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A New Numerical Method for Sumudu Transform Inversion

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Abstract: *In this paper, a new method for the numerical inversion of the Sumudu Transform is proposed. Using the Laplace-Sumudu duality, a method of series expansion is presented in which the expansion is done in terms of the Legendre polynomials.*
Key Words*: Sumudu Transform; Inverse Sumudu Transform; numerical inversion; Legendre Polynomials; Laplace-Sumudu duality.*

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I. Introduction

The Sumudu transform was introduced by Watugala in the early 1990's and was effectively used in solving ordinary differential equations. The fundamental properties of this transform which are thought to be an alternative to the Laplace transform have been established in many different articles. Though many functions can be inverted directly using the table of transforms given Watugala 1993[1] and a general formula for complex inversion of the Sumudu transform was given and applied in Weerakoon 1998[2], it is desirable to have numerical methods for approximating inverses.

The principal objective in this work is to showcase a new method that has been devised for the numerical inversion of the Sumudu transform. Using the Laplace-Sumudu duality as highlighted in Belgacem 2006[3], a method of series expansion is presented, where the expansion is done in terms of orthogonal polynomials using the Legendre polynomial. In order to understand this, we need to be familiar with some basic results about Sumudu transform and the inverse Sumudu transform.

II. Methods

Sumudu Transform

If $f(t)$ is a function of exponential order and is piecewise continuous, the Sumudu transformation is defined as follows:

$$G(u) = \mathbb{S}[f(t)] = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f(t)\, dt \quad u \in (-\tau, \tau) \tag{1}$$

over the set of functions defined by

$$A = \{f(t) | \exists M, \tau_1, \tau_2 > 0 | f(t)| < Me^{|t|/\tau_j}, if\ t \in (-1)^j \times [0, \infty)\}$$

Inverse Sumudu Transform

The complex inversion formula for Sumudu transform is given by

$$f(t) = \mathbb{S}^{-\prime}[G(s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{u} G\left(\frac{1}{u}\right) e^{ut} du \tag{2}$$

Here s is the radius of the circular region while all singularities of $\frac{1}{u} G\left(\frac{1}{u}\right)$ lie in $(s) < c$. In this case, the integral of $e^{ut} \frac{1}{u} G\left(\frac{1}{u}\right)$ around the circular region will approach zero. This result can be arrived at using the Cauchy's theorem.

The Laplace-Sumudu Duality

The well known Laplace transform is defined by $F(s) = \mathcal{L}\{f(t)\}$

The close connection between the Sumudu transform and the Laplace Transform is given in the following theorem

Theorem 1

Let the function $f(t) \in A$ with Laplace transform $F(s)$. Then the Sumudu transform G of $f(t)$is given by

$$G(u) = \frac{F(\frac{1}{u})}{u} \tag{3}$$

Proof
Let $f(t) \in A$, then for $-\tau_1 < 0 < \tau_2$,
$G(u) = \int_0^\infty e^{-t} f(ut)\, dt$
If we set $w = ut$ $(t = \frac{w}{u})$, then the right-hand side can be written as
$G(u) = \int_0^\infty e^{-w/u} f(w) \frac{dw}{u} = \frac{1}{u}\int_0^\infty e^{-w/u} f(w) dw$
The integral on the right-hand side is clearly $F(\frac{1}{u})$, thus yielding the result (3)

III. Results

Numerical method for Sumudu transform inverse

The close relation between the Laplace and Sumudu transforms will guide us in the numerical formulation of the Sumudu inversion method.

We consider the Laplace method of series expansion in terms of the Legendre polynomials as presented in Cohen 2007[4] and replicate the same process for the Sumudu method, taking note of the duality identity.

Recall the Sumudu transform method given by

$$G(u) = \mathbb{S}[f(t)] = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} f(t)\, dt = \bar{f}(u) \tag{4}$$

If we make a substitution for x as $x = e^{-\frac{t}{\sigma}}$ then $t = -\sigma \ln x$ so that

$e^{-\frac{t}{\sigma}} = e^{\sigma \frac{\ln x}{u}} = x^{\frac{\sigma}{u}}$

If $t = -\sigma \ln x$ then, $dt = -\frac{\sigma}{x} dx$

Then (4) becomes $\bar{g}(u) = \int_0^1 \frac{1}{u} x^{\frac{\sigma}{u}} f(-\sigma \ln x) \; -\frac{\sigma}{x} dx$

$$g(u) = \int_0^1 -\frac{\sigma}{u} x^{\frac{\sigma}{u}-1} f(-\sigma \ln x) dx \tag{5}$$

If we let $f(-\sigma \ln x) = h(x)$ we have (5) as

$\bar{g}(u) = -\frac{\sigma}{u}\int_0^1 x^{\frac{\sigma}{u}-1} h(x)\, dx$

If we now let $u = \frac{\sigma}{2r+1}$ then we get

$\int_0^1 -(2r+1) x^{2r} h(x)\, dx$

If we define the function $h(x) = f(-\sigma \ln x)$ in the interval $[0, 1]$ by $h(-x) = h(x)$, then $h(x)$ is an even function which can be expressed as a series of even Legendre polynomials. That is $h(x) = \sum_{k=0}^{\infty} \alpha_k P_{2k}(x)$ where $x = e^{-\frac{t}{\sigma}}$ or equivalently

$$f(t) = \sum_{k=0}^{\infty} \alpha_k P_{2k}(e^{-\frac{t}{\sigma}}) \tag{6}$$

This is our value for $f(t)$, but we need to determine values of α_k. To do that, we first note that since $P_{2k}(e^{-\frac{t}{\sigma}})$ is an even polynomial of degree $2k$ in $e^{-\frac{t}{\sigma}}$ and

$S\left[e^{-2r\frac{t}{\sigma}}\right] = \frac{1}{1+\frac{2r}{\sigma}t} = \frac{\sigma}{\sigma+2rt}$ for $r = 1, 2, 3, \dots, k$ we have

$\overline{\emptyset_{2k}}(u) = S\left[P_{2k}\left(e^{-\frac{t}{\sigma}}\right)\right] = \frac{A(u)}{(\sigma+2u)(\sigma+4u)\dots(\sigma+2ku)}$

Where $A(u)$ is a polynomial of degree $\leq k$

Furthermore, because of the orthogonality of the Legendre polynomials $\int_0^1 x^{2x} P_{2k}(x) dx = 0$ for $r < k$

Hence it follows that $\overline{\emptyset_{2k}}\left(\frac{\sigma}{2r+1}\right) = 0$, $r = 0, 1, \dots, k-1$ and thus the roots of $A(u)$ are $u = \frac{\sigma}{(2r+1)}$, $r = 0, 1, \dots, k-1$

So that the Sumudu transform of $P_{2k}\left(e^{-\frac{t}{\sigma}}\right)$ is

$\overline{\emptyset_{2k}}(u) = \frac{(\sigma-u)(\sigma-3u)(\sigma-5u)\dots(\sigma-(2k-1)u)}{(\sigma+2u)(\sigma+4U)\dots(\sigma+2ku)}$

The transform of equation (3) now yields

$\bar{g}(u) = \alpha_0 + \sum_{k=1}^{\infty} \frac{(\sigma-u)\dots(\sigma-(2k-1)u)}{(\sigma+2ku)} \alpha_k$

Substituting in turns $u = \sigma, \frac{\sigma}{3}, \frac{\sigma}{5}, \dots, \frac{\sigma}{2k-1}$ brings us to the triangular system of equations

$\bar{g}(\sigma) = \alpha_0$

$\bar{g}\left(\frac{\sigma}{3}\right) = \alpha_0 + \frac{2}{5}\alpha_1$

$\bar{g}\left(\frac{\sigma}{5}\right) = \alpha_0 + \frac{4}{7}\alpha_1 + \frac{4.2}{7.9}\alpha_2$

…

$\bar{g}\left(\frac{\sigma}{2k-1}\right) = \alpha_0 + \frac{(2k-2)}{2k+1}\alpha_1 + \cdots + \frac{(2k-2)(2k-4)\ldots2}{(2k+1)(2k+3)\ldots(4k-3)}\alpha_{k-1}$

Which can be solved sequentially.

References

[1]. Watugala, G. K. (1993). Sumudu Transform: A new integral transform to solve differential equations and control engineering problems. *International Journal of Mathematical Education in Science and Technology*, 24(1), 35-43

[2]. Weerakoon, S. (1998). Complex inversion formula for Sumudu transform. *International Journal of Mathematical Education in Science and Technology*, *29*(4), 618-620.

[3]. Belgacem, F. B. M., and Karaballi, A. A. (2006). Sumudu transform fundamental properties investigations and applications. *International Journal of Applied mathematics and Stochastic Analysis*, *2006*.

[4]. A. M. Cohen. Numerical Methods for Laplace Transform Inversion, Springer, 2007

Mrumun C. Soomiyol, et. al. "A New Numerical Method for Sumudu Transform Inversion." *IOSR Journal of Mathematics (IOSR-JM)*, 17(4), (2021): pp. 40-42.