IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 17, Issue 6 Ser. I (Nov. – Dec. 2021), PP 35-40
www.iosrjournals.org

Finding an Easy Way to Calculate Sum of Powers of Natural Numbers ($\sum n^k$) By Integration Method

(A Project done in 2014 for the mathematics fair)

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Abstract: *This particular method is for finding the fastest way to calculate the sum of powers of natural numbers by integration method*
Keywords: *integration, powers of natural numbers, integration, coefficients, algebraic method*

Date of Submission: 01-12-2021 Date of Acceptance: 15-12-2021

I. Introduction

We all know that the sum of powers of natural numbers has a crucial role in mathematics and natural sciences. It has comprehensive application in vast areas of simple mathematics problems to the analysis of black holes. For formulating the sum of powers of natural numbers, the conventional method is algebraic. It is tedious work. If the power is n, we have to solve n+1 equations to find the coefficients. This work is the solution for the ridiculous time-consuming algebraic method of formulating.

There will be an accidental understanding behind every invention. Because of that understanding, I have done this research. If $\langle f_k(n) \rangle = 1^K, 2^K, 3^K, \ldots\ldots\ldots, n^K$ then $\langle f_{k+1}(n) \rangle = 1^{K+1}, 2^{K+1}, 3^{K+1}, \ldots\ldots\ldots, n^{K+1}$. When I was Learning calculus, I saw a gripping formula in integration

$\int_0^n x^{th}\, term\ of\ \langle f_k(n) \rangle dx = \frac{n^{K+1}}{K+1} = \frac{1}{K+1} \times n^{th} term\ of\ \langle f_{K+1}(n) \rangle$, and this relation tells us that there exists an integral relationship between the nth terms of $\langle f_K(n) \rangle$ and $\langle f_{K+1}(n) \rangle$. That fact forced me to think about the possibility of an interrelation between the sum of n terms of $\langle f_{K+1}(n) \rangle$ using the sum of n terms of $\langle f_K(n) \rangle$ by integration.

This method itself becomes the proof for the vanishing of some terms of polynomial in the formulae of $\sum n^k$

II. Data required

- $f_1(n) = \frac{n(n+1)}{2}$
- $f_2(n) = \frac{n(n+1)(2n+1)}{6}$
- $f_3(n) = [\frac{n(n+1)}{2}]^2$
- $f_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
- $f_5(n) = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

- $$f_6(n) = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$$

III. Learning Activity

(1)Finding the formulae for $1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots + n^2 = f_2(n)$ *from*

$1 + 2 + 3 + 4 + \ldots\ldots. + n = f_1(n)$

$$f_1(n) = 1 + 2 + 3 + 4 + \ldots\ldots. + n = \frac{n}{2}(n+1)$$

Writing it as polynomial = $\frac{n^2}{2} + \frac{n}{2}$

Find $\int_0^n f_1(x) \cdot dx$

$$\int_0^n f_1(x) \cdot dx = \int_0^n \left(\frac{x^2}{2} + \frac{x}{2}\right).dx = \frac{n^3}{6} + \frac{n^2}{4}$$

Existing formulae for $f_2(n) = 1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Comparing coefficient of n^3 and n^2 in $\int_0^n f_1(x) \cdot dx$

(a) Coefficient of n^3 in $2 \times \int_0^n f_1(x) \cdot dx$ = coefficient of n^3 in $f_2(n) = \frac{1}{3}$

(b) Coefficient of n^2 in $2 \times \int_0^n f_1(x) \cdot dx$ = coefficient of n^2 in $f_2(n) = \frac{1}{2}$

And linear term in $f_2(n)$ = 1 – [sum of coefficient of nonlinear terms]

$$= 1 - \left[\frac{1}{3} + \frac{1}{2}\right] = \frac{1}{6}$$

$\therefore f_2(n) = \frac{n(n+1)(2n+1)}{6}$

(2)Finding the formulae for $f_3(n) = 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots + n^3$ *from*

$1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots + n^2 = f_2(n)$

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots\ldots + n^2 = f_2(n) = \frac{n(n+1)(2n+1)}{6}$$

Writing it as polynomial $= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

Find $\int_0^n f_2(x)\cdot dx$

$$\int_0^n f_2(x)\cdot dx = \int_0^n (\frac{x^3}{3}+\frac{x^2}{2}+\frac{x}{6}).dx = \frac{n^4}{12}+\frac{n^3}{6}+\frac{n^2}{12}$$

Existing formulae for = $f_3(n) = 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots + n^3 = [\frac{n(n+1)}{2}]^2$

$$= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

Comparing coefficients of n^4, n^3, and n^2 in

(a) Coefficient of n^4 in $3\times\int_0^n f_2(x)\cdot dx$ = coefficient of n^4 in $f_3(n) = \frac{1}{4}$

(b) Coefficient of n^3 in $3\times\int_0^n f_2(x)\cdot dx$ = coefficient of n^3 in $f_3(n) = \frac{1}{2}$

(c) Coefficient of n^2 in $3\times\int_0^n f_2(x)\cdot dx$ = coefficient of n^2 in $f_3(n) = \frac{1}{4}$

And the coefficient of the linear term in $f_3(n)$ = 1 – [sum of coefficient of nonlinear terms]

$$=1 - [\frac{1}{4}+\frac{1}{2}+\frac{1}{4}] = 0$$

$\therefore\ f_3(n) = [\frac{n(n+1)}{2}]^2$

(3) Finding the formulae for $f_4(n) = 1^4 + 2^4 + 3^4 + \ldots\ldots + n^4$ *from* $f_3(n) = 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots + n^3$

$$f_3(n) = 1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots + n^3 = [\frac{n(n+1)}{2}]^2$$

Writing it as polynomial $= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$

Find $\int_0^n f_3(x)\cdot dx$

$$\int_0^n f_3(x)\cdot dx = \int_0^n (\frac{x^4}{4}+\frac{x^3}{2}+\frac{x^2}{4}).dx = \frac{n^5}{20}+\frac{n^4}{8}+\frac{n^3}{12}$$

Existing formulae for $f_4(n) = 1^4 + 2^4 + 3^4 + \ldots\ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

$$= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

Comparing coefficient of n^5, n^4, $n^3$and n^2 in $\int_0^n f_3(x)\cdot dx$

(a) Coefficient of n^5 in $4 \times \int_0^n f_3(x) \cdot dx$ = coefficient of n^5 in $f_4(n) = \frac{1}{5}$

(b) Coefficient of n^4 in $4 \times \int_0^n f_3(x) \cdot dx$ = coefficient of n^4 in $f_4(n) = \frac{1}{2}$

(c) Coefficient of n^3 in $4 \times \int_0^n f_3(x) \cdot dx$ = coefficient of n^3 in $f_4(n) = \frac{1}{3}$

(d) Coefficient of n^3 in $4 \times \int_0^n f_3(x) \cdot dx$ = coefficient of n^2 in $f_4(n)$ =0

And the coefficient of the linear term in $f_3(n)$ = 1 – [sum of coefficient of nonlinear terms]

$$= 1 - [\frac{1}{5} + \frac{1}{2} + \frac{1}{3}] = \frac{1}{30}$$

$$\therefore \; f_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

(4)Finding the formulae for $1^5 + 2^5 + 3^5 + \ldots\ldots + n^5 = f_5(n)$ from $f_4(n) = 1^4 + 2^4 + 3^4 + \ldots\ldots + n^4$

$$f_4(n) = 1^4 + 2^4 + 3^4 + \ldots\ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Writing it as polynomial $= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

Find $\int_0^n f_4(x).dx$

$$\int_0^n f_4(x).dx = \int_0^n [\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30}].dx = \frac{n^6}{30} + \frac{n^5}{10} + \frac{n^4}{12} - \frac{n^2}{60}$$

Existing formulae for $f_5(n) = 1^5 + 2^5 + 3^5 + \ldots\ldots + n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

$$= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

Comparing coefficient of n^6, n^5, n^4, n3 and n^2 in $\int_0^n f_4(x).dx$

(a) Coefficient of n^6 in $5 \times \int_0^n f_4(x).dx$ = coefficient of n^6 in $f_5(n) = \frac{1}{6}$

(b) Coefficient of n^5 in $5 \times \int_0^n f_4(x).dx$ = coefficient of n^5 in $f_5(n) = \frac{1}{2}$

(c) Coefficient of n^4 in $5 \times \int_0^n f_4(x).dx$ = coefficient of n^4 in $f_5(n) = \frac{5}{12}$

(d) Coefficient of n^3 in $5 \times \int_0^n f_4(x).dx$ = coefficient of n^3 in $f_5(n) = 0$

(e) Coefficient of n^2 in $5 \times \int_0^n f_4(x).dx$ = coefficient of n^2 in $f_5(n) = -\frac{1}{12}$

And the coefficient of the linear term in $f_3(n)$ = 1 – [sum of coefficient of nonlinear terms]

$$= 1 - \left[\frac{1}{6} + \frac{1}{2} + \frac{5}{12} - \frac{1}{12}\right] = 0$$

$$\therefore\ f_5(n) = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

(5)Finding the formulae for $1^6 + 2^6 + 3^6 + \ldots\ldots + n^6 = f_6(n)$ *from* $1^5 + 2^5 + 3^5 + \ldots\ldots + n^5 = f_5(n)$

$$1^5 + 2^5 + 3^5 + \ldots\ldots + n^5 = f_5(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Writing it as a polynomial $= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$

Find $\int_0^n f_5(x).dx$

$$\int_0^n f_5(x).dx = \int_0^n [\frac{x^6}{6} + \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12}].dx = \frac{n^7}{42} + \frac{n^6}{12} + \frac{n^5}{12} - \frac{n^3}{36}$$

Existing formulae for $f_6(n) = 1^6 + 2^6 + 3^6 + \ldots\ldots + n^6 = \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42}$

$$= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

Comparing coefficient of n^7, n^6, n^5,n^4, n^3 and n^2 in $\int_0^n f_5(x).dx$

(a) Coefficient of n^7 in $6 \times \int_0^n f_5(x).dx$ = coefficient of n^7 in $f_6(n) = \frac{1}{7}$

(b) Coefficient of n^6 in $6 \times \int_0^n f_5(x).dx$ = coefficient of n^6 in $f_6(n) = \frac{1}{2}$

(c) Coefficient of n^5 in $6 \times \int_0^n f_5(x).dx$ = coefficient of n^5 in $f_6(n) = \frac{1}{2}$

(d) Coefficient of n^4 in $6 \times \int_0^n f_5(x).dx$ = coefficient of n^4 in $f_6(n) = 0$

(e) Coefficient of n^3 in $6 \times \int_0^n f_5(x).dx$ = coefficient of n^3 in $f_6(n) = -\frac{1}{6}$

(f) Coefficient of n^2 in $6 \times \int_0^n f_5(x).dx$ = coefficient of n^2 in $f_6(n) = 0$

And the coefficient of the linear term in $f_3(n)$ = 1 – [sum of coefficient of nonlinear terms]

$$= 1 - \left[\frac{1}{7} + \frac{1}{2} + \frac{1}{2} - \frac{1}{6}\right] = \frac{1}{42}$$

$$\therefore\ f_6(n) = \frac{n(n+1)(2n+1)(3n^4 + 6n^3 - 3n + 1)}{42}$$

IV. Tabular Analysis

1	2	3	4	5	6	7
K	$f_K(n)$	Sum of n terms of $f_K(n)$ as polynomial	$F_{K+1}(n)$	Sum of n terms of $F_{K+1}(n)$ as polynomial	$\int_0^n f_K(x).dx$	Relation between $f_K(n)$ and $F_{K+1}(n)$
1	$f_1(n)$	$\frac{n^2}{2}+\frac{n}{2}$	$f_2(n)$	$\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$	$\frac{n^3}{6}+\frac{n^2}{4}$	Nonlinear terms of column 5 = 2×nonlinear terms of column 6
2	$f_2(n)$	$\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$	$f_3(n)$	$\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2$	$\frac{n^4}{12}+\frac{n^3}{6}+\frac{n^2}{12}$	Nonlinear terms of column 5 = 3×nonlinear terms of column 6
3	$f_3(n)$	$\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2$	$f_4(n)$	$\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$	$\frac{n^5}{20}+\frac{n^4}{8}+\frac{n^3}{12}$	Nonlinear terms of column 5 = 4×nonlinear terms of column 6
4	$f_4(n)$	$\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$	$f_5(n)$	$\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2$	$\frac{n^6}{30}+\frac{n^5}{10}+\frac{n^4}{12}-\frac{n^2}{60}$	Nonlinear terms of column 5 = 5×nonlinear terms of column 6
5	$f_5(n)$	$\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2$	$f_6(n)$	$\frac{1}{7}n^7+\frac{1}{2}n^6+\frac{1}{2}n^5-\frac{1}{6}n^3+\frac{1}{42}n$	$\frac{n^7}{42}+\frac{n^6}{12}+\frac{n^5}{12}-\frac{n^3}{36}$	Nonlinear terms of column 5 = 6×nonlinear terms of column 6

V. Conclusion

- $f_{K+1}(n) = (K+1)\int_0^n f_K(x).dx$ + Linear term

  Such that, 1-[Sum of coefficients of nonlinear terms] = The coefficient of linear term

- We know, $\int_0^n ax^m = \frac{an^{m+1}}{K+1}$ if $a = 0$ then the term n^{m+1} vanishes

When we consider the integration method, if the formulae $f_K(n)$ don't have the term n^m then n^{m+1} never exist in the formulae of $f_{K+1}(n)$

Acknowledgment

I express my sincere thanks to
Mr. Unni Krishnan, SSMHSS Theyyalingal, Kerala, India
Ms. Santhi, SSMHSS Theyyalingal, Kerala, India
And to all who helped me to complete this project

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Silpadas N. "Finding An Easy Way To Calculate Sum of Powers Of Natural Numbers ($\sum n^k$) By Integration Method." *IOSR Journal of Mathematics (IOSR-JM)*, 17(6), (2021): pp. 35-40.