IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 17, Issue 3 Ser. III (May – June 2021), PP 62-72
www.iosrjournals.org

A Symbolic Continuous Time Markov Chain Model for Degrading Systems Analysis

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Abstract: *A symbolic model, based on a Continuous Time Markov Chain, is proposed for degrading systems analysis. Three types of degrading systems, namely completely recoverable, partially recoverable and non-recoverable ones, and two types of critical sets of states are defined in terms of this model. It is shown that for the proposed model, analytical expressions for probabilities of being in any possible state at each instant can be derived explicitly. Therefore, analytical expressions for the probability of reaching any set of target states at any instant can also be derived. The problems of computing the probability of reachability the target sets of states, if the values of the parameters are given, are solved.*
Key Word*: degrading systems, finite time horizon, Continuous Time Markov Chains, critical sets of states, reachability of target sets of states.*

Date of Submission: 08-06-2021 Date of Acceptance: 21-06-2021

I. Introduction

The importance of developing an effective maintenance policy [1] for an analyzed degrading system (DS) has increased significantly due to the widespread introduction of Cyber-Physical Systems into practice [2]. At the same time, along with completely recoverable DS, there was an urgent need to investigate partially recoverable and non-recoverable DS, which have numerous applications, in particular in the healthcare [3-5].

A maintenance policy, as a rule, is based on the results of simulation the temporal variability of deterioration for the analyzed DS. For this simulation, deterministic as well as stochastic models can be used [6]. It is well-known that stochastic models more accurately represent the degradation process for the analyzed DS. Among such models, stochastic models based on Markov processes with a finite set of states [7] are often used [8-11]. However, they are usually built for specific problems with fixed numeric parameters values. Taking this into account, a symbolic model of DS, based on Finite Markov Chains, which enables to analyze completely recoverable, partially recoverable and non-recoverable DS has been proposed in [12, 13]. The problem of bounded probabilistic analysis [14, 15] for this model has been solved in [13].

The model proposed in [13] is intended for analysis of DS, when the parameters measurements are carried out after the expiration of fixed time intervals and the probabilities of state transitions are constant. This model can also be used when state transition probabilities change over time. Indeed, it is sufficient to divide the time horizon into disjoint intervals, and on each of them apply the corresponding symbolic model proposed in [13]. This approach is inherently equivalent to a piece-wise constant approximation of a continuous process.

In view of the above, a symbolic model intended for DS analysis and based on the Continuous Time Finite Markov Chain (CT FMC) is proposed and investigated in the given paper. This model makes it possible to derive explicitly analytical expressions for the probabilities of being DS in any possible state, as well as for the probability of a set of target states reachability at each instant.

The rest of the paper is organized as follows. Section 2 consists of the preliminary information necessary for understanding further constructions and results. In Section 3 the proposed symbolic CT FMC model intended for DS analysis is presented. The structure of this model for completely recoverable, partially recoverable and non-recoverable DS is defined. Examples of the proposed model for these three types of DS are given. In Section 4 the analysis of the proposed symbolic CT FMC model is illustrated via the non-recoverable DS presented in Section 3. The properties of the probabilities as functions of time are characterized via relations between the parameters. The rates of these probabilities with the variations of the parameters are derived. In Section 5 the problems of the target sets of states reachability for the proposed CT FMC model are solved when the numeric values of the parameters are given. Section 6 is some discussion of obtained results. Section 7 contains concluding remarks.

II. Preliminary Information

It is known that any CT FMC $P_n \quad (n \geq 2)$ with the set of the states $S_n = \{s_1, \cdots, s_n\}$ can be defined by the transition rate matrix

$$\Lambda_{P_n} = \begin{bmatrix} -\sum\limits_{j \in \{1,\cdots,n\} \setminus \{1\}} \lambda_{1j} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & -\sum\limits_{j \in \{1,\cdots,n\} \setminus \{2\}} \lambda_{2j} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n1} & \cdots & -\sum\limits_{j \in \{1,\cdots,n\} \setminus \{n\}} \lambda_{nj} \end{bmatrix},$$

where $\lambda_{ij} \quad (i, j = 1, \cdots, n;\ i \neq j)$ is the rate of the departing from the state s_i and arriving in the state s_j. It should be noted that $\lambda_{ij} \geq 0 \quad (i, j = 1, \cdots, n;\ i \neq j)$.

Let $p_i(t) \quad (i = 1, \cdots, n)$ be the probability that the CT FMC P_n is in the state s_i at instant t. Then the vector $\mathbf{p}(t) = (p_1(t), \cdots, p_n(t))$ is the solution of the Chapman–Kolmogorov system of equations

$$\frac{d\mathbf{p}(t)}{dt} = \mathbf{p}(t)\Lambda_{P_n}.$$

Therefore, for any initial probability distribution

$$\mathbf{v} = (v_1, \cdots, v_n) \quad \left(0 \leq v_i \leq 1 \quad (i = 1, \cdots, n),\ \sum_{i=1}^{n} v_i = 1\right)$$

of the CT FMC P_n states, the vector $\mathbf{p}(t) = (p_1(t), \cdots, p_n(t))$ is the solution of the system of differential equations

$$\begin{cases} \dfrac{d\mathbf{p}(t)}{dt} = \mathbf{p}(t)\Lambda_{P_n} \\ \mathbf{p}(0) = \mathbf{v} \end{cases}. \tag{1}$$

It should be noted that with respect to the variables $p_i(t) \quad (i = 1, \cdots, n)$, the systems (1) is a system of linear differential equations with constant coefficients.

III. Proposed symbolic CT FMC P_n model

By analogy with [13], we define a symbolic model for the analyzed DS S_n with n stages of functionality as a CT FMC P_n that satisfy to the following three assumptions:

Assumption 1. The CT FMC P_n states represent the functionality stages of the DS S_n as follows:

- The state s_1 represents the analyzed DS S_n in the completely functional stage.
- The state s_n represents the analyzed DS S_n in the inoperable stage.
- The states $s_2, \cdots, s_{n-1}$ represent the analyzed DS S_n in all possible stages of partial functioning.

Assumption 2. For a given positive integer $k \quad (2 \leq k \leq n)$ some partition

$$\pi = \{B_1, \cdots, B_k\}$$

of the set S_n is fixed such that

$$B_1 = \{s_{i_1}\},$$

$$B_j = \{s_{i_{j-1}+1}, \cdots, s_{i_j}\} \quad (j = 2, \cdots, k-1),$$

$$B_k = \{s_{i_{k-1}+1}\},$$

where $i_1 = 1$, $i_{k-1} = n-1$ and $i_{j-1} < i_j$ for all $j = 2, \cdots, k-1$.

Remark 1. The interpretation of the partition π is as follows: each its block consists of all states $s \in S_n$ representing the stages the same functionality level for the DS S_n.

Assumption 3. The elements of the transition rate matrix Λ_{P_n} satisfy to the following six conditions:

Condition 1. The equality $\lambda_{nj} = 0$ holds for all $j = 1, \cdots, n-1$.

Condition 2. For all $j = 2,\cdots,k-1$ the equality $\lambda_{rh} = 0$ holds for all states $s_r, s_h \in B_j$ $(r \neq h)$.

Condition 3. For each state $s_r \in B_j$ $(j = 1,\cdots,k-1)$ there exists some subset $S_n^{dsc}(r)$ $\left(\varnothing \neq S_n^{dsc}(r) \subseteq \bigcup_{m=j+1}^{k} B_m\right)$ such that $\lambda_{rh} > 0$ for all states $s_h \in S_n^{dsc}(r)$, and $\lambda_{rh} = 0$ for all states $s_h \in \left(\bigcup_{m=j+1}^{k} B_m\right) \setminus S_n^{dsc}(r)$.

Condition 4. For all $j = 1,\cdots,k-1$ holds the equality $\bigcup_{s_r \in B_j} (S_n^{dsc}(r) \cap B_{j+1}) = B_{j+1}$.

Condition 5. For each state $s_r \in B_j$ $(j = 2,\cdots,k-1)$ there exists some subset $S_n^{anc}(r)$ $(S_n^{anc}(r) \subseteq \bigcup_{m=1}^{j-1} B_m)$ such that $\lambda_{rh} > 0$ for all states $s_h \in S_n^{anc}(r)$, and $\lambda_{rh} = 0$ for all states $s_h \in \left(\bigcup_{m=1}^{j-1} B_m\right) \setminus S_n^{anc}(r)$.

Condition 6. For all $j = 2,\cdots,k-1$, if $\lambda_{rn} = 0$ for all states $s_r \in B_j$, then $\lambda_{hn} = 0$ for all states $s_h \in B_{j-1}$.

Remark 2. Similarly to [13], we note that if the analyzed DS S_n is a technical system, and deteriorating in its functionality is carried out due to the appearance of faults in it, then it is usually assumed that in one step either one new fault can appear, or one of the existing faults can be eliminated. In this case, Conditions 3-5 can be simplified as follows:

- In Condition 3 formula $\left(\varnothing \neq S_n^{dsc}(r) \subseteq \bigcup_{m=j+1}^{k} B_m\right)$ can be changed by $(\varnothing \neq S_n^{dsc}(r) \subseteq B_{j+1})$, and formula $s_h \in \left(\bigcup_{m=j+1}^{k} B_m\right) \setminus S_n^{dsc}(r)$ can be changed by $s_h \in B_{j+1} \setminus S_n^{dsc}(r)$.
- In Condition 4 formula $\bigcup_{s_r \in B_j} (S_n^{dsc}(r) \cap B_{j+1}) = B_{j+1}$ can be changed by $\bigcup_{s_r \in B_j} S_n^{dsc}(r) = B_{j+1}$.
- In Condition 5 formula $(S_n^{anc}(r) \subseteq \bigcup_{m=1}^{j-1} B_m)$ can be changed by $(S_n^{anc}(r) \subseteq B_{j-1})$, and formula $s_h \in \left(\bigcup_{m=1}^{j-1} B_m\right) \setminus S_n^{anc}(r)$ can be changed by $s_h \in B_{j-1} \setminus S_n^{anc}(r)$.

Remark 3. Assumptions 1-3 imply that for any CT FMC P_n being a symbolic model of the analyzed DS S_n with n stages of functionality, the state s_n is the single absorbing state. Since the last equation of the Chapman–Kolmogorov system of equations is

$$\frac{dp_n(t)}{dt} = \sum_{i=1}^{n-1} \lambda_{in} p_i(t) ,$$

we get $\frac{dp_n(t)}{dt} > 0$, i.e. $p_n(t)$ is a strictly increasing function.

In the sequel it is supposed that for any CT FMC P_n being a symbolic model for the analyzed DS S_n with n stages of functionality, the initial probability distribution of CT FMC P_n states is

$$\mathbf{p}(0) = (1, \underbrace{0,\cdots,0}_{n-1 \text{ times}}) .$$

Assumptions 1-3 directly imply the correctness of the following two definitions.

Definition 1. *For a CT FMC P_n being the symbolic model for the analyzed DS S_n with n stages of functionality, we distinguish the following two types of the critical set of states:*

1. The critical set of states in the weak sense S_n^{ws-cr} consists of all states $s_r \in \bigcup_{j=1}^{k-1} B_j$ such that $\lambda_{rn} > 0$.

2. The critical set of states in the strong sense S_n^{ss-cr} *consists of all states* $s_r \in \bigcup_{j=1}^{k-1} B_j$ *such that* $S_n^{dsc}(r) = \{s_n\}$

.

Definition 2. *Let a CT FMC* P_n *be the symbolic model for the DS* S_n *with* n *stages of functionality. Then* P_n *is a model of:*

1. The completely recoverable DS S_n*, if for each integer* $j = 2, \cdots, k-1$ *the disequality* $S_n^{anc}(r) \neq \varnothing$ *holds for all states* $s_r \in B_j$*.*

2. The partially recoverable DS S_n*, if such an integer* l $(2 \le l \le k-1)$ *exists that for each integer* $j = l+1, \cdots, k-1$ *the disequalities* $S_n^{anc}(r) \neq \varnothing$ *hold for all states* $s_r \in B_j$*, and for each integer* $j = 2, \cdots, l$ *the equalities* $S_n^{anc}(r) = \varnothing$ *hold for all states* $s_r \in B_j$*.*

3. The non-recoverable DS S_n*, if for each integer* $j = 2, \cdots, k-1$ *the equality* $S_n^{anc}(r) = \varnothing$ *holds for all states* $s_r \in B_j$*.*

Let us illustrate the introduced concepts by the following examples.

Example 1. Consider the dynamics of a chronic disease with two stages. The deterioration of Patients' health is associated with the occurrence of the disease, staying at the first stage of the disease, transition to the second stage of the disease and staying there, and, finally, with the death of the Patient. So we deal with the DS $S_4^{(1)}$. Its symbolic model is the CT FMC $P_4^{(1)}$ with the set of states $S_4^{(1)} = \{s_1, s_2, s_3, s_4\}$, where:

1. The state s_1 represents the stage, when the Patient is healthy.

2. The state s_2 represents the situation, when the Patient is staying in the first stage of the disease.

3. The state s_3 represents the situation, when the Patient is staying in the second stage of the disease.

4. The state s_4 represents situation, when the Patient is dead.

The transition rate matrix of the CT FMC $P_4^{(1)}$ is as follows

$$\Lambda_{P_4^{(1)}} = \begin{bmatrix} -a_1 & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ 0 & -a_2 & \lambda_{23} & \lambda_{24} \\ 0 & 0 & -\lambda_{34} & \lambda_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where all $\lambda_{ij} > 0$, $a_1 = \lambda_{12} + \lambda_{13} + \lambda_{14}$ and $a_2 = \lambda_{23} + \lambda_{24}$.

For the CT FMC $P_4^{(1)}$ we get:

1. The partition of the set $S_4^{(1)}$ is $\pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{s_1\}$, $B_2 = \{s_2\}$, $B_3 = \{s_3\}$, and $B_4 = \{s_4\}$.

2. Due to Definition 1, $S_4^{ws-cr} = B_1 \cup B_2 \cup B_3$ since $\lambda_{i4} > 0$ $(i = 1, 2, 3)$, and $S_4^{ss-cr} = B_3$ since the equality $S_4^{dsc}(3) = \{s_4\}$ holds.

3. Due to Definition 2, the CT FMC $P_4^{(1)}$ is a symbolic model of the non-recoverable DS $S_4^{(1)}$ since $S_4^{anc}(r) = \varnothing$ for all states $s_r \in B_2 \cup B_3$.

Example 2. Consider a network consisting of three pairwise connected computers C_1, C_2 and C_3. Deteriorating in the functionality of this network is carried out due to the appearance of faults in the computers, and recovery consists of eliminating these faults. So we deal with the DS $S_8^{(1)}$. Its symbolic model is the CT FMC $P_8^{(1)}$ with the set of states $S_8^{(1)} = \{s_1, \cdots, s_8\}$, where:

1. The state s_1 represents the stage when the considered network is fault-free.

2. The state s_2 represents the stage when the computer C_1 is faulty and the computers C_2 and C_3 are fault-free.

3. The state s_3 represents the stage when the computer C_2 is faulty and the computers C_1 and C_3 are fault-free.

4. The state s_4 represents the stage when the computer C_3 is faulty and the computers C_1 and C_2 are fault-free.

5. The state s_5 represents the stage when the computers C_1 and C_2 are faulty and the computer C_3 is fault-free.

6. The state s_6 represents the stage when the computers C_1 and C_3 are faulty and the computer C_2 is fault-free.

7. The state s_7 represents the stage when the computers C_2 and C_3 are faulty and the computer C_1 is fault-free.

8. The state s_8 represents the inoperable stage, i.e. when all three computers C_1, C_2 and C_3 are faulty.

The CT FMC $P_8^{(1)}$ the transition rate matrix is as follows

$$\Lambda_{P_8^{(1)}} = \begin{bmatrix} -a_{1;2,3,4} & \lambda_{12} & \lambda_{13} & \lambda_{14} & 0 & 0 & 0 & 0 \\ \lambda_{21} & -a_{2;1,5,6} & 0 & 0 & \lambda_{25} & \lambda_{26} & 0 & 0 \\ \lambda_{31} & 0 & -a_{3;1,5,7} & 0 & \lambda_{35} & 0 & \lambda_{37} & 0 \\ \lambda_{41} & 0 & 0 & -a_{4;1,6,7} & 0 & \lambda_{46} & \lambda_{47} & 0 \\ 0 & \lambda_{52} & \lambda_{53} & 0 & -a_{5;2,3,8} & 0 & 0 & \lambda_{58} \\ 0 & \lambda_{62} & 0 & \lambda_{64} & 0 & -a_{6;2,4,8} & 0 & \lambda_{68} \\ 0 & 0 & \lambda_{73} & \lambda_{74} & 0 & 0 & -a_{7;3,4,8} & \lambda_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where all $\lambda_{ij} > 0$, and $a_{i;j_1,j_2,j_3} = \lambda_{ij_1} + \lambda_{ij_2} + \lambda_{ij_3}$.

For the CT FMC $P_8^{(1)}$ we get:

1. The partition of the set $S_8^{(1)}$ is $\pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{s_1\}$, $B_2 = \{s_2, s_3, s_4\}$, $B_3 = \{s_5, s_6, s_7\}$, and $B_4 = \{s_8\}$.

2. Due to Definition 1, $S_8^{ss-cr} = S_8^{ws-cr} = B_3 = \{s_5, s_6, s_7\}$ since $\lambda_{i8} > 0 \ \ (i = 5,6,7)$ and $\lambda_{i8} = 0 \ \ (i = 1,2,3,4)$.

3. Due to Definition 2, the CT FMC $P_8^{(1)}$ is a symbolic model of the completely recoverable DS $S_8^{(1)}$ since $S_8^{anc}(r) \neq \varnothing$ for all states $s_r \in B_2 \cup B_3$.

Example 3. Suppose the DS $S_8^{(2)}$ differs from the DS $S_8^{(1)}$ only in that the eliminating the faults in the network can be carried out only when two computers are faulty, and, at the same time, the fault can be eliminated in only one computer. So we deal with the DS $S_8^{(2)}$. Its symbolic model is the CT FMC $P_8^{(2)}$, whose the transition rate matrix $\Lambda_{P_8^{(2)}}$ differs from the transition rate matrix $\Lambda_{P_8^{(1)}}$ only in that $\lambda_{21} = \lambda_{31} = \lambda_{41} = 0$.

For the CT FMC $P_8^{(2)}$, as for the CT FMC $P_8^{(2)}$, $\pi = \{B_1, B_2, B_3, B_4\}$, where $B_1 = \{s_1\}$, $B_2 = \{s_2, s_3, s_4\}$, $B_3 = \{s_5, s_6, s_7\}$, and $B_4 = \{s_8\}$, and $S_8^{ss-cr} = S_8^{ws-cr} = B_3 = \{s_5, s_6, s_7\}$. But, due to Definition 2, the CT FMC $P_8^{(2)}$ is a symbolic model of the partially recoverable DS $S_8^{(2)}$ since $S_8^{anc}(r) \neq \varnothing$ for all states $s_r \in B_3$ and $S_8^{anc}(r) = \varnothing$ for all states $s_r \in B_2$.

II. Analysis of the symbolic CT FMC P_n model

Let a CT FMC P_n be a symbolic model for the analyzed recoverable or partially recoverable DS S_n. Then the Chapman–Kolmogorov system of equations is a parametric system of linear differential equations of a general form with constant coefficients. Its solution satisfying the initial condition $\mathbf{p}(0) = (1, \underbrace{0, \cdots, 0}_{n-1 \text{ times}})$ can be found by reducing it either to a parametric system of algebraic equations (using direct and inverse Laplace transformations) or to a parametric linear homogeneous differential equation of the n-th order with constant coefficients.

The situation is significantly simplified when a CT FMC P_n is a symbolic model for a non-recoverable DS S_n. In this case, the first $n-1$ equations of the Chapman–Kolmogorov system of equations are a triangular system of differential equations with constant coefficients. Let's illustrate this by the following example.

Example 4. Consider the symbolic model of Example 1. The Chapman–Kolmogorov system of equations has the form

$$\begin{cases} \dfrac{dp_1(t)}{dt} = -a_1 p_1(t) \\ \dfrac{dp_2(t)}{dt} = \lambda_{12} p_1(t) - a_2 p_2(t) \\ \dfrac{dp_3(t)}{dt} = \lambda_{13} p_1(t) + \lambda_{23} p_2(t) - \lambda_{34} p_3(t) \\ \dfrac{dp_4(t)}{dt} = \lambda_{14} p_1(t) + \lambda_{24} p_2(t) + \lambda_{34} p_3(t) \end{cases}. \tag{2}$$

Its solution satisfying the initial condition

$$\begin{cases} p_1(0) = 1 \\ p_2(0) = p_3(0) = p_4(0) = 0 \end{cases}$$

can be derived as follows.

Solving the 1-st equation of the system (2), we obtain

$$p_1(t) = \exp(-a_1 t). \tag{3}$$

Substituting (3) in the 2-nd equation of the system (2), and solving it, we obtain

$$a_2 = a_1 \Rightarrow p_2(t) = \lambda_{12} t \exp(-a_1 t), \tag{4}$$

$$a_2 \neq a_1 \Rightarrow p_2(t) = \lambda_{12}(a_2 - a_1)^{-1}(\exp(-a_1 t) - \exp(-a_2 t)). \tag{5}$$

Let’s solve the 3-rd equation of the system (1).

Substituting (3) and (4) in the 3-rd equation of the system (2), and solving it, we obtain

$$a_2 = a_1 = \lambda_{34} \Rightarrow p_3(t) = (\lambda_{13} t + 0.5\lambda_{12}\lambda_{23} t^2)\exp(-\lambda_{34} t), \tag{6}$$

$$a_2 = a_1 \;\&\; a_1 \neq \lambda_{34} \Rightarrow p_3(t) = (\lambda_{13}(\lambda_{34} - a_1)^{-1} - \lambda_{12}\lambda_{23}(\lambda_{34} - a_1)^{-2})(\exp(-a_1 t) - \exp(-\lambda_{34} t)) +$$
$$+\lambda_{12}\lambda_{23}(\lambda_{34} - a_1)^{-1} t \exp(-a_1 t). \tag{7}$$

Substituting (3) and (5) in the 3-rd equation of the system (2), and solving it, we obtain

$$a_1 \neq a_2 \;\&\; a_1 = \lambda_{34} \Rightarrow p_3(t) = ((\lambda_{13} + \lambda_{12}\lambda_{23}(a_2 - \lambda_{34})^{-1} - \lambda_{12}\lambda_{23}(a_2 - \lambda_{34})^{-2})t)\exp(-\lambda_{34} t) +$$
$$+\lambda_{12}\lambda_{23}(a_2 - \lambda_{34})^{-2}\exp(-a_2 t), \tag{8}$$

$$a_1 \neq a_2 \;\&\; a_2 = \lambda_{34} \Rightarrow p_3(t) = ((\lambda_{13}(\lambda_{34} - a_1)^{-1} + \lambda_{12}\lambda_{23}(\lambda_{34} - a_1)^{-2})(\exp(-a_1 t) - \exp(-\lambda_{34} t)) -$$
$$-\lambda_{12}\lambda_{23}(\lambda_{34} - a_1)^{-1} t \exp(-\lambda_{34} t), \tag{9}$$

$$a_1 \neq a_2 \;\&\; \lambda_{34} \neq a_1 \;\&\; \lambda_{34} \neq a_2 \Rightarrow$$
$$\Rightarrow p_3(t) = (\lambda_{34} - a_1)^{-1}(\lambda_{13} + \lambda_{12}\lambda_{23}(a_2 - a_1)^{-1})(\exp(-a_1 t) - \exp(-\lambda_{34} t)) -$$
$$-\lambda_{12}\lambda_{23}(\lambda_{34} - a_2)^{-1}(a_2 - a_1)^{-1}(\exp(-a_2 t) - \exp(-\lambda_{34} t)). \tag{10}$$

Substituting $p_1(t)$, $p_2(t)$ and $p_3(t)$ in the 4-th equation of the system (2), and solving it, we obtain

$$a_2 = a_1 = \lambda_{34} \Rightarrow p_4(t) = 1 - (1 + (\lambda_{12} + \lambda_{13})t + 0.5\lambda_{12}\lambda_{23} t^2)\exp(-\lambda_{34} t), \tag{11}$$

$$a_2 = a_1 \;\&\; a_1 \neq \lambda_{34} \Rightarrow p_4(t) = 1 - (a_1^{-1}(\lambda_{14} + (\lambda_{13}\lambda_{34} + \lambda_{12}(\lambda_{34} - \lambda_{24})(\lambda_{34} - a_1)^{-1}) -$$
$$-\lambda_{12}(\lambda_{34} - \lambda_{24})(\lambda_{34} - a_1)^{-1} t)\exp(-a_1 t) + (\lambda_{13}(\lambda_{34} - a_1)^{-1} - \lambda_{12}\lambda_{23}(\lambda_{34} - a_1)^{-2})\exp(-\lambda_{34} t), \tag{12}$$

$$a_1 \neq a_2 \;\&\; a_1 = \lambda_{34} \Rightarrow p_4(t) = 1 - \lambda_{34}^{-1}(\lambda_{13} + \lambda_{14} + \lambda_{12}\lambda_{23}(a_2 - \lambda_{34})^{-1} + a_2\lambda_{12}(\lambda_{24} - \lambda_{34})(a_2 - \lambda_{34})^{-2} +$$
$$+\lambda_{34}(\lambda_{13} + \lambda_{12}\lambda_{23}(a_2 - \lambda_{34})^{-1})t)\exp(-\lambda_{34} t) + \lambda_{12}(\lambda_{24} - \lambda_{34})(a_2 - \lambda_{34})^{-2}\exp(-a_2 t), \tag{13}$$

$$a_1 \neq a_2 \;\&\; a_2 = \lambda_{34} \Rightarrow p_4(t) = 1 - a_1^{-1}(\lambda_{14} + (\lambda_{12}\lambda_{24} + \lambda_{13}\lambda_{34})(\lambda_{34} - a_1)^{-1} +$$

$$+\lambda_{12}\lambda_{23}\lambda_{34}(\lambda_{34}-a_1)^{-2})\exp(-a_1 t)+\lambda_{34}^{-1}((\lambda_{12}\lambda_{24}+\lambda_{13}\lambda_{34}+\lambda_{12}\lambda_{23})(\lambda_{34}-a_1)^{-1}+$$

$$+\lambda_{12}\lambda_{23}\lambda_{34}(\lambda_{34}-a_1)^{-2}+\lambda_{12}\lambda_{23}\lambda_{34}(\lambda_{34}-a_1)^{-1}t)\exp(-\lambda_{34}t)\,, \quad (14)$$

$$a_1 \neq a_2 \;\&\; \lambda_{34} \neq a_1 \;\&\; \lambda_{34} \neq a_2 \Rightarrow$$

$$\Rightarrow\; p_4(t) = 1 - a_1^{-1}(\lambda_{14}+\lambda_{12}\lambda_{24}(a_2-a_1)^{-1}+\lambda_{13}\lambda_{34}(\lambda_{34}-a_1)^{-1}+$$

$$+\lambda_{12}\lambda_{23}\lambda_{34}(\lambda_{34}-a_1)^{-1}(a_2-a_1)^{-1})\exp(-a_1 t)+a_2^{-1}(\lambda_{12}\lambda_{24}(a_2-a_1)^{-1}+$$

$$+\lambda_{12}\lambda_{23}\lambda_{34}(\lambda_{34}-a_1)^{-1}(\lambda_{34}-a_2)^{-1})\exp(-a_2 t)-$$

$$-(\lambda_{12}\lambda_{23}(\lambda_{34}-a_1)^{-1}(\lambda_{34}-a_2)^{-1}-\lambda_{13}(\lambda_{34}-a_1)^{-1})\exp(-\lambda_{34}t)\,. \quad (15)$$

Example 4 illustrates that the solving of the Chapman–Kolmogorov system of equations for a CT FMC P_n, which is a symbolic model of a DS S_n, requires the analysis of all possible relationships between parameters (these cases are represented by formulae (3)-(15)), and leads to rather cumbersome analytical expressions.

The main arguments for deriving these parametric expressions are the following:

- These parametric expressions are derived only once.
- These parametric expressions give the possibility to characterize the probabilities $p_i(t)$ $(i=1,\cdots,n)$ as functions of time via the relations between the parameters.
- These parametric expressions give the possibility to find the rates of change

$$p_h^{-1}(t)\frac{\partial p_h(t)}{\partial \lambda_{ij}} \quad (h=1,\cdots,n)$$

for the probabilities $p_i(t)$ $(i=1,\cdots,n)$ relatively to variations of the parameters λ_{ij}.

Let us illustrate the above via the following example.

Example 5. For the symbolic model $P_4^{(1)}$ of Example 1, we examine the probabilities $p_1(t)$, $p_2(t)$ and $p_3(t)$ defined by formulae (3)-(6).

Let's characterize these probabilities as functions of time.

The probability $p_1(t)$ is a strictly decreasing function.

If $a_1 = a_2$ then the probability $p_2(t)$ is a strictly increasing function when $0 < t < a_1^{-1}$, and a strictly decreasing function when $t > a_1^{-1}$. If $a_1 \neq a_2$ then the probability $p_2(t)$ is a strictly increasing function when

$$0 < t < (\ln\max\{a_1,a_2\} - \ln\min\{a_1,a_2\})(\max\{a_1,a_2\}-\min\{a_1,a_2\})^{-1},$$

and a strictly decreasing function when

$$t > (\ln\max\{a_1,a_2\} - \ln\min\{a_1,a_2\})(\max\{a_1,a_2\}-\min\{a_1,a_2\})^{-1}.$$

If $a_1 = a_2 = \lambda_{34}$ then the probability $p_3(t)$ is a strictly increasing function when

$$0 < t < (\lambda_{12}\lambda_{23}-\lambda_{13}\lambda_{34}+\sqrt{\alpha})(\lambda_{12}\lambda_{23}\lambda_{34})^{-1},$$

and a strictly decreasing function when

$$t > (\lambda_{12}\lambda_{23}-\lambda_{13}\lambda_{34}+\sqrt{\alpha})(\lambda_{12}\lambda_{23}\lambda_{34})^{-1},$$

where $\alpha = (\lambda_{13}\lambda_{34}-\lambda_{12}\lambda_{23})^2 + 2\lambda_{12}\lambda_{23}\lambda_{34}\lambda_{13}$.

Now we derive the rates of change for these probabilities relatively to variations of the parameters λ_{ij}.

For the probability $p_1(t)$ we get

$$p_1^{-1}(t)\frac{\partial p_1(t)}{\partial \lambda_{1i}} = -t \quad (j=2,3,4)\,.$$

Let's analyze the probability $p_2(t)$.

If $a_1 = a_2$ then

$$p_2^{-1}(t)\frac{\partial p_2(t)}{\partial \lambda_{1j}} = \gamma_j\lambda_{12}^{-1} - t \quad (j=2,3,4)\,,$$

where $\gamma_2 = 1$ and $\gamma_3 = \gamma_4 = 0$.

If $a_1 \neq a_2$ then

$$p_2^{-1}(t)\frac{\partial p_2(t)}{\partial \lambda_{1j}} = \gamma_j \lambda_{12}^{-1} + (a_2 - a_1)^{-1} - t\exp(-a_1 t)(\exp(-a_1 t) - \exp(-a_2 t))^{-1} \quad (j = 2,3,4),$$

where $\gamma_2 = 1$ and $\gamma_3 = \gamma_4 = 0$, and

$$p_2^{-1}(t)\frac{\partial p_2(t)}{\partial \lambda_{2j}} = -(a_2 - a_1)^{-1} + t\exp(-a_2 t)(\exp(-a_1 t) - \exp(-a_2 t))^{-1} \quad (j = 3,4).$$

For the probability $p_3(t)$, when $a_2 = a_1 = \lambda_{34}$ we get

$$p_3^{-1}(t)\frac{\partial p_3(t)}{\partial \lambda_{12}} = 0.5\lambda_{23} t(\lambda_{13} + 0.5\lambda_{12}\lambda_{23} t)^{-1},$$

$$p_3^{-1}(t)\frac{\partial p_3(t)}{\partial \lambda_{13}} = (\lambda_{13} + 0.5\lambda_{12}\lambda_{23} t)^{-1},$$

$$p_3^{-1}(t)\frac{\partial p_3(t)}{\partial \lambda_{23}} = 0.5\lambda_{12} t(\lambda_{13} + 0.5\lambda_{12}\lambda_{23} t)^{-1},$$

and

$$p_3^{-1}(t)\frac{\partial p_3(t)}{\partial \lambda_{34}} = -t.$$

It should be noted that for a CT FMC P_n, which is a symbolic model of a DS S_n, the analysis of parametric expressions for probabilities $p_i(t)$ $(i = 1,\cdots,n)$ and their rates of change relatively to variations of the parameters λ_{ij} forms some base for statistical modeling of the DS S_n behavior under parameters' variation.

III. Target set reachability for the symbolic CT FMC P_n model.

Let a CT FMC P_n be a symbolic model for the DS S_n, the analysis of which is carried out on the finite time horizon $[0,T]$.

We call the target set of states S_n^{trgt} for the CT FMC P_n any element of the set $\{\{s_n\}, S_n^{cr-ws}, S_n^{cr-ss}\}$.

Let

$$\mathbf{P}_{s_1, S_n^{trgt}}(t) = \sum_{s_i \in S_n^{trgt}} p_i(t).$$

We consider the reachability problem for the CT FMC P_n target set of states S_n^{trgt} when the numeric values of the parameters λ_{ij} are given.

Remark 4. Substituting the numeric values of λ_{ij} into the parametric probability expressions $p_i(t)$ $(i = 1,\cdots,n)$ we get formulae that determine these probabilities as functions of t only.

Let $S_n^{trgt} = \{s_n\}$. We formulate the problem of analysis of the target set reachability as follows.

Problem 1. The numeric values of the parameters λ_{ij} and the numbers ε $(0 < \varepsilon < 1)$ and τ $(0 < \tau < 0.5T)$ are given. It is necessary to find $t_0 \in (0,T)$ such that

$$\begin{cases} p_n(\max\{0, t_0 - \tau\}) < \varepsilon \\ p_n(\min\{T, t_0 + \tau\}) \geq \varepsilon \end{cases}.$$

The solution of this problem can be obtained by using the following algorithm.

Algorithm 1.

Step 1. If $p_n(T) < \varepsilon$ then print "The required value $t_0 \in (0,T)$ does not exist" and HALT, else go to Step 2.

Step 2. $\alpha := 0$, $\beta := T$.

Step 3. $\gamma := 0.5(\alpha + \beta)$.

Step 4. If $p_n(\gamma) < \varepsilon$, then $\alpha := \gamma$, else $\beta := \gamma$.

Step 5. If $\beta - \alpha \leq 2\tau$, then $t_0 := 0.5(\alpha + \beta)$ and HALT, else go to Step 3.

Theorem 1. Algorithm 1 is sound. The number of iterations of steps 3-5 is $\lceil \log T - \log \sigma - 1 \rceil$.

Proof. The correctness of Step 1 of Algorithm 1 follows from the fact that $p_n(0) = 0$ and $p_n(t)$ is a strictly increasing function.

Steps 2-5 of Algorithm 1 are performed if and only if the required value of $t_0 \in [0, T]$ exists. These steps implement the bisection method. In so doing, Algorithm 1 terminates if and only if the length of the current considered interval $[\alpha, \beta]$ does not exceed 2τ, $p_n(\alpha) < \varepsilon$ and $p_n(\beta) \geq \varepsilon$. These conditions occur after the finite number of iterations of Steps 3-5, since $p_n(0) = 0$, $p_n(t)$ is a strictly increasing function and $p_n(T) \geq \varepsilon$. This implies the correctness of Algorithm 1 when performing the iterations defined by Steps 2-5 of Algorithm 1.

So, Algorithm 1 is sound.

Let k be the number of Steps 3-5 iterations for Algorithm 1. Then

$$2^{-k} T \leq 2\tau \Leftrightarrow 2^k \geq 0.5 T \tau^{-1} \Leftrightarrow k \geq \log T - \log \tau - 1 \Rightarrow k = \lceil \log T - \log \tau - 1 \rceil .$$

Q.E.D.

Let $S_n^{trgt} \in \{S_n^{cr-ws}, S_n^{cr-ss}\}$. There is no guarantee that

$$\mathbf{P}_{s_1, S_n^{trgt}}(t) = \sum_{s_i \in S_n^{trgt}} p_i(t)$$

is a strictly monotone function on the interval $[0, T]$. Therefore, the problem of analysis of the target set reachability can be formulated as follows.

Problem 2. The numeric values of the parameters λ_{ij} and the number ε $(0 < \varepsilon < 1)$ are given. It is necessary to find the set

$$\mathrm{T}_{s_1, S_n^{trgt}}(\varepsilon) = \{t_0 \in [0, T] \mid \mathbf{P}_{s_1, S_n^{trgt}}(t_0) \geq \varepsilon\}$$

in the explicit form.

It was noted above that there is no guarantee that $\mathbf{P}_{s_1, S_n^{trgt}}(t)$ $(S_n^{trgt} \in \{S_n^{cr-ws}, S_n^{cr-ss}\})$ is a strictly monotone function on the interval $[0, T]$. Besides, the analytical representation of this function is usually rather cumbersome. Moreover, since this expression contains transcendent functions of the form $t^m \exp(-at)$, there is no algorithm for solving problem 2 (it is well-known that there is no algorithm for finding an exact solution even for the equation $t \exp(-t) = b$, where $b > 0$).

Therefore, it is natural to confine yourself to solving the following problem.

Problem 3. The numeric values of the parameters λ_{ij}, the number ε $(0 < \varepsilon < 1)$ and the positive integer k are given. It is necessary to find the set

$$\mathrm{T}_{s_1, S_n^{trgt}}(k, \varepsilon) = \{h 2^{-k} T \mid h \in \{0, 1, \cdots, 2^k\} \;\&\; \mathbf{P}_{s_1, S_n^{trgt}}(h 2^{-k} T) \geq \varepsilon\}$$

in the explicit form.

The solution of this problem can be obtained by using the following algorithm.

Algorithm 2.

Step 1. $h := 0$, $\mathrm{T}_{s_1, S_n^{trgt}}(k, \varepsilon) := \varnothing$.

Step 2. If $\mathbf{P}_{s_1, S_n^{trgt}}(h 2^{-k} T) < \varepsilon$, then go to Step 3, else $\mathrm{T}_{s_1, S_n^{trgt}}(k, \varepsilon) := \mathrm{T}_{s_1, S_n^{trgt}}(k, \varepsilon) \cup \{h 2^{-k} T\}$.

Step 3. $h := h + 1$.

Step 4. If $h \leq 2^k$, then go to Step 3, else HALT.

Theorem 2. Algorithm 2 is sound. The number of iterations of Steps 2-4 is $2^k + 1$.

Proof. The soundness of Algorithm 2 follows from the fact that the values $\mathbf{P}_{s_1, S_n^{trgt}}(h 2^{-k} T)$ $(h = 0, 1, \cdots, 2^k)$ are sequentially calculated, and each of the these values is compared with the number ε.

The number of calculated values of the function $\mathbf{P}_{s_1, S_n^{trgt}}$ (it is the same as the number of iterations of Steps 2-4) is equal to $2^k + 1$.

Q.E.D.

It should be noted that the results obtained in this Section form some base for simulation of the DS S_n behavior when the parameters are varied.

VI. Discussion

The main aim of the given paper was to define and investigate a symbolic model based on a CT FMC and intended for analysis of recoverable, partially recoverable, and non-recoverable DSs within the finite time horizon.

The proposed model gives the possibility to derive explicitly analytical expressions for the probabilities of being the analyzed CT FMC in any of the possible states at any instant. Therefore, it is possible to obtain the analytical expressions for the sets of target states reachability at any instant in the explicit form.

Moreover, the pointed analytical expressions also provide the possibility to investigate the probabilities of the DS being at this or another functionality stage as time functions and the parameter functions, both. The latter is especially important in the process of designing DSs.

Obviously, the above mentioned analytical probability expressions can be effectively used in the process of statistical modeling and simulation [16] of the analyzed DS behavior. Significantly, that through Monte Carlo simulation [17], these analytical probability expressions can also be used for searching the most acceptable numerical values of the parameters for the designed DC.

However, it should be noted that when using the Monte Carlo Method, it is often difficult to select a set of simulated vectors of numerical parameter values, since for real DS the parameter values are not independent, as a rule. One possible approach to identify and effectively use these dependencies of numeric parameters values is the development and application of corresponding logical Model-Checking formalisms [18] in the process of DS analysis.

VII. Conclusion

In the given paper, a symbolic CT FMC model P_n intended for unified analysis of any class of DSs S_n with the same transition structure is defined and investigated. This model is designed by analogy with the symbolic Finite Markov Chains model C_n intended for unified analysis of any class of all DSs S_n with the same transition structure that has been defined and investigated in [13].

Both of these models provide some mathematical base for the development of logical Model-Checking formalisms for the process of DS analysis. Development of these formalisms forms a possible direction for further research.

It was shown that the proposed CT FMC model P_n makes it possible to derive parametric expressions for the probabilities $p_i(t)$ $(i = 1, \cdots, n)$ and their rates of change relatively to variations of parameters λ_{ij}. The Problem of the target set S_n^{trgt} reachability for DS S_n, when the numeric parameters values are given, was solved.

These results provide some mathematical base for the development and implementation of statistical simulation methods for analysis DS behavior relatively to the parameters variation. This is another possible direction for further research.

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