (1969). *Ramanujan: Twelve Lectures on subjects suggested by his Life and Work*. AMS CHELSEA PUBLISHING, American Mathematical Society. Providence, Rhode Island.
- [7]. E.W. Weisstein, "Glaisher-Kinkelin Constant." From MathWorld--A Wolfram Web Resource. Retrieved 20 October, 2020. URL: <https://mathworld.wolfram.com/Glaisher-KinkelinConstant.html>.
- [8]. Y. Matiyasevich. "Yet another Representation for the Reciprocals of the Non trivial Zeros of the Riemann Zeta Function", *Mat. Zametki*, 97:3 (2015).
- [9]. G. Vacca. "A new series for the Eulerian constant $\gamma=0.577...$ ", *Quart. J. Pure Appl. Math.*, 41 (1909-1910), pp. 363-366.
- [10]. J. Sondow. "New Vacca-type rational series for Euler's constant γ and its "alternating" analog $\ln 4/\pi$ ", *Additive number theory*, Springer, New York (2010), pp. 331-340.

Appendix1

For $n=7$ we have $(7, -21, +35, -35, +21, -7, +1)$ and the expression:

$$\lambda_7 = 7 \cdot \varphi_7 - 7 \cdot (1 \cdot \varphi_1 \cdot \varphi_6 + 1 \cdot \varphi_2 \cdot \varphi_5 + 1 \cdot \varphi_3 \cdot \varphi_4) + 7 \cdot (2 \cdot \varphi_1 \cdot \varphi_2 \cdot \varphi_4 + 1 \cdot \varphi_1 \cdot \varphi_1 \cdot \varphi_5 + 1 \cdot \varphi_1 \cdot \varphi_3 \cdot \varphi_3 + 1 \cdot \varphi_2 \cdot \varphi_2 \cdot \varphi_3) - 7 \cdot (3 \cdot \varphi_1^2 \cdot \varphi_2 \cdot \varphi_3 + 1 \cdot \varphi_1^3 \cdot \varphi_4 + 1 \cdot \varphi_1 \cdot \varphi_2^3) + 7 \cdot (1 \cdot \varphi_1^4 \cdot \varphi_3 + 2 \cdot \varphi_1^3 \cdot \varphi_2^2) - 7 \cdot \varphi_1^5 \cdot \varphi_2 + 1 \cdot \varphi_1^7$$

that is:

$$\begin{aligned} \lambda_7 &= 1.230448804543289 - 0.1096622863027495 + \\ &+ 0.003734023309666 - 0.00006093378862279656 + \\ &+ 5.119411153565860 \cdot 10^{-7} - 2.136204147398473 \cdot 10^{-9} + \\ &+ 3.505250725720914 \cdot 10^{-12} = \\ &= 1.120786518240540 + \\ &+ 0.00367089521043532 + \\ &+ 5.098049112091875 \cdot 10^{-7} + \\ &+ 3.505250725720914 \cdot 10^{-12} = \end{aligned}$$

= 1.120786518240540 to be compared with the true value 1.12446011757 [2].

The first two terms of the alternating decaying sequence for $n=8$ are given by $(8, -28, +56, -70, \dots)$

$$\lambda_8 = 8 \cdot \varphi_8 - (8 \cdot \varphi_1 \cdot \varphi_7 + 8 \cdot \varphi_2 \cdot \varphi_6 + 8 \cdot \varphi_3 \cdot \varphi_5 + 4 \cdot \varphi_4 \cdot \varphi_4) + \dots$$

and those for $n = 9$ by $(9, -36, +84, -126, \dots)$

$$\lambda_9 = 9 \cdot \varphi_9 - (9 \cdot \varphi_1 \cdot \varphi_8 + 9 \cdot \varphi_2 \cdot \varphi_7 + 9 \cdot \varphi_3 \cdot \varphi_6 + 9 \cdot \varphi_4 \cdot \varphi_5)$$

The bounds for these two Li-Keiper coefficients are given on the Table in the text.

To conclude, for $n=10$ we have $1+9, 2+8, 3+7, 4+6, 5+5$ and still for the bounds alone

$$10 \cdot \varphi_{10} - (10 \cdot \varphi_1 \cdot \varphi_9 + 10 \cdot \varphi_2 \cdot \varphi_8 + 10 \cdot \varphi_3 \cdot \varphi_7 + 10 \cdot \varphi_4 \cdot \varphi_6 + 10 \cdot \varphi_5 \cdot \varphi_5 / 2!)$$

where the sum of the coefficients equal

$$4 \cdot 10 + \left(\frac{1}{2}\right) \cdot 10 = \binom{10}{2} = \frac{10 \cdot 9}{2}$$

Appendix 2

We now give the Equation to obtain φ_n' from the upper bound $\lambda_n < n^2 \cdot \lambda_1$ (notice $\varphi_0'=1$)

$$\exp(\sum \lambda_n \cdot z^n / n) = \exp(\sum \lambda_1 \cdot z^n \cdot n) = \sum \varphi_n' \cdot z^n = \dots$$

$$\begin{aligned} &= 1 + \lambda_1 \cdot z + 2 \cdot \lambda_1 \cdot (1 + \lambda_1 / 4) \cdot z^2 + 3 \cdot \lambda_1 \cdot (1 + (2/3) \cdot \lambda_1 + (1/18) \cdot \lambda_1^3) \cdot z^3 + \\ &+ 4 \cdot \lambda_1 \cdot (1 + (5/4) \cdot \lambda_1 + (1/4) \cdot \lambda_1^2 + (1/96) \cdot \lambda_1^3) \cdot z^4 + \dots = \end{aligned}$$

$$= 1 + \varphi_1' \cdot z_1 + \varphi_2' \cdot z^2 + \varphi_3' \cdot z^3 + \varphi_4' \cdot z^4 + \dots$$

Thus:

$$\varphi_2' = 2 \cdot \lambda_1 \cdot (1 + \lambda_1 / 4) = (\frac{1}{2}) \cdot (\lambda_2 + \lambda_1^2) \rightarrow \lambda_2 = 4 \cdot \lambda_1 = 2^2 \cdot \lambda_1$$

The lower bound λ'_2 is given by $4 \cdot \lambda_1 - \lambda_1^2 = 0.09184942..$ while λ''_2 is given by:
 $\lambda''_2 = 2 \cdot \varphi_2 - \lambda_1^2 = 2 \cdot (0.0464395736) - (0.0230957089)^2 = 0.0923457...$

Appendix 3

Here we are concerned with a calculation of the constant c related to the Glaisher-Kinkelin constant not given in [1].

In the next Table 4 we give the values of the partial sums from $n=1$ to N with Eq.(25) and Eq.(27) together with the plots.

N	$c_1(N)$ from Eq.(15)	$c_2(N)$ from Eq.(16)
1		0.00604644
2	0.06597384	0.01813447
3		0.03172440
4	0.06888272	0.04379304
5		0.05321034
6	0.06936221	0.05998085
7		0.06458058
8	0.07000387	0.06757850
9	0.07031007	0.06947135
10	0.07053674	0.07063684
11	0.07062297	0.07134000
12	0.07075722	0.07175715
13	0.07081050	0.07200113
14	0.070853493	0.07214211
15		0.07222273
16	0.070970159	
17	0.071043492	
18	0.071105560	
19	0.071140890	
20	0.071186500	
21	0.071212856	
22	0.071235851	
23	0.071245942	
24	0.071272657	
25	0.071288451	
26		
27		
28		
29		
30		
31		
32	0.071348919	

Table 4

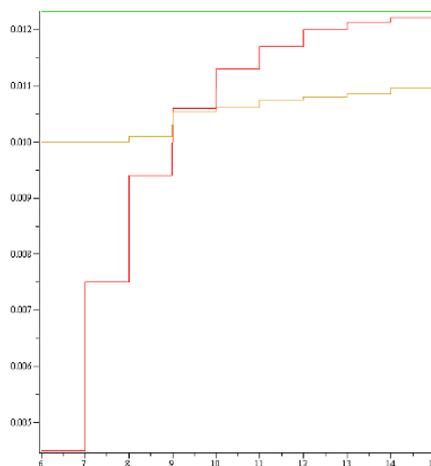


Fig.4. Plot of the two Sequences $c_1(n)-0.06$ of the Table (Eq.(24)) (in maroon) of the Sequence $c_2(n)-0.06$ of the Table (Eq.(26)(in red) and the constant also c decreased by 0.06 in the range (6-15)· $c_1(n)$ appears as an increasing “lower bound” to $c_2(n)$ ($c=0.072325....$).

Discussion: We note that $c_1(n)$ and $c_2(n)$ are equal around $n=11$ for c_1 and $n=10$ for c_2 . Then c_2 grows faster than c_1 .

Appendix 4

Another formula is the following:

$$\log \xi(z) = -\log 2 + \sum_{n=1}^{\infty} \frac{\lambda_n}{n} z^n$$

We have:

$$\log \xi(2) = \log \left(\frac{1}{2} \cdot 2 \cdot (2-1) \cdot \pi^{-1} \cdot \frac{\pi^2}{6} \right) = \log \frac{\pi}{6}$$

thus

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \cdot \left(\frac{1}{2}\right)^n = \log \frac{\pi}{3}$$

A4.1

Eq. A4.1 and Eq. (27) i.e.

$$c = \frac{\pi}{3} \cdot \sum_{n=1}^{\infty} \lambda_n \cdot \left(\frac{1}{2}\right)^{n+1}$$

are investigated using the primes and the equation for $\log \pi$ and γ of Vacca, Sondow and Matiyasevich, i.e.:

$$\log \frac{\pi}{3} = \log \frac{2}{\pi} - \sum_{p=2}^{\infty} \log(1 - p^{-2}) = \log \frac{4}{3} - \gamma + 2 \sum_{n=1}^{\infty} \frac{N_0(n)}{2n \cdot (2n+1)}$$

where p is a prime and $N_0(n)$ is the number of zeros in the binary representation of n [IRFIA, Internal Note, 2021].

Danilo Merlini, et. al. " A Possible New Upper and Lower Bound From The Infinite Set Of Nonlinear Equations For The Li- Keiper Coefficients." *IOSR Journal of Mathematics (IOSR-JM)*, 17(2), (2021): pp. 01-12.