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Effects of Perturbation on Complete Controllability of Control Systems

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ABSTRACT. *In this paper, the effect of perturbation on complete controllability of both linear and non-linear control systems were analysed. It was seen that if a linear system is completely controllable, then the perturbed system is also completely controllable provided that the sum of norm differences of the concerned matrices is as small as possible. For non-linear systems, it was seen that if the system is completely controllable,and the linearization of the perturbing function is stable at the origin, then the perturbed control system is also completely controllable.*

Key words and phrases. *Controllability, Complete Controllability, Linearization, Perturbation.*

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I. Introduction

In this paper, the effect of perturbation on complete controllability of both linear and non-linear controllable systems were analysed. We are interested in the equations of the form

$$\dot{x} = Ax(t) + Bu \tag{1.1}$$

where A and B are respectively $n \times n$ and $n \times m$ constant matrices; and x and u are n–vector and m– vector respectively and

$$\dot{x} = f(x, t, u) \tag{1.2}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a mapping satisfying regularity conditions and the control u is taken from admissible subclass of the set of measurable mappings of $\mathbb{R} \times \mathbb{R}^n$ and $f(.)$ is non-linear.

We are considering the effect of small perturbation on the complete controllability of the systems (1.1) and (1.2). The benefits of these knowledge is enormous because for the stability of any dynamical system to maintain the taste of time, one must have the knowledge of its complete controllability when subjected to conditions different from ones under which the system was made. Let us now see the notations and definitions which we wand to use in this paper.

C^0 = the set of continuous functions.
C^1 = the set of functions whose first derivative is continuous.
$\mathbb{R}$ = the one dimensional Euclidean space.
L^p = set of functions f whose $\|f\| = (\int_\Omega |f|^p d\mu)^{\frac{1}{\mu}} < \infty$.
$\mathbb{R}^+$ = the set of all non negative real numbers.
$\mathbb{R}^n$ = the n– finite dimensional Euclidean space.
C^∞ = the class of continuous differentiable functions defined on $\mathbb{R}^+$.
$W = \mathbb{R}^+ \times D$ where $D \subset \mathbb{R}^m$ is open set containing the origin.

Definitions:

Definition 1.1[1] For linear system (1.1) if there exists an imput u which transforms the initial state $x(t_0) = x_0$ to the final state $x(t_1) = x_1$ in a finite time t_1, the state x_0 is said to be controllable.

Definition 1.2: Small perturbation [2].
Let f: $\mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping satisfying regularity conditions. Given a control function $f(.)$ and its perturbed control system $f(.) + g(.)$; by small perturbation we mean that

$$|f(.) + g(.) - g(.)| < \epsilon, \quad \epsilon > 0.$$

Definition 1.3: Complete controllability [5].
Consider the linear time - varying system defined by

$$\dot{x} = A(t)x(t) + B(t)u(t)$$
$$y = C(t)x(t)) \qquad (1.3)$$

where A is $n \times n$, B is $n \times m$ and $C$is $r \times n$ matrices is said to be completely controllable ($C.C$) if for any $t_0 \geq 0$ the initial state $x(t_0) = x_0$ and any given final state x_f ,there exist a finite time $t_1 > t_0$ and control $u(t)$, $t_0 \leq t \leq t_1$, such that $x(t_1) = x_f$.

Note:
In definition 1.3, the qualifying term "Completely" implies that the definition holds for all x_0 and x_f.

Definition 1.4: Linearization [4].
This is an approximation of non- linear equation by a linear one

II. Propositions and Theorems:

We need the following propositions and Theorems for the completion of this paper.

Theorem 2.1: The constant System.

$$\dot{x} = Ax + Bu \qquad (2.1)$$

or the pair $[A, B]$ is completely controllable if and only if the $n \times m$ controllable matrix

$$U = [B, AB, A^2B, ..., A^{n-1}B] \qquad (2.2)$$

has rank equal to n.

Proof: For the proof of this theorem, see [2] pp $86 - 87$.

Proposition 2.1:
If the rank $B = p$ in Theorem 2.1, then the condition in the theorem reduces to

$$rank[B, AB, A^2B, ..., A^{n-1}B] = n$$

Proof: For the proof of this proposition, see [2] p.87.

3. Linear and Non-Linear Perturbation:

Linear Perturbation.

Firstly, let us consider linear perturbation. Consider a linear control system

$$\dot{x} = Ax + Bu \qquad (3.1)$$

where A and B are of dimensions mentions stated earlier, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.
We want to study the effect of perturbation on the complete controllability of the linear control system (3.1). That is, if

$$\dot{y} = C(t)y + D(t)u + g(x, t) \qquad (3.2)$$

where g is some linear perturbation, and C and D are also of appropriate dimensions and $y \in \mathbb{R}^n$
Deuer (1971) pointed out that if (3.1) is completely controllable, then (3.2) is also completely controllable provided that the sum of the norm difference of $A(t)$ and $C(t)$, $B(t)$ and $D(t)$ is as small as possible.

Proof:
The proof of this fact can be seen in Deuer 1971.

Theorem 3.1.[3]
Suppose $A \in L$ and $B \in L^p$, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and assume that the system (3.1) is completely controllable in L^q, there exist $\epsilon > 0$ such that if

$$||A - C||_p + ||B - D||_p \leq \epsilon$$

then

$$y^{\cdot} = C(t)y + D(t)u \qquad (3.3)$$

is completely controllable in L^q.

Proof:
The proof of this theorem can be found in Deuer (1971).

Remark:
From the above theorem, we note that complete controllability of linear systems is stable under small perturbation of such systems.

Non-Linear Perturbation.
Let $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a mapping satisfying regularity conditions and consider the systems

$$x^{\cdot} = f(t, x, u) \qquad (3.4)$$

where the control is taken from some admissible subclass of the set of measurable mappings of R into $\mathbb{R}^n$ and f (.) is non-linear. The general problem which is imposed is as follow " if the control system (3.4) has certain controllability property, then under what condition does the perturbed control system

$$x^{\cdot} = f(t, x, u)t, x, u + g(t, x, u) \qquad (3.5)$$

for sufficiently small mapping g(.) also share this controllability property?"
The solution to the above problem will be of immense benefits.We now remark that it is important to obtain results which will guarantee the controllability under small perturbations of the control systems on account of the following reasons.

1. Any such result can be interpreted as giving information about the well posedness of the controllability property in question and such information has a big physical significance.

2. If the results are formulated in a sufficiently constructive manner, then one can produce a new example of control systems which has the controllability property through perturbations of the system already known to have that property.

Let us now consider the control systems of the form

$$x^{\cdot} = f(x, u) \qquad (3.6)$$

where f(.) is a non-linear C^1- function and also its perturbation

$$x^{\cdot} = f(x, u) + g(x, u) \qquad (3.7)$$

Given that the control system (3.6) is completely controllable, we want to find conditions under which the perturbed system (3.7) will also be completely controllable.This will take us to the next theorem.

Theorem 3.2.
Suppose that the control system (3.6) is completely controllable and the linearization of $g(x, u)$ about the origin is stable, given a control function $u = Fx(t)$, $t > 0$, then the perturbed control system (3.7) is completely controllable.

Proof:
If the system (3.6) is controllable, this implies that there exists a control function $u = Fx(t)$, $t > 0$ such that

$$x^{\cdot} = Ax + Bu = (A + BF)x \qquad (3.8)$$

is controllable, where

$$A = \frac{\partial f}{\partial x}(0,0), \ \ and \ \ B = \frac{\partial f}{\partial u}(0,0)$$

This implies that the given eigenvalues of $(A + BF)$ are negative or have negative real parts. The linearization of $g(x, u)$ is $Cx + Du$
where

$$C = \frac{\partial g}{\partial x}(0,0), \ \ and \ \ D = \frac{\partial g}{\partial u}(0,0)$$

Applying the control law $u = Fx(t)$ to (3.7), we get

$$\dot{x} = (A + BF)x + (C + DF)x$$
$$= [(A + C) + (B + D)F]x.$$

Since $(C + DF)$ is stable, the eigenvalues of $(A + C) + (B + D)F$ have negative real parts. Therefore, the perturbed control system (3.7) is completely controllable.

IV. Conclusion.

The conditions for complete controllability for both linear and non-linear controllable systems to be stable undersmall perturbation have been stated.

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