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Some considerations relating γ and log(π)

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Abstract
We consider two Equations of Matiyasevich concerning the constant γ and log(π) (binary system) and two other Equations of Keiper (with the primes).
We connect the two systems of Equations and analyse with them the first Li-Keiper coefficient and two special values of integrals over the Zeta function connecting outside and inside of the critical strip (on the Riemann Hypothesis),with a numerical experiment.
***Key words**: Constants gamma and log(π), Binary system, Primes numbers, Critical strip.*

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I. Introduction

In this paper we present a relation between two Equations of Matiyasevich in the binary system and two others by Keiper given with the primes.
Different works
We start with the Equations obtained by Matiyasevich in his work[1],i.e.

$$\gamma - log\left(\frac{4}{\pi}\right) = 2 \cdot \sum_{n=1}^{\infty} \mathrm{N}_0(n) \cdot \frac{1}{2 \cdot n \cdot (2 \cdot n + 1)} \tag{1}$$

$$\gamma + log\left(\frac{4}{\pi}\right) = 2 \cdot \sum_{n=1}^{\infty} \mathrm{N}_1(n) \cdot \frac{1}{2 \cdot n \cdot (2 \cdot n + 1)} \tag{2}$$

where $\mathrm{N}_0$and N_1respectively are the number of 0 and of 1 in the binary representation of the integer n. It is known that

$$\mathrm{N}_0(n) + \mathrm{N}_1(n) = \left\lfloor \frac{\log(2 \cdot n)}{\log(2)} \right\rfloor = \left\lfloor 1 + \frac{\log(n)}{\log(2)} \right\rfloor$$

(where the symbol ⌊⌋ represents the floor).
Here the computations are carried out using the binary system.
We now consider two Equations obtained by Keiper in his pioneering work [2],i.e.

$$\sum_{k=2}^{\infty} [\zeta(\mathrm{k}) - 1] \cdot \left(\frac{1}{k \cdot 2^k}\right) = \frac{1 - \gamma - log\left(\frac{4}{\pi}\right)}{2} \tag{3}$$

$$\sum_{k=2}^{\infty} [\zeta(\mathrm{k}) - 1] \cdot \left(\frac{1}{2^k}\right) = \log(2) - \frac{1}{2} \tag{4}$$

where ζ is the Riemann Zeta function. For particular studies see [3, 4].

Here the computations may be carried out using the primes (n>1, the Euler product),for example the Zeta function in the region of absolute convergence.

It is interesting to combine the four above Equations for the computations of some constants.
We first use the four Equations to express the right hand side of Eq.(1) and Eq.(2),using Eq.(3) and Eq.(4).

$$\mathbf{N_1} = \sum_{n=1}^{\infty} \mathrm{N_1(n)} \cdot \left[\frac{1}{2\cdot n\cdot(2\cdot n+1)}\right] = \frac{1}{2} - \sum_{k=2}^{\infty}[\zeta(\mathrm{k})-1]\cdot\left(\frac{1}{k\cdot 2^k}\right) \tag{5}$$

$$\mathbf{N_0} = \sum_{n=1}^{\infty} \mathrm{N_0(n)} \cdot \left[\frac{1}{2\cdot n\cdot(2\cdot n+1)}\right] = \gamma - \frac{1}{2} + \sum_{k=2}^{\infty}[\zeta(\mathrm{k})-1]\cdot\left(\frac{1}{k\cdot 2^k}\right) \tag{6}$$

and $\mathbf{N_1} + \mathbf{N_0} = \gamma$ (7)

$$\mathbf{N_1} - \mathbf{N_0} \coloneqq 1 - \gamma - 2\cdot\sum_{k=2}^{\infty}[\zeta(\mathrm{k})-1]\cdot\left(\frac{1}{k\cdot 2^k}\right) = log\left(\frac{4}{\pi}\right) \tag{8}$$

Computations of the right hand side of Eq.(5) and Eq.(6) give:

$\mathbf{N_1}$ = 0.409390070086...
$\mathbf{N_0}$ = 0.167825594815...

The independent calculation of functions of the type $\sum \mathrm{N_0(n)}\cdot\mathrm{f(n)}$or of the type $\sum\mathrm{N_1(n)}\cdot\mathrm{f(n)}$ is given below, in particular for $\mathrm{N_0}$ and $\mathrm{N_1}$the left hand side of Eq.(5) and Eq.(6) where

$$f(n) = \frac{1}{2\cdot n\cdot(2\cdot n+1)}$$

We have:
$\mathbf{N_1} + \mathbf{N_0}$ = 0.577215....= γ.
$\mathbf{N_1}$- $\mathbf{N_0}$= 0.241564475270..= log(4/ □).

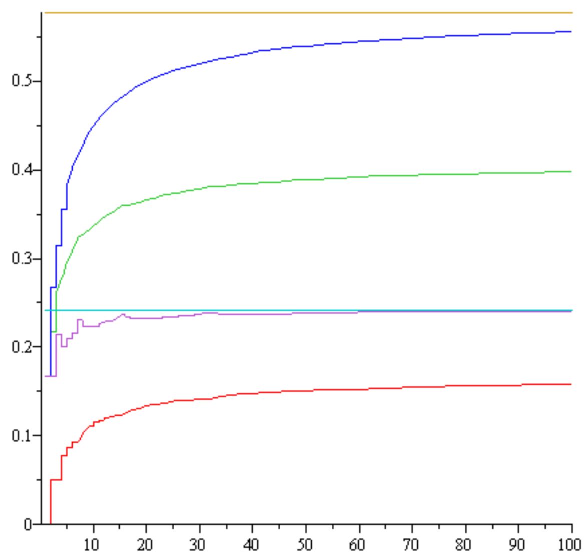

Fig. 1 In red $\mathbf{N_0}$, in green $\mathbf{N_1}$, in blue $\mathbf{N_1}$+$\mathbf{N_0}$, in maroon the limit γ; in pink $\mathbf{N_1}$-$\mathbf{N_0}$, in light hell the limit log(4/ π). ($\mathbf{N_1}$ and $\mathbf{N_0}$ are calculated with Eq.(1) and Eq.(2)).

Numerical experiment
We now present our numerical experiment.
1. First Li-Keiper coefficient λ_1.
It is known that λ_1 is independent of the truth of the RH [5,6].
RH is true if
$\lambda_1 = 1+ \gamma/2 - \log(4\cdot\pi)/2 = 0.0230957...$
exhausts the sum of the reciprocal values of all nontrivial zeros sitting on the critical line s=1/2.
We may express λ_1 using the above relations; we find

$$\lambda_1 = \mathbf{N_1} - 2\cdot(\log(2)-1/2) \tag{9}$$

Thus λ_1, from Eq.(9), is entirely given by a function on the binary system.
Moreover, with Eq.(5) and Eq.(4) we also have:

$$\lambda_1 = \frac{1}{2} - \sum_{k=2}^{\infty} [\zeta(k) - 1] \cdot \left(\frac{1}{2^k}\right) \cdot \left(\frac{1}{k} + 2\right) \tag{10}$$

Below the plot of the right hand side of Eq.(10).

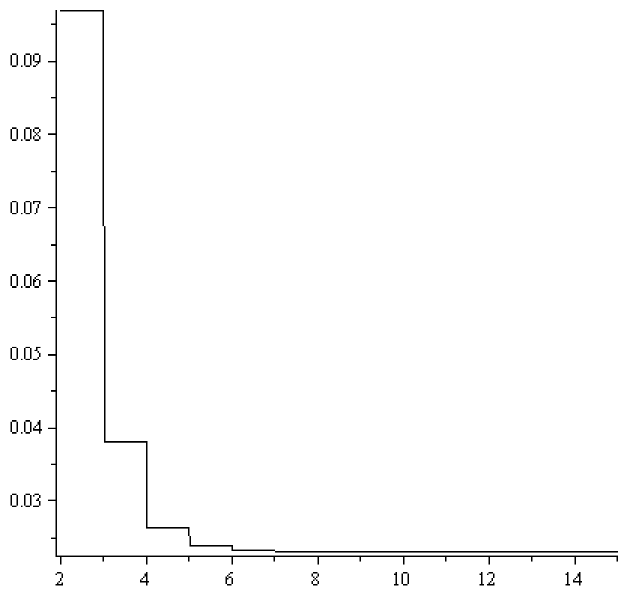

Fig. 2 Plot of the right hand side of Eq.(10)

With some few terms (up to k~12) we obtain $\lambda_1 = 0.0230957575...$ exact to 7 digits.
λ_1 may thus be calculated using the primes (i.e., with Zeta values outside the critical strip) or with N_1 or N_0, i.e. in connection with a function on the number of 1 or 0 in the binary expression of an integer n.

2.Critical strip
Another example concerns the relation between outside and inside of the critical strip. We take as an example the function φ given by:

$$\varphi(\rho) = log\left[\frac{\zeta(\rho + \frac{1}{2}) \cdot (\rho - \frac{1}{2})}{\left(|\rho - 1| + \frac{1}{2}\right)}\right] \tag{11}$$

Eq. (11) was obtained as an integration of log(| ζ(ρ +i·t)|) with a Lorentz measure $d\mu=1/(1/4+t^2)\cdot dt$ on vertical straight lines of abscissa ρ ≥1/2 and is equivalent to the RH [9]. For related works see [7, 8,10, 11, 12]. The function of Eq.(11) is not injective.
Notice also the presence of the shift of ½ due to such a measure. Outside the critical strip the function of Eq.(11) is given by φ (ρ) =log(ζ(ρ+1/2)), ρ ≥1.
We may now choose a value ρ' outside the critical strip. On the RH there exists the same value of the function inside the critical strip for a value ρ(ρ') < 1 and this for all values of ρ ≥1/2(see next Figure).

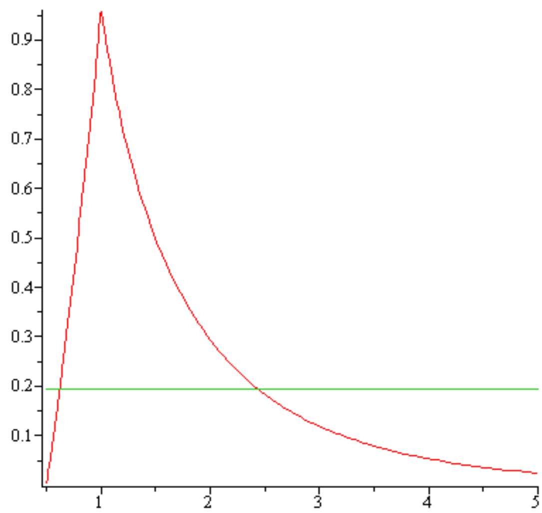

Fig.3 The function given by Eq.(11) with the constant log(2)-1/2 in green, the abscissa of intersection are given below with 10 digits.

We choose for example, the value at our disposal given by the right hand side of Eq.(4) i.e. φ (ρ) = log(2)-1/2 and the solutions with 9 digits are given by:

$\rho_0' = 2.446255389...$ and $\rho_0 = 0.618444186...$

We define here:

$$\varphi(\rho_0, \mathrm{N}) := \sum_{k=2}^{N} [\zeta(\mathrm{k}) - 1] \cdot \left(\frac{1}{2^k}\right)$$

Thus (on the RH):

$$f(\rho) = \left(\frac{1}{2\cdot\pi}\right)\int_R d\mu \cdot log|\zeta(\rho + i \cdot t)| \cdot \frac{1}{\left(\frac{1}{4} + t^2\right)} =$$

$$= \mathrm{g}(\rho') = \left(\frac{1}{2\cdot\pi}\right)\int_R d\mu \cdot log|\zeta(\rho + i \cdot t)| \cdot \frac{1}{\left(\frac{1}{4} + t^2\right)} =$$

$$= \log(2) - \frac{1}{2} = \lim_{N\to\infty} \varphi(\rho_0, \mathrm{N}) := \sum_{k=2}^{\infty} [\zeta(k) - 1] \cdot \left(\frac{1}{2^k}\right) \tag{12}$$

Thus, we may control the above integral on the straight vertical line inside the critical strip by means of the primes given by the righthand side of the above Equation, whose plot is given below.

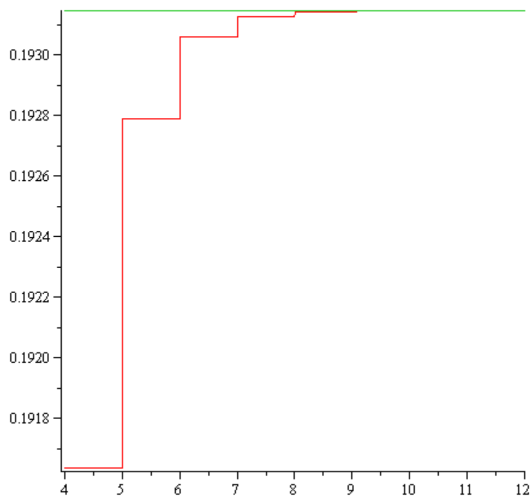

Fig.4 The right hand side of Eq.(12) as a function of Nand the constant log(2)-1/2.

We notice that the true value of Eq.(12) to 10 digits is: 0.1931471806.On the Table below we give the values of the function given by Eq.(11).φ (ρ) up to n=12 (first column) and f_1, g_1 as a function of t_k (values of t solution**s** off (t) = g(t) for t in the range t=(0..61.4)) where

$$f_1(t_k) = \int_0^{t_k} dt \cdot \left(\frac{1}{\pi}\right) \cdot log(|(\zeta(\rho_0 + i \cdot t|)) \cdot \frac{1}{\left(\frac{1}{4} + t^2\right)} \quad (13)$$

$$g_1(t_k) = \int_0^{t_k} dt \cdot \left(\frac{1}{\pi}\right) \cdot log(|(\zeta(\rho_0' + i \cdot t|)) \cdot \frac{1}{\left(\frac{1}{4} + t^2\right)} \quad (14)$$

N	φ(ρ, N)	t_k	f_1	g_1
2	0.**1**612335169	6.432	0.189489	0.191921
3	0.**1**864906298	12.796	0.195226	0.193603
4	0.**19**16358319	15.270	0.192979	0.193197
5	0.**193**0608096	21.943	0.192568	0.193027
6	0.**19**27898242	24.164	0.193211	0.193079
7	0.**193**0608096	25.767	0.192568	0.193027
8	0.**1931**260383	31.437	0.193180	0.193170
9	0.**19314**19655	31.946	0.193151	0.193137
10	0.**19314**58881	33.611	0.192755	0.193064
11	0.**19314**68593	36.987	0.193353	0.193179
12	0.**1931471**007	38.218	0.193170	0.193187
13	0.**1931471**608	40.258	0.193306	0.193172
		41.701	0.193079	0.193131
		42.582	0.193074	0.193109
		43.889	0.192917	0.193099
		47.424	0.193373	0.193193
		50.393	0.193027	0.193129
		52.448	0.193126	0.193124
		53.426	0.193058	0.193130
		55.955	0.193240	0.193166
		56.970	0.193178	0.193169
		58.729	0.193227	0.193157
		61.360	0.193005	0.193118

Table 1

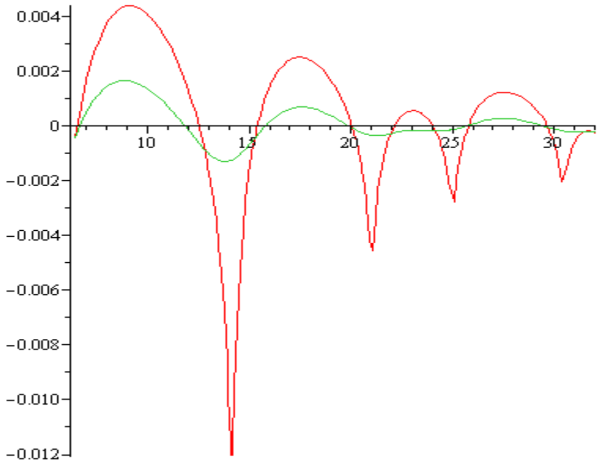

Fig.5 The integrand in Eq.(13) (in red) and the integrand in Eq.(14) (in green) in the range t= (6..32)

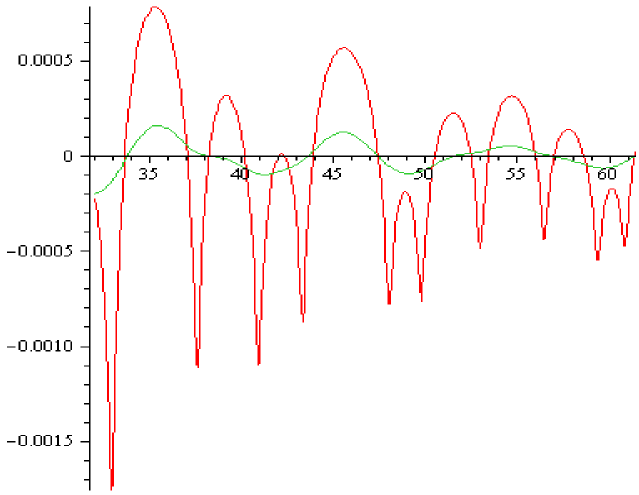

Fig. 6 The integrand in Eq.(13) (in red) and the integrand in Eq.(14) (in green) in the range t= (32..61.4).

We notice that $\varphi(\rho) = \varphi(\rho') = \log(2)-1/2$ "appears" as an attractor to the sequences above for f_1 and g_1: the values of f_1 and $g_1$alternatethe "fix point" $\log(2)-1/2\sim 0.193147$..

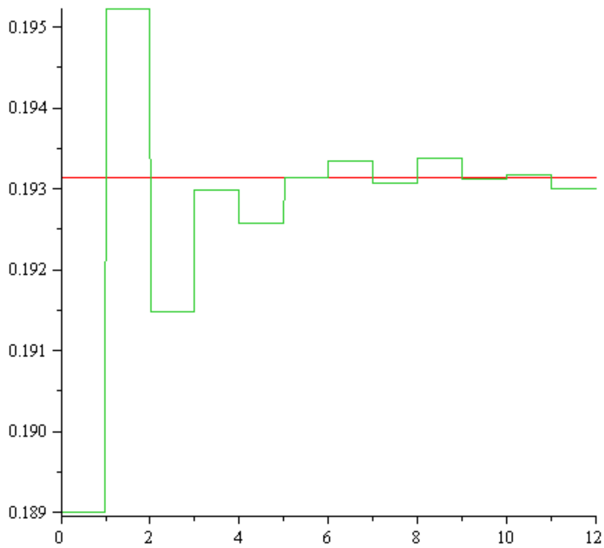

Fig.7 The function $f_1(t_k)$, k=1..12 (in reed the "fixpoint" log(2)-1/2).

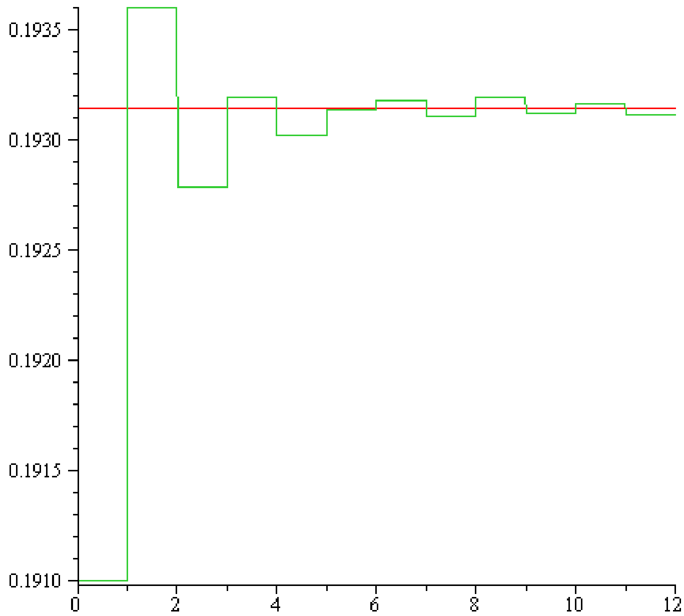

Fig.8 The function $g_1(t_k)$, k=1..12.In reed the "fixpoint" log(2)-1/2.

We now construct a graphic with more points, i.e. with all 23 points of Table 1 which illustrate more in details the alternating behaviour of the two sequences around the "attractor" given by log(2)-1/2 = 0.193147...

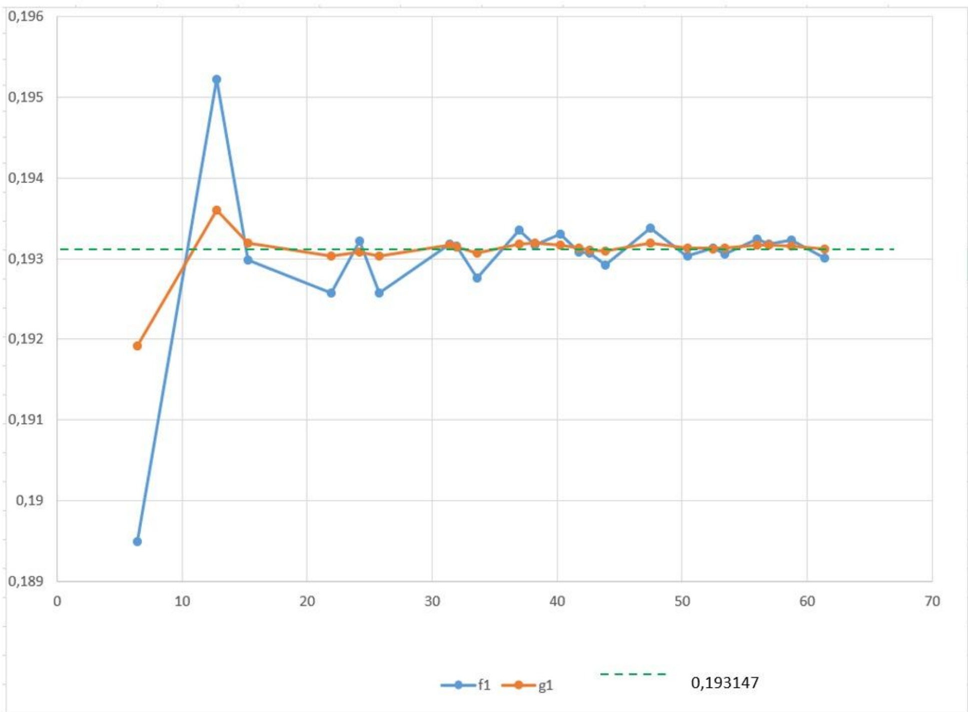

Fig.9 Graphic representation of the values in Table 1 for f_1 and g_1

3. Another computation concerns a value still in the critical strip but more near to the region of absolute convergence studied recently in another approach, i.e. the case ρ_0 =0.9 [7]; from Eq.(11), we obtain: ρ_0' = 1.191957288.

Then, $f(\rho_0) = g(\rho'_0)$ =0.7277248488 (see Fig.3).

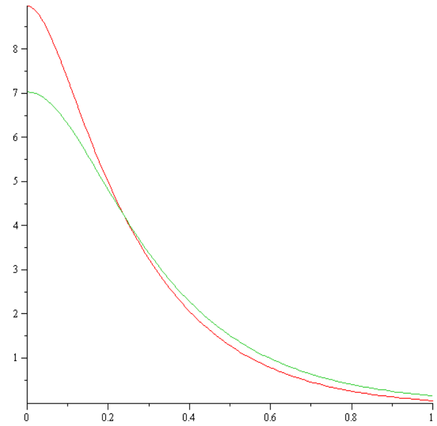

Fig.10 Plot of the integrands $\left(\frac{1}{\pi}\right)\cdot\frac{log(|\zeta(x+i\cdot t|)}{\left(\frac{1}{4}+t^2\right)}$ for x= ρ_0=0.9 (in red) and for x= ρ_0' = 1.1919.. (in green with the first intersection t_1 around t=0.25).

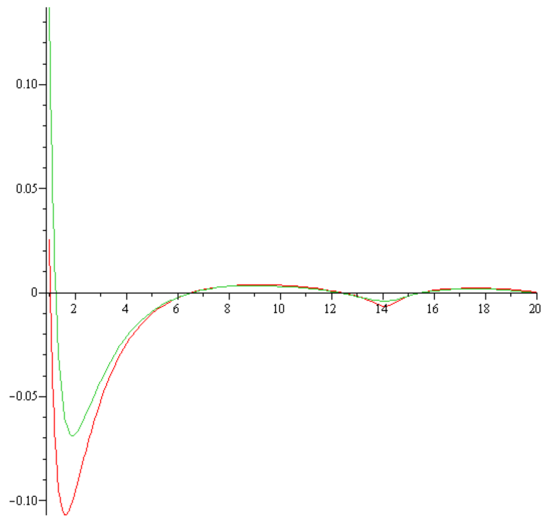

Fig.11 Plot of the same integrands in the range t=(1..20)

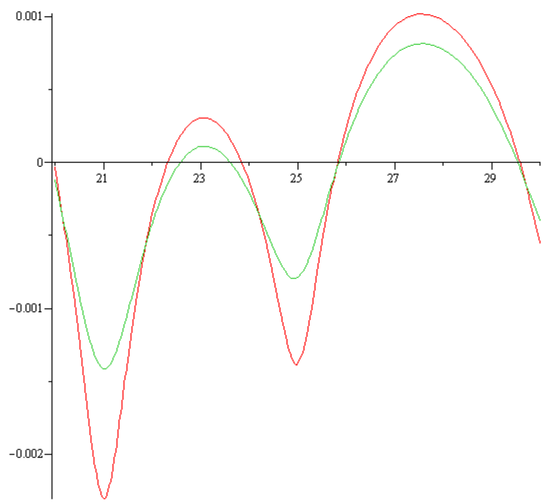

Fig.12 Plot of the same integrands in the range of t=(20..30) with 5 intersections.

As for the first case of computation above we may use Eq.(12), i.e.

$$0.7277248488 = \lim_{N\to\infty} \varphi(\rho_0, \mathrm{N}) \coloneqq$$

$$= \left[\frac{0.7277248488}{\left(\log(2) - \frac{1}{2}\right)}\right] \cdot \sum_{k=2}^{\infty} [\zeta(k) - 1] \cdot \left(\frac{1}{2^k}\right) \tag{15}$$

The right hand side of Eq.(15) is a function [0.727../(log(2)-1/2)] = 3.76..time the function of the right hand side of Eq.(12): for n=13 the value is 0.7277248441 exact to 8 digits.
Here too we note the same behaviour as in the first case: the alternance of points; below we give 22 intersection points t_k and the values of the corresponding integrals from zero to t_k on the Table 2.
The "attractor" has here the value $f(\rho_0) = g(\rho_0') = 0.7277248488..$

t_k	f_1	g_1
0.2419	0.520698	0.453033
6.3819	0.724621	0.725108
12.9914	0.729279	0.728916
15.1572	0.726455	0.726748
20.1625	0.728459	0.728289
21.8692	0.727665	0.727727
24.2112	0.727712	0.727670
25.7673	0.727286	0.727390
29.7036	0.728125	0.728033
32.1439	0.727694	0.727687
33.6191	0.727426	0.727496
36.9443	0.727780	0.727829
38.2378	0.727771	0.727781
41.6205	0.727682	0.727695
42.6408	0.727552	0.727651
47.4001	0.727896	0.727856
50.3877	0.727639	0.727663
52.3990	0.727692	0.727688
53.5223	0.727663	0.727680
57.0105	0.727763	0.727763
58.7472	0.727780	0.727763
61.3901	0.727617	0.727642

Table 2

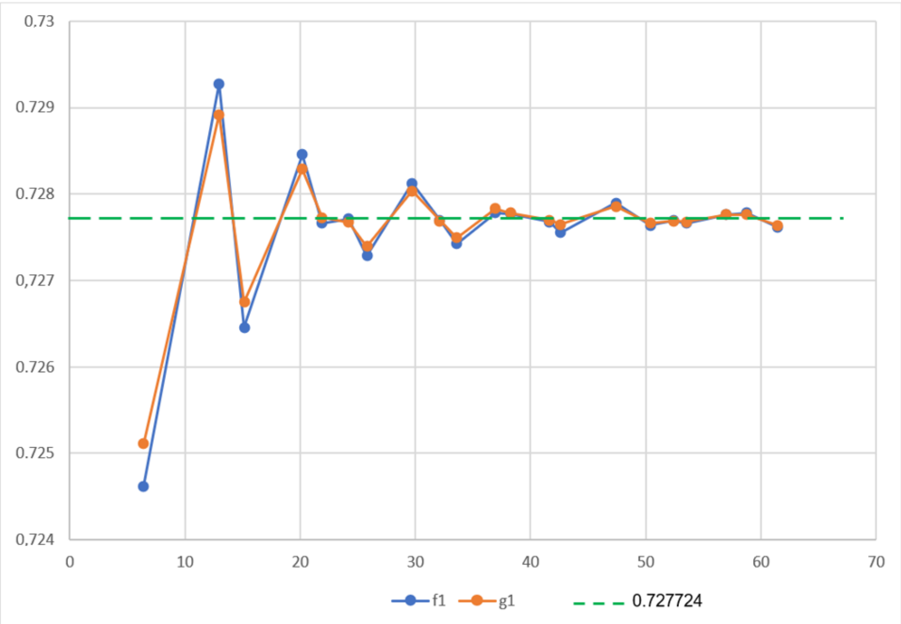

Fig.13 Graphic of the plots of (t,f_1) and of (t,g_1) and the "attractor" 0.727724....(first row of Table 2 is omitted for a better readable of the graphic).

II. Concluding remark

In this paper we have considered a set of four Equations relating the constants γ, log(π) from the pioneering works of Matiyasevich and of Keiper. As an application we have presented three numerical experiments concerning the first exact known coefficient $\lambda_1$and other two concerning the global symmetry (ρ, ρ') relating inside and outside of the critical strip (ρ ≥ 1/2).The computations concerning such quantities may be set with functions on the binary system or by the primes. We have considered the numerical findings are sequences whose values alternate around the corresponding "attractor" i.e. log(2)-1/2 =0.193147 in the first case, and0.727724...in the second case.

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