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On Epi α -Regular Spaces

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Abstract: *A topological space (X,τ) is said to be epi α-regular if a coarser topology τ' on X exists such that (X,τ') is T_1, α-regular. In this article we introduce and implement this property and give some examples to show the relationships between epi α -regular, epiregular, epinormal, submetrizable semiregular and Almost α-normal (almost β-normal).*
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I. Introduction

A topological space (X,τ) is said to be epiregular [4] if a coarser topology τ on X exists such that (X,τ') is T_1, regular . A topological space (X,τ) is said to be α -regular [10], [30] if for every closed subset F of X and $x \in X$ such that $x \notin F$ there exist disjoint open sets U and V such that $x \in U$ and $V \cap F$ is dense in F. We use these definitions to introduce another new topological property as a simultaneous generalization called *epi α -regularity*. The intent of this article is to implement this property. We show the relationship between epi α -regular space, α -normal, α -regular, epinormal, epiregular, semiregular and Almost α-normal (almost β-normal) spaces. Also we show that every epi α -regular space is Hausdorff. We prove that submetrizability or T_1, α -regularity imply epi α -regularity but the converse is not correct in general. We give some examples to show that epi α -regularity, α -regularity and semiregularity are not necessarily related.

II. Epi α -Regularity

***Definition* 1.1**. A topological space (X,τ) is said to be epi α -regular if a coarser topology τ' on X exists such that (X,τ') is T_1, α -regular.

Note that if we necessarily let (X,τ') to be just α -regular in the above definition, then any space will be epi α -regular since the indiscrete topology will satisfy the property.

Observe that if for any topological space (X,τ') which is $T_i, i \in \{0,1,2\}$ then any larger topology τ on X so is, and since every α -regular T_1 is Hausdorff [10], [30], then we can conclude the following.

***Theorem* 1.2.** Every epi α -regular space is Hausdorff.■

We note that if X is not T_i, where $i \in \{0,1,2\}$, then X is not epi α regular. For example, Sierpinski space and The closed extension topology see [9], are not Hausdorff, then they cannot be epi α -regular. Since every regular space is α -regular, then the next theorem is true.

***Theorem* 1.3.** Every epiregular space is epi α -regular.■

The opposite direction of the above statement is not always true, but we still have the following correct.

***Theorem* 1.4.** If (X,τ) is an epi α -regular space, and the witnesses of epi α -regularity (X,τ') is first countable, then (X,τ) is epiregular.■

Before proofing the above theorem, we need the following proposition which is proved by a similar argument found in [29].

***Proposition* 1.5**. [30] Every first countable α -regular Hausdorff space is regular.

Proof. Using a contradiction, we suppose that X is a first countable, Hausdorff and non regular space. Then there is an $x \in X$ and a closed subset A of X such that $x \notin A$ where there are no disjoint open sets that separate them.

Let $\{U_n : n \in \omega\}$ be an open base in x such that $U_{n+1} \subset U_n$ for all $n \in \omega$. Let $H = \{x_n : x_n \in \overline{U_n} \cap A, n \in \omega\}$. Note that x_n was chosen inductively and because the space X is Hausdorff, we can also suppose at each step of the induction that $x_n \notin \overline{U_{n+1}}$, it follows that $x_n \in \overline{U_m}$ if and only if $m \leq n$.

The set H is closed. Indeed, if $y \notin \overline{H^\circ}$, then $X \setminus (\overline{U_n} \cap A)$ is an open set containing y and not intersecting $\overline{H^\circ}$ which implies that $X \setminus (\overline{U_n} \cap A)$ is a neighborhood open set containing y and not intersecting H. Therefore $y \notin H$. Note that $x \notin H$. Since x and H can not be separated, so X is not an α-regular space. ■

***Proof of theorem* $(\mathbf{1.4})$**: It is straightforward by proposition 1.5 and theorem 1.3.■

Theorem 1.6. If (X, τ) is an epi α-regular space, and the witnesses of epi α -regularity $(X, \tau^{'})$ is first countable, then (X, τ) is completely Hausdorff.

Proof. Let (X, τ) be any epi α-regular space, and let x, y be any distinct points in X, then one can find a coarser topology $\tau^{'}$ on X such that $(X, \tau^{'})$ is T_1, α -regular, and then $(X, \tau^{'})$ is Hausdorff [10]. It follows that there exist two disjoint open sets $G, H \in \tau^{'}$ such that $x \in G, y \in H$,. Now since $(X, \tau^{'})$ is first countable then by proposition 1.5 $(X, \tau^{'})$ is regular, so there exist $U, V \in \tau^{'}$ such that $x \in U \subseteq \overline{U}^{\tau^{'}} \subseteq G$ and $y \in V \subseteq \overline{V}^{\tau^{'}} \subseteq H$, where $\overline{U}^{\tau^{'}} = \{x \in X: W \cap U \neq \emptyset, \forall$ open W in $\tau^{'}, x \in W\}$ similarly $\overline{V}^{\tau^{'}}$. Since $\bar{A}^{\tau} \subseteq \bar{A}^{\tau^{'}}$, for any $A \subseteq X$, this implies That $\overline{U}^{\tau} \subseteq \overline{U}^{\tau^{'}}$. As $\overline{U}^{\tau^{'}} \cap \overline{V}^{\tau^{'}} = 0$. Thus (X, τ) is completely Hansdorff. ■

Thus any space (X, τ) which is not completely Hausdorff, such that any coarser topology of it is T_2 first countable, cannot be epi α -regular.

Since any β -normal or α -normal [26] satisfying T_1 axiom is α-regular [30], [10], then we end to the following theorem

***Theorem* 1.7.** Every epi β -normal (epi α -normal) space is epi α -regular. ■

As every second countable T_3 space is metrizable, [[8],4.2.9], and since every second countable is first countable then by proposition 1.5 we have the following corollary.

***Corollary* 1.8.** If (X, τ) is epi α -regular and the witness of epi α-regularity $(X, \tau^{'})$ is second countable, then (X, τ) is submetrizable. ■

Note that corollary 1.8 is not correct in general. For example, the Tychonoft Plank $\left((\omega_1 + 1) \times (\omega_0 + 1)\right) \setminus \{\langle \omega_1, \omega_0 \rangle\}$ is Tychonoff being Hausdorff locally compact, and hence it is epi α -regular, but it is not submetrizable, because if it was, then $(\omega_1 + 1) \times \{0\} \subseteq \left((\omega_1 + 1) \times (\omega_0 + 1)\right) \setminus \{\langle \omega_1, \omega_0 \rangle\}$ is submetrizable, because submetrizablity is hereditary, but $(\omega_1 + 1) \times \{0\} \cong \omega_1 + 1$ and $\omega_1 + 1$ is not submetrizable.

It is well known that T_2 paracompact space is T_4, then we have the following result proved.

***Corollary* 1.9.** If (X, τ) is epi α-regular and the witness of epi α -regularity $(X, \tau^{'})$ is paracompact, then (X, τ) is T_4. ■

Also, we remind that any T_2 compact space is T_4, and we conclude.

***Corollary* 1.10**. Any epi α-regular compact space is T_4. ■

A Hausdorff space X is said to be H-*closed* if X is a closed subspace of every Hausdorff space in which it is contained [[8],3.12.5]. Since a regular space is H -closed if and only if it is compact [[8],3.12.5]. Then we can prove a similar argument for epi α -regularity.

***Corollary* 1.11**. If (X, τ) is epi α -regular compact space, then the witness of epi α -regularity $(X, \tau^{'})$ is H-closed. ■

***Theorem* 1.12**. If (X, τ) is an epi α -regular space, then for every compact subset F of X and every $x \in X$ such that $x \notin F$, there exist disjoint open sets U, W such that $\overline{F \cap U} = F$ and $x \in W$.

Proof. Let (X, τ) be an epi α -regular space, then a coarser topology $\tau^{'}$ on X exists such that $(X, \tau^{'})$ is α - regular, T_1. Let F be any compact set in (X, τ) and let $x \notin F$, hence F is closed in $(X, \tau^{'})$ and $x \notin F$, by α-regularity of $(X, \tau^{'})$, there exist $U, W \in \tau^{'}$ such that $F \cap U = F, x \in W$ and $U \cap W = \emptyset$. ■

***Corollary* 1.13**. If F and E disjoint compact sets in an epi α-regular space X, then there exist disjoint open sets U and W such that $\overline{F \cap U} = F, \overline{E \cap W} = E$.

Proof. Let (X, τ) be an epi α -regular space, then there exists a coarser topology $\tau^{'}$ on X such that $(X, \tau^{'})$ is α - regular, T_1. Let F, E be any disjoint compact subsets of (X, τ), hence they are disjoint compact subsets of $(X, \tau^{'})$ and by theorem 1.12 for each $a \in F$ and compact set E, there exist open sets U_a, W_a such that $a \in U_a, \overline{E \cap W_a} = E$ and $U_a \cap W_a = \emptyset$. Now consider F is an arbitrary compact set disjoint from E. For each a in F, by theorem 1.12 gives disjoint open sets U_{a_i} containing a and $\overline{E \cap W_{a_i}} = E$ and $U_{a_i} \cap W_{a_i} = 0$. The family $\{U_{a_i}: i \in I\}$ is an open cover of F, since F is compact, there is a finite subfamily $\{U_{a_1}, \dots, U_{a_n}\}$ which covers F and the corresponding $\{\overline{W_{a_1}}, \dots, \overline{W_{a_n}}\}$ is a closed cover of E. So that $U = \cup_{i=1}^{n} U_{a_i}$ is an open set containing F and disjoint from $W = \cap_{i=1}^{n} W_{a_i}$ which is an open set. such that $\overline{F \cap U} = F, \overline{E \cap W} = E$. Indeed, it is obvious that $\overline{E \cap W} \subseteq E$. On the other hand, let $x \in E$, and G is an open set containing x, we need to show $G \cap E \cap W \neq 0$. Let $G \cap E \cap W = 0$, then there is $1 \leq j \leq n$ such that $G \cap E \cap W_{a_j} = 0$, since G is open then $x \notin \overline{E \cap W_{a_j}}$ which is a contradiction. Therefore $G \cap E \cap W \neq 0$ which implies that $x \in \overline{E \cap W}$. Hence $\overline{E \cap W} = E$, and we are done.■

III. Properties of Epi α-Regularity

***Theorem* 2.1**. [10] Let X be an α-regular space, $f: X \to Y$ is an onto, continuous, open, and closed function. Then Y is α-regular.

Proof. Let X be an α-regular space, A be a closed subset of Y and $y \in Y$ such that $y \notin A$. Then $f^{-1}(A)$ is a closed subset of X and there exists $x \in X$ such that $f(x) = y$ and $x \notin f^{-1}(A)$. Since X is an α-regular space, there exist disjoint open subsets G and H of X such that $x \in H$ and $\overline{f^{-1}(A) \cap G} = f^{-1}(A)$, and so $x \notin \bar{G}$. Since $x \notin \bar{G}$, then $y \notin f(\bar{G})$. It is clear that $f(\bar{G})$ is a closed set containing the open set $f(G), \overline{f(G)} \subseteq f(\bar{G})$. Thus $y \notin \overline{f(G)}$ which implies $y \in f(H)$ and $f(G) \cap f(H) = 0$. Now it is sufficient to show that $\overline{A \cap f(G)} = A$. Let $z \in A$ and W is an open set containing z, then $f^{-1}(z) \subseteq f^{-1}(A) \cap f^{-1}(W)$. Since $\overline{f^{-1}(A) \cap G} = f^{-1}(A), f^{-1}(A) \cap G \cap f^{-1}(W) \neq \emptyset$. Hence by surjectivity of f, $A \cap f(G) \cap W = f(f^{-1}(A)) \cap f(G) \cap f(f^{-1}(W)) \supseteq f(f^{-1}(A) \cap G \cap f^{-1}(W)) \neq 0$ as required.■

***Corollary* 2.2**. Let (X, τ) be an epi α -regular space, $f: (X, \tau) \to (Y, \mathcal{S})$ is an onto, continuous, open, and closed function. Then Y is epi α -reqular.

Proof. Let (X, τ) be any epi α -regular space, let τ' be a coarser topology on X such that (X, τ') is α -regular, T_1. Since $f: X \to Y$ is an onto, continuous, open, and closed function then by theorem 2.13 $(Y, \mathcal{S}')$, where $\mathcal{S}' = \{f\{U\}: U \in \tau'\}$, is α -regular, and it is obviously T_1. Hence $(Y, \mathcal{S})$ is epi α -regular.■

***Corollary* 2.3**. Epi α -regularity is a topological property.■

The proof of the following theorem is due to Murtinová.

***Theorem* 2.4**. [30], [10] Every subspace of an α -regular space is α -regular.■

Proof. Let X be an α -regular space and A is a subspace $X, y \in A$ and $y \notin F \subset A, \bar{F} \cap A = F$ where $\bar{F}$ refers to the closure of F in X. Then $y \notin \bar{F}$ and X is α -regular, hence there are disjoint open sets U, V in X such that $y \in U$ and $\overline{F \cap V} = F$. The sets $U \cap A$ and $V \cap A$ are the sets witnessing α -regularity of A. Indeed, they are disjoint, open in A, $y \in U \cap A$. It remains to show that $F \cap V(\cap A)$ is dense in F in the space A. The A-closure of $F \cap V$ is $\overline{F \cap V} \cap A \subset \bar{F} \cap A = F$. On the other hand, let $x \in F$, W is an open subset in $X, x \in W$. We have to prove that $W \cap F \cap V \neq \emptyset$. Suppose for contradiction that $W \cap F \cap V = \emptyset$. Since $W \cap V$ is open, $W \cap \bar{F} \cap V = \emptyset$ as well. And since W is open, $\emptyset = W \cap \overline{\bar{F} \cap V} = W \cap \bar{F}$. But $x \in W \cap \bar{F}$ which is a contradiction.■

***Corollary* 2.5**. Epi α-regularity is a hereditary property.

Proof. Let (X, τ) be an epi α -regular space and let (A, τ_A) be a subspace of (X, τ). Let τ' be a coarser topology on X such that (X, τ') is α -regular, T_1. The subspace (A, τ_A') is α -regular, T_1 as α -regular [30], [10], T_1 is hereditary 2.5, and $\tau_A' \subseteq \tau_A$, therefore (A, τ_A) is epi α -regular.■

***Theorem* 2.6**. α-regularity are additive properties.

Proof. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of α -regular spaces, and A be a closed subset of the sum $\bigoplus_{\alpha \in \Lambda} X_\alpha, x \in \bigoplus_{\alpha \in \Lambda} X_\alpha$ such that $x \notin A$. By proposition 2.2.1 in[8] the intersections $A \cap X_\alpha$ is closed in X_α for every $\alpha \in \Lambda$ and $x \notin A \cap X_\alpha$ From α -regularity of X_α it follows that there are two open sets U_α and V_α in X_α and such that

$$\overline{A \cap X_\alpha \cap U_\alpha} = A \cap X_\alpha, x \in V_\alpha$$

and

$$\overline{U \cap A} = \emptyset$$

Let $U = \cup_{\alpha \in \Lambda} U_\alpha$ and $V = \cup_{\alpha \in \Lambda} V_\alpha$, then clearly

$$\overline{A \cap U} = \overline{U_{\alpha \in \Lambda}(A \cup U_\alpha)} = \cup_{\alpha \in \Lambda} \bar{A} = \bar{A} = A, x \in V$$

$$U \cap V = \cup_{\alpha \in \Lambda} U_\alpha \cap \cup_{\alpha \in \Lambda} V_\alpha = \cup_{\alpha \in \Lambda} (U_\alpha \cap V_\alpha) = 0$$

Since U and V are open in $\bigoplus_{\alpha \in \Lambda} X_{\dot{\alpha}}$, the sum $\bigoplus_{\alpha \in \Lambda} X_\alpha$ is α -regular.■

***Theorem* 2.7**. Epi α-regularity is an additive property.

Proof. Let (X_α, τ_α) be an epi α -regular space for each $\alpha \in \Lambda$ For each $\alpha \in \Lambda$, let τ_α' be a topology on X_α coarser than τ_α such that (X_α, τ_α') is α -regular, T_1. since T_1 is additive see [[8],2.2.7] and α -regularity is also additive by theorem 2.6. Then $\bigoplus_{\alpha \in \Lambda} (X_\alpha, \tau_\alpha')$ is α -regular, T_1, and its topology is coarser than the topology on $\bigoplus_{\alpha \in A} (X_a, \tau_\alpha)$.■

***Theorem* 2.8**. Let $\{(X_\alpha, \tau_\alpha): \alpha \in \Lambda\}$ be a family of epi-regular spaces, and let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Then (X, τ) is epi α -regular, where τ is the Tychonoff product topology, if only if (X_α, τ_α) is epi α -regular for each $\alpha \in \Lambda$.

***Proof*.** Let (X, τ) be an epi α -regular space, and let $\beta \in \Lambda$, by Theorem 2.5, every subspace of (X, τ) is epi α -regular. By [[31],2.39], there is a subspace of (X, τ) that is homeomorphic to X_β. since epi α-regularity is a topological property then (X_β, τ_β) is epi α -regular.

Now let (X_α, τ_α) be epiregular, epi α -regular space for each $\alpha \in \Lambda$. For each $\alpha \in \Lambda$, let τ_α' be a topology on X_α, coarser than τ_α such that (X_α, τ_α') is T_3. since T_3 is multiplicative [[8],2.3.11]. Then $\prod_{\alpha \in \Lambda} X_\alpha$ is T_3 with respect of the product topology of τ_α''s, which implies that $\prod_{\alpha \in \Lambda} X_\alpha$ is α -regular T_1 with respect of the product topology of τ_α''s and its topology is coarser than the topology on $\prod_{\alpha \in \Lambda} (X_\alpha, \tau_\alpha)$.■

Let $\mathbb{R}$ be the real line. Let $\mathbb{P}$ be the set of all irrational numbers and $\mathbb{Q}$ be the rational numbers. Let U be the usual topology of the real line $\mathbb{R}$. The real line with the topology generated by $\mathcal{B} = \{(x-\varepsilon, x+\varepsilon): x \in \mathbb{Q}\} \cup \{\{x\}: x \in \mathbb{P}\}$ is called the *Michael line* and is denoted by M. And $M \times P$, where $\mathbb{P}$ has the usual topology, is called the *Michael product* [8]. As the Michael line is α -regular, T_1, hence we have the following corollary.

***Corollary* 2.9**. The Michael line is epi α -regular space.■

The space $\mathbb{M} \times \mathbb{P}$ is α -regular, T_1 space being product of two (regular) α regular, T_1 spaces, so we have the following theorem.

***Theorem* 2.10**. The Michael product is an epi α -regular space. ■

Note that epiregularity is invariant under products, however, this is not the case for α -regularity as Murtinová in [30] proved that α -regularity is not preserved under products. Regarding Murtinová result in [30], the following theorem proves that epi α -regularity is not preserved by products and at the same time we construct a non epi α -regular space from a non epiregular space.

***Theorem* 2.11**. Let $A(\kappa)$ is the one-point compactification of a discrete set of cardinality κ. Then for every non-epiregular T_1 space X there is $\kappa \leq \chi(X)$ such that $X \times A(\kappa)$ is not epi α -regular.

Proof. By a similar argument used in theorem [7] in [30].■

It follows that product of an epi α -regular space and a compact zero dimensional space may fail to be epi α-regular. In particular it means that epi α-regularity is not preserved by products.

There are many ways of producing a new topological space from an old one. In 1929 , Alexandroff introduced his method by constructing the Double Circumference Space [1]. In 1968, R. Engelking generalized this construction to an arbitrary space as follows: Let X be any topological space. Let $X' = X \times \{1\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $(x,1)$ in X' by x' and for a subset $B \subseteq X$ let $B' = \{x': x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$ For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}): U$ is open in X with $x \in U\}$. Let τ denote the unique topology on $A(X)$ which has $\{\mathcal{B}(x): x \in X\} \cup \{\mathcal{B}(x'): x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate* of X [9]. The following is easy to prove.

***Lemma* 2.12**. If X is T_1 then its Alexandroff Duplicate $A(X)$ is also T_1.■

***Theorem* 2.13**. [10] If X is α-regular satisfying T_1 axiom, then its Alexandroff Duplicate $A(X)$ is also α-regular.

Proof. Let E be aclosed set in $A(X)$ and $x \in A(X)$ such that $x \notin E$. Write $E = E_1 \cup E_2$, where $E_1 = E \cap X, E_2 = E \cap X'$. So $x \notin E_1$ in X and $x' = (x,1) \notin E_2$. By α -regularity of X, there exist two disjoint open sets U and V of X such that $E_1 \cap U$ is dense in E_1 and $x \in V$. Since X is T_1 we can choose $W_1 = (U \cup U' \cup E_2) \setminus \{x\}$ and $W_2 = (V \cup V' \cup \{x'\}) \setminus E$. Then W_1 and W_2 are disjoint open sets in $A(X)$, and $x' \in W_2$. Now, we prove $W_1 \cap E$ is dense in E. Note that $W_1 \cap E = (W_1 \cap E_1) \cup (W_1 \cap E_2) = (U \cap E_1) \cup E_2$, so $\overline{(W_1 \cap E)} = \overline{(U \cap E_1) \cup E_2} = (\overline{U \cap E_1}) \cup (\overline{E_2}) \supset E_1 \cup \overline{E_2} \supset E$. Therefore, $W_1 \cap E$ is dense in E. Then $A(X)$ is α-regular. Hence α-regular is preserved by the Alexandroff Duplicate space.■

***Theorem* 2.14**. If $(X, \mathcal{K})$ is epi α-regular, then so is its Alexandroff Duplicate $(A(X), \tau)$.

Proof. Let $(X, \mathcal{K})$ be an epi α -regualr space, then a coarser topology $\mathcal{K}'$ on X exists such that $(X, \mathcal{K}')$ is T_1, α -regular. Let $(A(X), \tau')$ be the Alexandroff Duplicate of $(X, \mathcal{K}')$. Since by theorem 2.13 α-regularity is preserved by the Alexandroff Duplicate space and also T_1, then $(A(X), \tau')$ is also T_1, α -regular, and it is obviously coarser than $(A(X), \tau)$ by the topology of the Alexandroff Duplicate. Hence, $A(X)$ is epi α -regualr.■

In 1951, Bing [5] and Hanner [14] introduced a new topological space by generating it from an old topological space. This new space is called *discrete extension*.

***Definition* 2.15**. Let M be a non-empty proper subset of a topological space (X, τ). Define a new topology $\tau_M = \{U \cup K: U \in \tau$ and $K \subseteq X \setminus M\}$. The space (X, τ_M) is called *discrete extension*, and donated by X_M, see [8], [21]. In [21], properties such as countable tightness, Fréchet, and weaker types of normality were investigated for discrete extension. Here we study the relationship between a space X and a discrete extension X_M of X according to epi α -regularity.

For any epi α -regular space (X, τ) we have $\tau' \subseteq \tau \subseteq \tau_M$ where τ' is Tychonoff, so we have the following proved.

***Theorem* 2.16.** If (X, τ) is an epi α -regular space, then also is X_M.■

Since any Hausdorff locally compact is Tychonoff and hence epi α -regular, then by theorem 2.16 the following is easy to prove

***Corollary* 2.17**. If (X, τ) is a Hausdorff locally compact space, then X_M is epi α -regular.■

IV. Epi α -Regularity And Some Other Separation Axioms

A topological space (X, τ) is called *submetrizable* if there exists a metric d on X such that the topology τ_d on X generated by d is coarser than τ, i.e., $\tau_d \subseteq \tau$, see [13], since, by definitions, any submetrizable space is epi α-regular. The converse of the last statement is not true in general. For example, $\omega_1 + 1$ is epi α-regular being T_2 compact, hence α-regular T_1 and therefore epi α-regular. But it is not submetrizable, because if $\omega_1 + 1$

was submetrizable, then there would be a metric d on $\omega_1 + 1$ such that the topology τ_d on $\omega_1 + 1$ generated by d is coarser than the usual ordered topology. This means that $(\omega_1 + 1, \tau_d)$ is perfectly normal. So, the closed set $\{\omega_1\}$ is a G_δ −set in $(\omega_1 + 1, \tau_d)$. i.e., $\{\omega_1\} = \cap_{n\in N} U_n$, where $U_n \in \tau_d$, hence U_n is open in the usual ordered topology on $\omega_1 + 1$, which is a contradiction.

Obviously, any α-regular, T_1 space is epi α-regular, just by taking $\tau' = \tau$. However epi α -regularity and α -regularity do not imply each other. For example, the Half-Disc space [33] is epi α-regular which is not α-regular by proposition 1.5, since the space is Hausdorrf first countable not regular. Similarly, Deleted Diameter topology [32] is epi α -regular being submetrizable, but it is not α -regular. Any indiscrete space which has more than one element is an example of an α -regular space which is not epi α -regular.

Semiregularization topologies were studied in [27], a *Semiregular* space is T_2 space in which the regular open sets form a basis for the topology [33]. Epi α -regularity and semiregularity are independent, for example the Half-Disc space [33], is epi α-regular but not semiregular, It is epi α -regular because it is submetrizable. and any indiscrete space which has more than one element is an example of a semiregular space which is not epi α-regular.

Recall that a topological space (X, τ) is called *extremally disconnected* if it is T_1 and the closure of any open set is open [18]. Since every α -regular, extremely disconnected space is regular [10], then we have the following correct.

***Corollary* 3.1**. If X is an epi α-regular space and the attested of epi α-regularity is extremely disconnected, then X is epiregular.■

Recall that a topological space (X, τ) is called *Zero-dimensional* if it is a non-empty T_1 space and has a base consisting of open-and-closed sets [8].

Clearly, every zero-dimensional space is Tychonoff space, and hence T_3, so we conclude.

***Corollary* 3.2**. Any zero-dimensional space is epi α-regular.■

The converse of the above result is not always correct. For example, The Euclidean topology on the set of real numbers is epi α -regular since it is T_3 but not zero dimensional. The following example [22] is a modified example of *Mysior's example* from [28].

***Example* 3.3**. Let $A \subseteq \mathbb{R}$ be such that the intersection $A_k = A \cap [k, k + 1)$ is uncountable for every integer $k \in \mathbb{Z}$. Let $\Delta = \{(a, a): a \in A\}$ be the diagonal of $X = A^2$ and define the following sets

$$U_k = \{(a, b) \in X: a > k\}$$

for $k \in \mathbb{Z}$

$$\Gamma_a = \{(a + \varepsilon, a) \in X: \varepsilon \in [0,3]\} \cup \{(a, a - \varepsilon) \in X: \varepsilon \in [0,3]\}$$

for $a \in$ A. Consider a topology τ on $X = A^2$ generated by a basis consisting of all singletons $\{x\}$ with $x \in X \setminus \Delta$ and all sets $\Gamma_a \setminus F$, where $a \in A$ and F is finite. Clearly X is Hausdorff and zero-dimensional, and so is epi α-regular.

The following example is constructed by Murtinová in [28] as she showed that it is an example of an α -normal Hausdorff, hence α -regular, non regular space.

***Example* 3.4.**[29] Let $X = \omega_1 + 1$ and define a topology τ such that: ω_1 with the ordinal topology is an open subspace and a base in the point ω_1 will be the collection:

$$U_C = \{\omega_1\} \cup \{\alpha + 1: \alpha \in C\}$$

where C is a closed unbounded subset of ω_1 (Club).

The topology τ is Hausdorff since it is stronger than the onder topology on $\omega_1 + 1$. This space is epi α -regular since it is α -regular Hausdorff and it is epinegular since it is stronger that the order topology on $\omega_1 + 1$ but it is not regular nor first countable.

Note that *the right order topology* defined on the set of real numbers $\mathbb{R}$[33] is an example of β -normal, α-normal since there are no disjoint closed sets on it and it is not epi α-regular since it is not Hansdorff.

Recall that a topological space (X, τ) is called *epicompletely regular* [12] if there is a coarser topology τ' on X such that (X, τ') is Tychonoff. Note that if a topological space X is epicompletely regular, then the space is epi α-regular. But the converse of the above statement is not always true. however, the following theorem is correct since epiregularity implies epicompletely regularity.

***Corollary* 3.5**. If (X, τ) is an epi α-regular space, and the witnesses of epi α-regularity (X, τ') is first countable, then (X, τ) is epicompletely regular.■

It is well known that every compact second countable topological space satisfying T_2 axiom is metrizable, [[8],4.2.8] and this induces another result.

***Corollary* 3.6**. If a topological space X is epi α-regular, compact, and the attested of epi α-regular is second countable then the space is submetrizable.■

Remind that a topological space (X, τ) is *C_2-paracompact* if there is a T_2, paracompact space (Y, δ) and a bijective map $f: (X, \tau) \longrightarrow (Y, \delta)$ such that the restriction $f_{|A}: A \longrightarrow f(A)$ is a homeomorphism for every

compact subspace $A \subseteq X$. For more details see [15]. A space X is called *Fréchet* if for every $A \subseteq X$ and every $x \in \bar{A}$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of A
such that $x_n \to x$, see [8].

***Theorem* 3.7.** Let (X,τ) be a C_2 -paracompact and Fréchet, then (X,τ) is epi α-regular.

Proof. Let (X,τ) be a C_2 -paracompact and Fréchet, then (X,τ) is epinormal by theorem 2.16 in [15], then it is epi α -normal. Hence (X,τ) is epi α regular.■

***Theorem* 3.8**. If (X,τ) is Lindelöff epi α-regular space and the attasted of epi α-regularity $(X,\tau^{'})$ is first countable, then (X,τ) is C_2 -paracompact.

Proof. Let (X,τ) be a Lindelöff epi α-regular space, then there exists a coarser topological space $(X,\tau^{'})$ that is T_1, α-regular first countable, and hence is regular by propostion 1.5. since (X,τ) is a Lindelöff space, then $(X,\tau^{'})$ is also Lindelöff and regular which implies that $(X,\tau^{'})$ is T_2 and paracompact, and therefore the identity map id : $(X,\tau) \longrightarrow (X,\tau^{'})$ is the required map to have our space (X,τ) to be C_2 -paracompact.■

Since any regular Lindelöff space is normal, then this is not hard to show

***Corollary* 3.9**. Let (X,τ) be an epi α-regular Lindeloff space, and the attested of epi α-regularity is first countable, then (X,τ) is epinormal.■

Remind that a topological space (X,τ) is called *nearly compact* [23] if every open cover of X has a finite subfamily the interiors of the closures of whose members covers X.

***Theorem* 3.10**. If (X,τ) is a Hausdorff nearly compact space, then (X,τ) is epi α-regular.

Proof. Let (X,τ) be a Hausdorff nearly compact space, and let τ_A be the semi regularization of τ, then τ_s is a Hausdorff nearly compact space. Therefore τ_A is T_4, and hence T_2, α-regular. Therefore (X,τ) is epi α-regular.■

Remind that a topological space (X,τ) is called *partially normal* [18] if for any two disjoint subsets A and B of X, where A is regularly closed and B is π-closed, there exist two disjoint open subsets U and V of X containing A and B respectively.

***Theorem* 3.11**. If (X,τ) is a semi regular partial normal space and τ_s is T_1, then (X,τ) is epi α-regular.

Proof. It is enough to show that (X,τ_s) is α-regular. Let U be any open set containing x in (X,τ_s). By semiregularity, there is an open set W such that $x \subseteq W \subseteq \text{int}(\overline{W}) \subseteq U$. Since int $(\overline{W})$ is regularly open and using the same idea of theorem 2.11 in [2] there exists an open set V in (X,τ_s) such that $x \subseteq V \subseteq \bar{V} \subseteq \text{int}(\overline{W}) \subseteq U$. Therefore $\overline{A \cap V} \subseteq \bar{V} \subseteq \text{int}(\bar{U}) \subseteq B$. Hence (X,τ_s) is α-regular, and then (X,τ) is epi α-regular.■

Epi α-regularity and α-normality do not imply each other. For example, the Dieudonné topology and The deleted Tychonoff Plank, see [26] and [33], are not normal space nor α -normal, but they are epi α-regular because they are zero dimensional.

Remind that a space (X,τ) is called *almost α-normal* [11] if for any two disjoint closed subsets A and B of X one of which is regularly closed there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B. That is, $\overline{A \cap U} = A$ and $\overline{B \cap V} = B$. and a space (X,τ) is called *almost β-normal* [11] if for any two disjoint closed subsets A and B of X one of which is regularly closed there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B. That is, $\overline{A \cap U} = A$, $\overline{B \cap V} = B$ and $\bar{U} \cap \bar{V} = \emptyset$.

Note that almost α-normality (almost β-normality) and epi α-regularity are not related to each other. For example, $\mathbb{R}$ with the particular point topology τ_p, see [8], [33], where the particular point is $p \in \mathbb{R}$, is not normal nor β-normal nor α-normal. But the space is almost β-normal and almost α-normal since the only regularly closed sets are $\mathbb{R}$ and $\emptyset$. However this space is not Hausdorff and then it is epi α-regular. Conversely, Any indiscrete space which has more than one element is an example of an almost α-normal (almost β-normal) space which is not epi α-regular. However, every almost α normal extremely disconnected space is epi α-regular.

A β-normal epi α -regular non normal space example found in [29] which as follows:

***Example* 3.12**. Let $S = \{\alpha < \omega_2 : cf(\alpha) = \omega_1\}$, and consider the set $X = \{(\alpha,\beta) : \beta \leq \alpha \leq \omega_2, (\alpha,\beta) \neq \omega 2, \omega 2$ and its partition into

$$A = \{(\alpha,\alpha) : \alpha < \omega_2\}$$
$$B = \{(\omega_2,\beta) : \beta < \omega_2\}$$
$$D = \{(\alpha,\beta) : \beta < \alpha < \omega_2\}$$

Topologize X as follow: Let each $(\alpha,\beta) \in D$ be isolated, and let an open base in $(\alpha,\alpha) \in A$ consists of all sets of type

$$\{(\gamma,\gamma) : \alpha_0 < \gamma \leq \alpha\} \cup \bigcup\{\{\gamma\} \times C_\gamma : \alpha_0 < \gamma \leq \alpha, \gamma \in S\}$$

where $\alpha_0 < \alpha$ and every C_γ is a closed and unbounded (club) subset of γ, and let an open base in $(\omega_2,\beta) \in B$ consists of all sets

$$\{(\alpha,\gamma) : \beta_0 < \gamma \leq \beta, \alpha_\gamma < \gamma \leq \omega_2\}$$

where $\beta_0 < \beta, \beta \leq \alpha_\gamma < \omega_2$.
All above defined basic open neighborhoods are closed. That is, X is zero dimensional hence it is epi α -regular. Murtinová in [29] proved that this space is β -normal non normal.

Remind that a toplogical space (X, τ) is called *epi-mildly normal* [17] if there exists a coarser topology τ' on X such that (X, τ') is Hausdorff, mildly normal.

The following theorem induced by theorem 2.4 in [26] shows a relationship between epi-mildly normality, β -normal and epi α -regular.

***Theorem* 3.13**. If a topological space is epi-mildly normal and the witness of epi-mildly normality is β-normal then (X, τ) is epi α-regular.

Proof. Let (X, τ) be a topological epi-mildly normal space and the attested of epi-mildly normal epinormal is β -normal, then there is a coarser topological space (X, τ') that is Hausdorff, mildly normal and β-normal, so by theorem (2.4) in [26] then (X, τ') is Hausdorrf and normal, and therefore it is Hansdorff α -normal, and so (X, τ') is T_1, α -regular [10]. Hence (X, τ) is epi α -regular.■

Epi α -regularity does not imply mildly normality.

***Example* 3.14**. [3] Let $\mathbb{P}$ denote the irrationals and $\mathbb{Q}$ denote the rationals. For each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $p_n = \left(p, \frac{1}{n}\right) \in \mathbb{R}^2$. For each $p \in \mathbb{P}$, choose a sequence $(p_n^*)_{n \in \mathrm{N}}$ of rationals such that $p_n' = (p_n^*, 0) \to (p, 0)$ where the convergence is taken in $\mathbb{R}^2$ with its usual topology $\mathcal{U}$. For each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $A_n((p,0)) = \{p_k' : k \geq n\}$ and $B_n((p,0)) = \{p_k : k \geq n\}$. Now,for each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $U_n((p,0)) = \{(p,0)\} \cup A_n((p,0)) \cup B_n((p,0))$. Let $X = \{(x;0) \in \mathbb{R}^2 : x \in \mathbb{R}\} \cup \{p_n = \left(p, \frac{1}{n}\right) \in \mathbb{R}^2 : p \in \mathbb{P}$ and $n \in \mathbb{N}\}$. For each $q \in \mathbb{Q}$, let $B((q,0)) = \{\{(q,0)\}\}$. For each $p \in \mathbb{P}$, let $B((p,0)) = \{U_n((p,0)) : n \in \mathbb{N}\}$. For each $p \in \mathbb{P}$ and each $n \in \mathbb{N}$, let $B(p_n) = \{\{p_n\}\}$. Denote by τ the unique topology on X that has $\{B((x,0)), B(p_n) : x \in \mathbb{R}, p \in$ Pandn $\in \mathbb{N}\}$ as its neighbomood system. Let $Z = \{(x,0) : x \in \mathbb{R}\}$. That is, Z is the x -axis. Then $(Z, \tau) \cong (\mathbb{R}, \mathcal{RS})$, where $\mathcal{RS}$ is the Rational Sequence Topology, see [33]. Since Z is closed in X and $(\mathbb{R}, \mathcal{RS})$ is not normal, then X is not normal, but, since any basic open set is closed-and-open and X is T_1, then X is zero-dimensional, hence epi α-regular. Now, Let $A \subseteq \mathbb{P}$ and $B \subseteq \mathbb{P}$ be closed disjoint subsets that cannot be separated in $(\mathbb{R}, \mathcal{RS})$. Let $G = \cup \{B_1((p,0)) : p \in A\}$ and $H = \cup \{B_1((p,0)) : p \in B\}$. Then G and $H$are both open in (X, τ) and $\bar{G}$ and $\bar{H}$ are disjoint closed domains that cannot be separated, hence X is not mildly normal.

Epi α-regularity does not imply epinormality, and here is an example.

***Example* 3.15**. Let $G = D^{\omega_1}$ where $D = \{1,2\}$ with the discrete topology. Let H be the subspace of G consisting of all points of G with at most countabl many zero coondinates. Put $X = G \times H$. Raushan Buzyakova proved that X cannot be mapped onto a normal space Y be a bijective continuous function [7]. Using Buzyakova's and the fact that X is k -space [[8],3.3.27], then this implies that X is Tychonoff and so is epi α -regular and it cannot be C-normal see 3], and since epinormality implies C-normailty, then X cannot be epinormal.

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