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On The Existence of Invariant Measures for Stochastic Evolution Equations Driven By Lévy Process

Onwukwe Ijioma
Department of Mathematics, Abia state University, Uturu,Abia State, Nigeria.

Abstract
We consider a stochastic evolution equation on a separable Hilbert space H *defined by* $dX = (AX(t) + F\big(X(t)\big)dt + B\big(X(t)\big)dZ(t)$

$X(0) = \eta,$ *(1)*

where $\eta \in H, A$ *is a linear operator ,* F *is a bounded mapping from* H *into* H, Z *takes values in a separable Hilbert space,* U *and* $B,$ *is a bounded mapping from* H *into space of linear continuous operators from* U *into* $H.$

We derive a sufficient condition for the existence of an invariant measure in a global case when Z *is a Lévy process.*
Keywords: *Brownian Motion, Evolution Equation, Hilbert space, Lévy Process, Stochastic Evolution Equation.*

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I. Introduction

The study on invariant measures for (stochastic) dynamical systems is an important topic in the theory of (stochastic) dynamical systems. These measures provide certain invariant characterization such as ergodicity, strong or weak asymptotic stability for the processes described by the systems. The research on the existence, uniqueness and regularity of invariant measures for stochastic evolution equations in Hilbert spaces has received a lot of attention (see [4,5,6,11]). Some useful methods have been developed in dealing with the invariant measures of stochastic evolution equations in Hilbert spaces in terms of different conditions on the coefficients of the stochastic evolution equations (SEEs). For the existence of invariant measures, the Krylov–Bogoliubov criterion is a powerful tool. Some efficient methods (so-called compactness method and dissipativity method) have been used by Da Prato, Gatarek and Zabczyk [3] and were further applied to study some specific equations. There are mainly three ways to show the uniqueness of invariant measures. The first one is to verify the strong Feller property (SFP) and irreducibility (I) (see [1,2,10,12]). In this case, one usually had to assume that the noise term is nondegenerate. The second one is the so-called Lyapunov approach (see [7,8,9]). In this case, the SEE's admit degenerate noise. The third one is the dissipative method developed by Da Prato, G¸atarek and Zabczyk [3]. In this case, the drift coefficients can satisfy some dissipativity (for example, having certain polynomial growth). There are also some other methods of proving uniqueness of invariant measures (see [11]). The regularity of invariant measures in the Hilbert space setting has been studied by Da Prato and Zabczyk.

In recent times, the growing interest in stochastic evolution equations with white noise driven by various dimensional Wiener process has received robust research. Extensive work of the existence and uniqueness have been studied see [13 and 15]. The research of infinite dimensional equations driven by not well developed, especially in the area of invariant measure in the case of Gaussian noise [15]. This paper is to prove and extend the work on the existence of invariant measure in infinite dimensional wiener process to the case of levy process. We considered a stochastic evolution equation in a separable Hilbert space driven by Hilbert space valued process.

Let $H \, and \, E$ be separable real Hilbert spaces, let A be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0} \; in \; E, let \; F: E \to E \; and \; G: E \to L(H, E)$ be defined as $\|F(x) - F(y)\| \leq L_F \|x - y\|,$

$\|G(x) - G(y)\| \leq L_G \|x - y\|, \; \forall x, y \in E, for \; some \; constants \; L_F, L_G > 0.$

$Let\ L(H,E)$ denotes the space of all bounded linear operators from $H\ to\ F\ and\ \|\cdot\|$ be the norm. Let $Let\ (\Omega,\mathscr{F},\mathcal{P})$ be a probability space and $(\mathscr{F}_t)_{t\geq 0}$ be a filtration in $\mathscr{F}$, $let\ (Z(t))_{t\geq 0}$ be a family of random variables on $(\Omega,\mathscr{F},\mathcal{P})$ such that:

$Z(t)\ is\ \mathscr{F}_t$ measurable, $Z(t+u)-Z(t)$ is independent of $\mathscr{F}_t$.

$Z(t+u)-Z(t)\ and\ Z(s+u)-Z(s)$ be of the same distribution $\forall\ s,t,u\geq 0$

such that $Z(0)=0\ and\ sup_{0\leq t\leq T}E\|Z(t)\|^2<\infty\ \forall\ T\geq 0$.

Theorem: Assume condition in (1) and assume that there exist $M\geq 0\ and\ \alpha>0$ such that
$\|S(t)\|\leq Me^{-\alpha t}\ \forall\ t\geq 0.\ Let\ \Lambda=((\mathbb{E}\|Z(1)-\mathbb{E}Z(1)\|^2)^{1/2}\ if\ L_F\ and\ L_G\ are\ small\ that$

$6M^2(L_F^2/\alpha\ +L_G^2\Lambda)<\infty$, then there exist a unique invariant measure $\mu\ of\ (1)\ with\ \int_E\|x\|^2d\mu(x)<\infty$

Consider processes on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$. Let $Z(t)$ be a Lévy process taking values in a separable Hilbert space $(U,\|\cdot\|_u)$.

Let $Z(t)$ be associated with two measures on U and Let the measure of jumps of Z be μ and Lévy measure of Zbe v.

Let $\mu([0,t],\Gamma)=\sum_{0\leq s\leq t\leq}1_\Gamma\big(Z(s)-Z(s^-)\big)$,

$$tv(\Gamma)=\mathbb{E}(\mu([0,t],\Gamma)) \quad (2)$$

Where Γ is a Borel subset of U such that $\overline{\Gamma}\subseteq U\setminus\{0\}$.

It implies that $v(\{0\})=0$ and $\int_u \min(\|\cdot\|_u^2,1)\ v(dy)<\infty$ (3)

The Lévy process $Z(t)$ is represented by

$$Z(t)=at+W(t)+\int_0^t\int_{\|y\|_{u\leq 1}}y(\mu(dy,ds)-v(dy)(ds)+\int_0^t\int_{\|y\|_{u>1}}y\mu(dy,ds) \quad (4)$$

where $a\in U,W$ is Wiener process with value points in U with covariance operator Q.

Let $L(H)$ be the space of linear continuous operator from another separable Hilbert space $(H,\|\cdot\|)$. Also Let $L(U,H)$ be the space of linear continuous operator from $U\ into\ H$.

Consider a semi-linear stochastic evolution equation in a separable real Hilbert space H written as $dX(t)=[AX(t)+\mathscr{F}(X(t)]dt+G(X(t)dZ\ ,\ t\geq 0,\ X(0)=\eta.$ (5)

where $\eta\in H$, the noise driven by Hilbert space valued process $(Z(t))_{t\geq 0}$ with stationary and independent increments and locally bounded second moments, A is a linear operator with dense domain, $\mathscr{F}$ is a bounded mapping from $H\ to\ L(U,H)$.

We state the following conditions:

(1) $C=\int_u\|y\|_u^2v(dy)<\infty$

(2) A is the infinitesimal generator of a strongly continuous semi-group on H,

(3) $\exists\ \mathscr{L}_\alpha>0$ such that $\|\alpha(x)-\alpha(y)\|\leq\mathscr{L}_\alpha\|x-y\|,\forall x,y\in H\ for\ some\ \mathscr{L}_\alpha>0$

(4) $\exists\mathscr{L}_\beta>0\ such\ that\ \|\beta(x)-\beta(y)\|_{\mathscr{L}(U,H)}\leq\mathscr{L}_\beta\|x-y\|$

Now, condition (1) implies the existence of $\int_{\|y\|_u>1}yv(dy)$ u

We have, $\int_{\|y\|_u>1} \|y\|_u v(dy) \leq \int_{\|y\|_u>1} \|y\|_u^2 v(dy) \leq \int_u \|y\|_u^2 v(dy) < \infty$ (6)

$\exists \bar{\beta} = \int_{\|y\|_u>1} yv(dy) \in U$

Then $Z(t) = at + W(t) + \int_0^t \int_{\|y\|_u \leq 1} y(\mu(dy, ds) - v(dy)(ds) + \int_0^t \int_{\|y\|_{u>1}} y\mu(dy, ds)$

$-\int_0^t \int_{\|y\|_u>1} yv(dy, ds) + bt$ (7)

$= (a+b)t + W(t) + \int_0^t \int_u y(\mu(dy, ds) - v(dy)ds)$ (8)

So that, $EZ(1) = a + b \; and \; VarZ(1) = Var\, W(1) + \int_0^1 \int_u \|y\|^2 v(dy)ds = Tr\Lambda + k$ (9)

For process$Z(t), t \geq 0, Let\ \hat{Z}(t)$ be a process defined by

$$\hat{Z}(t) = \begin{cases} Z(t) & , \quad t \geq 0 \\ Z_L(-t), & \quad t < 0 \end{cases} \quad , t \in \mathbb{R} \tag{10}$$

where $(Z_L(t))$ is a Lévy process with same distribution as $(Z\,(t))$ and independent of $(Z\,(t))\,_{t\geq 0}$.

DEFINITION 1: Let H be a separable real Hilbert space. Let a probability space be given as $(\Omega, \mathscr{F}, \mathbb{P})$. Let $Z = (Z(t))_{t\geq 0}$ be a family of H-valued random variables defined on $(\Omega, \mathscr{F}, \mathbb{P})$.

Then Z is a process with independent increments.

If (a) for every $t \geq s \geq 0$ the increment $Z(t) - Z(s)$is independent of the $\sigma-$algebra generated by $\{Z(u): 0 \leq u \leq s\}, Z$ has stationary increment,

If (b) for every $s, t, u \geq 0$, the increments $\big(Z(t+u) - Z(u)\big)\; and\; (Z(s+u) - Z(s))$ have the same distribution.

The family $(Z(t))$ is called a Lévy process if in addition to (a) and (b), we have

(c) $Z(0) = 0,\ \mathbb{P} - a.s.$

(d) $t \to Z(t)$ is continuous in probability; if $t \to t_0, then\ \mathbb{P}(\|Z(t) - Z(t_0)\| > \epsilon)\ \forall \epsilon > 0$.

The process Z is called $(\mathscr{F}_t)_t$ stationary independent increments process if Z is $(\mathscr{F}_t)_t$ adapted.

The Z has stationary increments, and for every $t \geq s \geq 0$ the increment $Z(t) - Z(s)$ is independent of $(\mathscr{F}_s)$.

LEMMA 1: Let $(H, \|\cdot\|)$ be a normed space. Let $T > 0$ and Let$u: [0, T] \to H$ be additive, that is, $u(s+t) = u(s) + u(t)\quad for\ every\ s, t \geq 0\ with\ s + t \leq T$. If u is bounded on $[0, T]$, then $u(t) = \left({}^{t}/_{T}\right)u(T)\ \ \forall\ t \in [0, T]$. (11)

LEMMA 2: If $sup_{0\leq t\leq T}\mathbb{E}\|Z(t)\| < \infty\ \ \forall\ T > 0\ , then\ \mathbb{E}Z(t) = t\mathbb{E}Z(1)\ \forall\ T \geq 0$. (12)
$\mathbb{E}\|Z(t)\|^2 < \infty\ \ \forall\ t > 0$, then $\mathbb{E}\big\|\hat{Z}(t)\big\|^2 = t\mathbb{E}\big\|\hat{Z}(1)\big\|^2\ \ \forall\ t \geq 0$,
where $\hat{Z}(t) = Z(t) - \mathbb{E}Z(t), t \geq 0$. (13)

II. Invariant Measure:

We now consider the existence of invariant measures to the problem of stochastic evolution equations driven by Lévy process, more general noise processes than Wiener processes.

Consider the processes with parameters such as , zero means by transformation.

PROPOSITION 1. Assume that $\mathbb{E}\|Z(t)\|^2 < \infty \,\forall t \geq 0$, Let $m(t) = \mathbb{E}Z(t), \hat{Z}(t) = Z(t) - m(t), t \geq 0$

The statements are equivalent:

(1) $Sup_{0\leq t\leq T}\, \mathbb{E}\|Z(t)\|^2 < \infty \;\forall\, T \geq 0,$
(2) $Sup_{0\leq t\leq T}\, \mathbb{E}\|Z(t)\| < \infty \;\forall\, T \geq 0,$
(3) $m(t) = tm(1) \;\forall\, t \geq 0,$
(4) $Sup_{0\leq t\leq T}\, \|m(t)\| < \infty \;\forall\, T \geq 0,$
(5) $t \to Z(t)$ is continuous in probability.

Now, if $(Z(t)_{t\geq 0}$ is Lévy process such that $\mathbb{E}\|Z(t)\|^2 < \infty \;\forall\, t \geq 0,$ $then\; \mathbb{E}Z(t) = t\mathbb{E}Z(1)$ and $\mathbb{E}\|Z(t) -$ *EZ(t)2=tEZ1−EZ(1)2 , t≥0* (14)

We conclude that the process $Z = (Z(t))_{t\geq 0}$ has locally bounded second moments if $Sup_{0\leq t\leq T}\, \mathbb{E}\|Z(t)\|^2 < \infty \;, \forall\, T \geq 0$

***Proof*:** (1)⇒ (2) since $\mathbb{E}\|Z(t)\| \leq (\mathbb{E}\|Z(t)\|^2)^{1/2} \;\forall\, t \geq 0$

(2)⇒ (3)

(3) ⇒(4) is trivial

We need to show that (4) ⇒(1),

$$\mathbb{E}\left\|\hat{Z}\,(t)\right\|^2 = \mathbb{E}[\|Z(t)\|^2 - 2\langle Z(t), m(t)\rangle + \|m(t)\|^2]$$

$$= \mathbb{E}[\|\mathrm{Z(t)}\|^2 - 2\langle \mathbb{E}\mathrm{Z(t)}, \mathrm{m(t)}\rangle + \|\mathrm{m(t)}\|^2]$$

$$= \mathbb{E}[\|Z(t)\|^2 - \|m(t)\|^2] \;\;\forall\, t \geq 0$$

Hence , $Sup_{0\leq t\leq T}\, \mathbb{E}\|Z(t)\|^2 \leq T\mathbb{E}\left\|\hat{Z}(1)\right\|^2 + (Sup_{0\leq t\leq T}\|m(t)\|)^2.$

III. Existence Of An Invariant Measure

The existence and the uniqueness of an invariant measure is considered in this section.
$Let\; (\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, we consider a multidimensional diffusion process $X(t) = X_t$ which is a solution of the following stochastic differential problem

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

$$X_0 = u_0 \in \mathbb{R}^d. \tag{15}$$

Where $(W_t)_{t\in[0,T]}$ is the standard d-dimentional Brownian motion,$d \in N$, the drift coefficient $b{:}\,\mathbb{R}^d$ and $\sigma{:}\,\mathbb{R}^d \to \mathscr{L}(\mathbb{R}^m, \mathbb{R}^d)$, $m \in N$ are Lipschitz continuous function. The problem of (15) admits the following stochastic representation:

$S_t\; \varphi(x) = \mathbb{E}[\varphi(X(t,x)], x \in \mathbb{R}^d \;, \varphi \in \mathcal{C}_b\left(\mathbb{R}^d\right), where\; S_t, t \geq 0$, is the corresponding transition semi group, $\mathcal{C}_b(\mathbb{R}^d)$ denotes the space of all functions from $\mathbb{R}^d$ into $\mathbb{R}$ that are uniformly continuous and bounded, $\mathbb{E}$ denotes the conditional expectation.

If $u_0{:}\,\mathbb{R}^d \to \mathbb{R}$ is a regular function, then the following function

$$u(t,x) = (S_t u_0)(x) = \mathbb{E}[u_0(X_t)]. \tag{16}$$

Is the unique solution of the following problem:

$$\frac{\partial U}{\partial t} + \mathscr{L}u = 0 \;in\; [0, +\infty\,) \times \mathbb{R}^d \;, u(0,x) = u_0 \;in\; \mathbb{R}^d. \tag{17}$$

where $\mathscr{L}$ is the linear , second-order uniformly elliptic operator associated with a diffusion process in the defined space.

The infinitesimal generator of the process (15) is given as

$$\mathscr{L} = -a(x): \mathcal{D}^2 - b(x)\nabla. \tag{18}$$

Where the matrix $a(x) = (a_{ij}(x))$ is defined as follows:

$$a_{ij}(x) = \frac{1}{2}\sum_{\gamma=1}^{k} \sigma_{i\gamma}(x)\sigma_{j\gamma}(x) \quad and \quad a(x): \mathcal{D}^2 = trace[a\mathcal{D}^2] = \sum_{i,j=1}^{d} a_{ij}\, \partial_{ij} \tag{19}$$

∇ *and* $\mathcal{D}^2$ denote the gradient and the Hessian operators with respect to the spatial variable x respectively.

Furthermore, state below the proposition of the auxiliary mapping of Lyapunov function to establish the existence of invariant measure.

PROPOSITION 2:

Let $V: H \to [0, +\infty]$ be a Borel function whose level sets
$K_a = \{x \in H: V(x) \leq a\}, a > 0$ are compact for any $a > 0$. Assume that $\exists\, x_0 \in \mathbb{R}^n$ *and* $\mathcal{C}(x_0) > 0$

Such that $[V(X(t, x_0))] \leq \mathcal{C}(x_0)$, $\forall\, t \geq 0$. (20)

Then Λ is nonempty.

If in addition $\exists\, \mathcal{C} > 0$ such that $\mathbb{E}[V(X(t,x))] \leq \mathcal{C}\ \forall\, t \geq 0, x \in H$ (21)

Then Λ is tight and $\int_{R^n} V(x)\mu(dx) \leq \mathcal{C}\ \forall\ \mu \in \Lambda$. (22)

IV. Conclusion

We have examined the invariant measures for stochastic evolution equations driven by Lévy process in Hilbert space and derive a sufficient condition for the existence of an invariant measure in a global case for a more corrupted noise than Wiener processes. The conditions can be extended in the solution to the problems of classical heat equation and functional differential equation.

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