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Analysis of an Inventory System for Items with Price-Ramp type Dependent Demand and Time Dependent Three-Parameter Weibull Deterioration Function

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Abstract
This article focuses on the development of a mathematical model describing an inventory system with time dependent three-parameter Weibull deterioration, and witha Ramp-type demand rate, which is a quadratic function of time in the beginning of a cycle, and subsequently becomes linear over time. The model incorporates shortages which are partially backlogged.We derived the optimal inventory policy for the model and established the necessary and sufficient conditions for the optimal policy. The objective of the model is to minimize the total inventory costs. For the numerical solution of the model, MathCAD14 numerical software was employed. We obtained satisfactory results. We then proceeded to performed sensitivity analysis of our model. The sensitivity analysis showed that the optimal solution of the model is affected by slight changes or errors in its input parameter values.
Keywords: Inventory System, Price-Ramp, Dependent Demand and Deterioration Function

I. Introduction

The classical inventory modelsuch as presented by Harris-Wilson [11] considers the ideal situation in which depletion of inventory is caused by a constant demand rate alone. However, it was observedafterwards that depletion of inventory may take place due to deterioration. Virtually all items deteriorate over time with the exception of items such as hardware, glassware, steel etc. On the other hand, all perishables such as food items, chemicals etc. deteriorate quite rapidly over time and become unsuitable for consumption. This loss must be taken into account when analyzing inventory systems. In recent times several research articles on the above subject have appeared in the literature including Chakrabarti et al [1]. Covert and Philip [4] developed a two parameter Weibull distribution deterioration for an inventory model. This investigation was followed by Datta and Pal [5], Jalan et al. [12], Dixit and Shah [7], Giri et al. [10], Shah et al. [15] and more recently Nwoba et al. [19], [20] analyzed an inventory system for items with stochastic demand andtime dependent three-parameter Weibull deterioration function and Analysis of Inventory system for items with price-dependent demand and time dependent three-parameter weibull deterioration function. Some other researchers have also presented inventory control studies where the demand is inverse Gaussian and the lead time is gamma[2].

The main objective of inventory management for deteriorating items is to obtain optimal returns during the useful lifetime of the product [18]. This leads to three main issues: determining reasonable and appropriate methods for issuing inventory, replenishing inventory and allocating inventory. The choice of inventory valuation methods adopted in issuing inventory (i.e. the order in which the items are to be issued), such as methods based on time sequence including FIFO (first-in, first-out) and LIFO (last-in, first-out), depends on both the intrinsic characteristics of the inventory (e.g. lifetime, quantity, variety, issuing frequency etc.) and the influence on the company (e.g. inventory balance, cost of goods sold etc.) [21].In the present paper, we consider an EOQ model for inventory of items that deteriorate following a time dependent three-parameter Weibull deterioration and price-dependent demand rate.

1.1 Mathematical formulation

A rich literature on modelling of deteriorating inventory shows how the deterioration of products has been captured in the research problem up till now. To integrate deterioration into mathematical models, the model type (deterministic or stochastic) and the considered time horizon (infinite or finite) lead to specific methods [8]. In what follows we consider the basic characteristics of EOQ inventory models.

1.2Ramp-type demand function

It is observed that, the demand rate of an item is influenced by the selling price of an item, as, whenever the selling price of an item increases, the demand decreases and vice-versa. Generally, this type of demand is seen for finished goods. Several authors have investigated this type of inventory model. According to the market research, it is observed that time to time advertisement of an item can also affect its demand. The demand rates of these items may be dependent on displayed stock level. Such types of demand in different forms were considered by Maiti [16], Chung et al [3]. All these models considered either linear or non-linear form of demand and derived results. In this research work we propose a demand rate that is a non-linear function of time*t*, and given by

$$D(t) = a + bt + c\left[t - (t-\mu)H(t-\mu)\right]t, \qquad (1)$$

where

$$H(t-\mu) = \begin{cases} 0, & t < \mu \\ 1, & t \geq \mu \end{cases}$$

Thus,

$$D(t) = \begin{cases} a + \kappa t, & t \geq \mu \\ a + bt + ct^2, & t < \mu \end{cases} \qquad (2)$$

where $\kappa = b + c\mu$. The Heaviside function is depicted in the graph shown in Fig. 1.

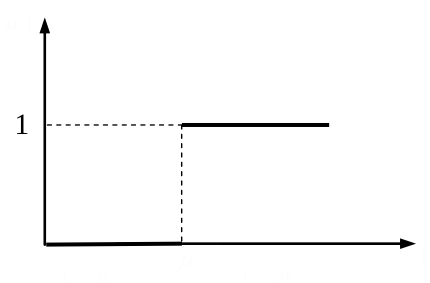

Fig.1 The Heaviside function

1.3 Proposed Deterioration Model

Threeparameter Weibull distribution applied in this study is shown in equation (2);

$$W(t) = \alpha\beta(t-\gamma)^{\beta-1}\exp\left(-\alpha(t-\gamma)^{\beta}\right),\ t > 0, \qquad (3)$$

is often used for modeling lifetime data. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shape. Rinne [10] suggested that a three-parameter generalization of the Weibull distribution deals with general situations in modeling survival process with various shapes in the hazard function. Chakrabarty*et al*. [1] provided rationale for considering three-parameter Weibull deterioration rate. They discovered that many products that start deteriorating appreciably only after a certain period (e.g. after they are produced) and for which the rate of deterioration increases over time have a deterioration rate best described by a Weibull distribution. We consider the following inventory data adapted from Ghosh and Chaudhuri [9] and Sahaand Chakrabarti [22].

1.4 Notations and Assumptions of the Model

We adopt the following notations and assumptions in the derivation of our model.
Notations:
(i) Replenishment size is constant.
(ii) Lead time is zero.
(iii) T is the fixed length of each production cycle (cycle time)
(iv) c_1 is the inventory holding cost per unit per unit time.
(v) c_2 is the deterioration cost per unit per unit time.
(vi) c_3 is the shortage cost of each deteriorated unit.
(vii) c_4 is the unit cost of lost sales.
(viii) c' is the ordering cost
(ix) I_0 is initial inventory size.

1.5Assumptions

(i) The inventory system under consideration deals with single item.
(ii) The planning horizon is infinite.
(iii)The demand rate is non-linear as a function of time$t$, and specified by equation (1)
(iv) Shortages in the inventory are allowed and completely backlogged.
(v) The supply is instantaneous and the lead time is zero.
(vi) Deteriorated unit is not repaired or replaced during a given cycle.
(vii) The holding cost, ordering cost, shortage cost and unit cost remain constant overtime.
(viii) There are no quantity discounts.
(ix) The distribution of the time to deterioration of the items follows the three-parameter Weibull distribution, given in equation (2). The instantaneous rate function is $\theta(t) = \alpha\beta(t-\gamma)^{\beta-1}$.
(x) $P(t) = \sigma D(t)$ is the production rate where $\sigma > 1$ is a constant.
(xi) Unsatisfied demand is backlogged at a rate $\exp(-\lambda t)$, where $t$is the time up to next replenishment and λ is a positive constant.

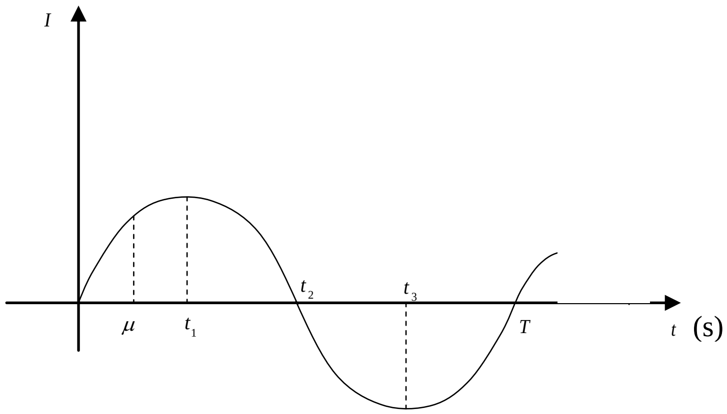

Fig.2Stock level as a function of time

II. The Mathematical Model

We will take the initial stock to be zero. Production starts just after t=0and continues up to $t=t_1$when the stock reaches a level S. Production is stopped at $t=t_1$. Inventory accumulated in [0, t_1] after meeting the demands is used in[t_1,t_2]. The stock reaches the zero level at time $t2$. Now shortages start to develop and accumulate to the level L at$t=t_3$. Production starts at time $t3$. The running demands as well as the backlog for [$t_{2,3}$] are satisfied in [t_3,T]. The inventory again reaches the zero level at time T. The cycle then repeats itself after a scheduling time T. Our objective is to determine the optimum values of K,$_1$,t_2,$t_3$and $T$with the assumptions stated above. The situation is depicted in figure 2.

The differential equations governing the instantaneous states of $I(t)$ in the interval [0,] are as follows:

$$\frac{dI(t)}{dt} + \theta(t)I(t) = \sigma D(t) - D(t), \quad 0 \le t < \mu; \; I(0) = 0. \tag{4}$$

$$\frac{dI(t)}{dt} + \theta(t)I(t) = \sigma D(t) - D(t), \quad \mu \le t < t_1; \; I(t_1) = S. \tag{5}$$

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -D(t), \quad t_1 \le t < t_2; \; I(t_2) = 0. \tag{6}$$

$$\frac{dI(t)}{dt} = \sigma D(t) - D(t), \quad t_2 \le t < t_3; \; I(t_3) = -L. \tag{7}$$

$$\frac{dI(t)}{dt} = \sigma D(t) - D(t), \quad t_3 \le t < T; \; I(T) = 0. \tag{8}$$

Substituting the relevant parameters, we obtain the required differential equations

$$\frac{dI(t)}{dt} + \alpha\beta(t-\gamma)^{\beta-1} I(t) = (\sigma-1)(a+bt+ct^2), \quad 0 \le t < \mu; \; I(0) = 0. \tag{9}$$

$$\frac{dI(t)}{dt} + \alpha\beta(t-\gamma)^{\beta-1} I(t) = (\sigma-1)(a+\kappa t), \quad \mu \le t < t_1; \; I(t_1) = S. \tag{10}$$

$$\frac{dI(t)}{dt} + \alpha\beta(t-\gamma)^{\beta-1} I(t) = -(a+\kappa t), \quad t_1 \le t < t_2; \; I(t_2) = 0. \tag{11}$$

$$\frac{dI(t)}{dt} = -(a+\kappa t)e^{-\lambda t}, \quad t_2 \le t < t_3; \; I(t_3) = -L. \tag{12}$$

$$\frac{dI(t)}{dt} = (\sigma-1)(a+\kappa t), \quad t_3 \le t < T; \; I(T) = 0. \tag{13}$$

We begin by solving the simple ODE's (12) and (11) following the order of difficulty.

$$\frac{dI(t)}{dt} = (a+\kappa t)e^{-\lambda t}$$

$$\Rightarrow \quad I(t) = \int (a+\kappa t)e^{-\lambda t} = \left\{ -\left(\frac{a\lambda + \kappa\lambda t + \kappa}{\lambda^2} \right) e^{-\lambda t} \right\} + C$$

$$\therefore \quad I(t) = -\left(\frac{a\lambda + \kappa\lambda t + \kappa}{\lambda^2} \right) e^{-\lambda t} + C$$

$$I(t_3) = -\left(\frac{a\lambda + \kappa\lambda t_3 + \kappa}{\lambda^2} \right) e^{-\lambda t_3} + C = -L \quad \Rightarrow \quad C = \left(\frac{a\lambda + \kappa\lambda t_3 + \kappa}{\lambda^2} \right) e^{-\lambda t_3} - L$$

$$\therefore \quad I(t) = \left(\frac{a\lambda + \kappa\lambda t + \kappa}{\lambda^2} \right) e^{-\lambda t} + \left(\frac{a\lambda + \kappa\lambda t_3 + \kappa}{\lambda^2} \right) e^{-\lambda t_3} - L \tag{14}$$

Next we consider;

$$\frac{dI(t)}{dt} = (\sigma-1)(a+\kappa t)$$

$$\therefore \quad I(t) = (\sigma-1)\int (a+\kappa t) + D = (\sigma-1)\left(at + \tfrac{1}{2}\kappa t^2\right) + D$$

$$I(T) = (\sigma-1)\left(aT + \tfrac{1}{2}\kappa T^2\right) + D = 0 \quad \Rightarrow \quad D = -(\sigma-1)\left(aT + \tfrac{1}{2}\kappa T^2\right)$$

$$\therefore \quad I(t) = (\sigma-1)\left(at + \tfrac{1}{2}\kappa t^2\right) - (\sigma-1)\left(aT + \tfrac{1}{2}\kappa T^2\right)$$

$$\Rightarrow \quad I(t) = \tfrac{1}{2}(\sigma-1)(t-T)\left[2a + \kappa(t+T)\right], \quad t_3 \le t \le T \tag{15}$$

The next ODE is

$$\frac{dI(t)}{dt} + \alpha\beta(t-\gamma)^{\beta-1} I(t) = -(a+\kappa t), \quad t_1 \le t < t_2; \; I(t_2) = 0.$$

The above equation is a first order differential equation and its integrating factor is:

$$\exp\left[\alpha\beta\int (t-\gamma)^{\beta-1} dt\right] = e^{\alpha(t-\gamma)^\beta}$$

$$\frac{d}{dt}\left[I(t) e^{\alpha(t-\gamma)^\beta} \right] = -(a+\kappa t)e^{\alpha(t-\gamma)^\beta}$$

$$\therefore \quad \left[I(t) e^{\alpha(t-\gamma)^\beta} \right]_t^{t_2} = -\int_t^{t_2} (a+\kappa t)e^{\alpha(t-\gamma)^\beta} dt$$

$$\Rightarrow \qquad \left[I(t) e^{\alpha(t-\gamma)^{\beta}} \right]_{t}^{t_2} = -\int_{t}^{t_2} (a + \kappa t) e^{\alpha(t-\gamma)^{\beta}} dt$$

$$\Rightarrow \qquad I(t_2) e^{\alpha(t_2-\gamma)^{\beta}} - I(t) e^{\alpha(t-\gamma)^{\beta}} = -\int_{t}^{t_2} (a + \kappa t) e^{\alpha(t-\gamma)^{\beta}} dt$$

$$\therefore \qquad I(t) e^{\alpha(t-\gamma)^{\beta}} = I(t_2) e^{\alpha(t_2-\gamma)^{\beta}} + \int_{t}^{t_2} (a + \kappa t) e^{\alpha(t-\gamma)^{\beta}} dt$$

$$\Rightarrow \qquad I(t) = I(t_2) e^{\alpha(t_2-\gamma)^{\beta} - \alpha(t-\gamma)^{\beta}} + e^{-\alpha(t-\gamma)^{\beta}} \int_{t}^{t_2} (a + \kappa t) e^{\alpha(t-\gamma)^{\beta}} dt$$

Since $I(t_2) = 0$, we have finally;

$$I(t) = e^{-\alpha(t-\gamma)^{\beta}} \int_{t}^{t_2} (a + \kappa t) e^{\alpha(t-\gamma)^{\beta}} dt, \qquad t_1 \le t < t_2 \text{ .(16)}$$

Next we consider the differential equation (9)

$$\frac{dI(t)}{dt} + \alpha\beta (t-\gamma)^{\beta-1} I(t) = (\sigma - 1)(a + \kappa t), \quad \mu \le t < t_1; \quad I(t_1) = S.$$

As in the previous case, its integrating factor is: $\exp\left[\alpha\beta \int (t-\gamma)^{\beta-1} dt \right] = e^{\alpha(t-\gamma)^{\beta}}$

$$\frac{d}{dt}\left[I(t) e^{\alpha(t-\gamma)^{\beta}} \right] = (\sigma - 1)(a + \kappa t) e^{\alpha(t-\gamma)^{\beta}}$$

$$\therefore \quad \left[I(t) e^{\alpha(t-\gamma)^{\beta}} \right]_{t}^{t_1} = (\sigma - 1) \int_{t}^{t_1} (a + \kappa t) \left\{ 1 + \alpha (t-\gamma)^{\beta} \right\} dt$$

Here we employed the first order approximation of the integrand as before.

$$\therefore \quad I(t_1) e^{\alpha(t_1-\gamma)^{\beta}} - I(t) e^{\alpha(t-\gamma)^{\beta}} = (\sigma - 1) \int_{t}^{t_1} (a + \kappa t) e^{\alpha(t-\gamma)^{\beta}} dt$$

$$\Rightarrow \quad I(t) e^{\alpha(t-\gamma)^{\beta}} = I(t_1) e^{\alpha(t_1-\gamma)^{\beta}} - (\sigma - 1) \int_{t}^{t_1} (a + \kappa t) e^{\alpha(t-\gamma)^{\beta}} dt$$

$$\therefore \quad I(t) = I(t_1) e^{\alpha(t_1-\gamma)^{\beta} - \alpha(t-\gamma)^{\beta}} - (\sigma - 1) e^{-\alpha(t-\gamma)^{\beta}} \int_{t}^{t_1} (a + \kappa t) e^{\alpha(t-\gamma)^{\beta}} dt$$

Since $I(t_1) = S$, it then follows that;

$$I(t) = S e^{\alpha(t_1-\gamma)^{\beta} - \alpha(t-\gamma)^{\beta}} - (\sigma - 1) e^{-\alpha(t-\gamma)^{\beta}} \int_{t}^{t_1} (a + \kappa t) e^{\alpha(t-\gamma)^{\beta}} dt, \qquad \mu \le t < t_1. \qquad (17)$$

Finally, we solve equation (8), i.e.

$$\frac{dI(t)}{dt} + \alpha\beta (t-\gamma)^{\beta-1} I(t) = (\sigma - 1)\left(a + bt + ct^2\right), \quad 0 \le t < \mu; \quad I(0) = 0.$$

Again we employ the integrating factor: $\exp\left[\alpha\beta \int (t-\gamma)^{\beta-1} dt \right] = e^{\alpha(t-\gamma)^{\beta}}$

$$\frac{d}{dt}\left[I(t) e^{\alpha(t-\gamma)^{\beta}} \right] = (\sigma - 1)\left(a + bt + ct^2\right) e^{\alpha(t-\gamma)^{\beta}} \quad ,.$$

$$\therefore \quad \left[I(t) e^{\alpha(t-\gamma)^{\beta}} \right]_{t}^{\mu} = (\sigma - 1) \int_{t}^{\mu} \left(a + bt + ct^2\right) e^{\alpha(t-\gamma)^{\beta}} dt$$

$$\Rightarrow \quad I(\mu) e^{\alpha(\mu-\gamma)^{\beta}} - I(t) e^{\alpha(t-\gamma)^{\beta}} = (\sigma - 1) \int_{t}^{\mu} \left(a + bt + ct^2\right) e^{\alpha(t-\gamma)^{\beta}} dt$$

But $I(0) = 0$, hence

$$I(\mu) e^{\alpha(\mu-\gamma)^{\beta}} - I(0) e^{\alpha(0-\gamma)^{\beta}} = (\sigma - 1) \int_{0}^{\mu} \left(a + bs + cs^2\right) e^{\alpha(s-\gamma)^{\beta}} ds$$

$$\Rightarrow \qquad I(\mu) e^{\alpha(\mu-\gamma)^{\beta}} = (\sigma - 1) \int_{0}^{\mu} \left(a + bs + cs^2\right) e^{\alpha(s-\gamma)^{\beta}} ds$$

$$\therefore \qquad I(\mu) = (\sigma - 1) e^{-\alpha(\mu-\gamma)^{\beta}} \int_{0}^{\mu} \left(a + bs + cs^2\right) e^{\alpha(s-\gamma)^{\beta}} ds$$

Therefore;

$$I(t) = I(\mu)e^{\alpha(\mu-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}} - (\sigma-1)e^{-\alpha(t-\gamma)^{\beta}}\int_t^{\mu}\left(a+bs+cs^2\right)e^{\alpha(s-\gamma)^{\beta}}ds, \quad \mu \le t < t_1. \quad (18)$$

The solutions to the differential equations (8) to (12) are summarized respectively below;

Hence, the inventory level at any time $t \in [0,T]$ is given by the solutions to the differential equations (8) to (12) are summarized below;

$$\begin{cases} I(t) = I(\mu)e^{\alpha(\mu-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}} - (\sigma-1)e^{-\alpha(t-\gamma)^{\beta}}\int_t^{\mu}\left(a+bs+cs^2\right)e^{\alpha(s-\gamma)^{\beta}}ds, & 0 \le t < \mu \\ I(t) = Se^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}} - (\sigma-1)e^{-\alpha(t-\gamma)^{\beta}}\int_t^{t_1}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}}ds, & \mu \le t < t_1 \\ I(t) = e^{-\alpha(t-\gamma)^{\beta}}\int_t^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}}ds, & t_1 \le t < t_2 \\ I(t) = \left(\dfrac{a\lambda+\kappa\lambda t+\kappa}{\lambda^2}\right)e^{-\lambda t} + \left(\dfrac{a\lambda+\kappa\lambda t_3+\kappa}{\lambda^2}\right)e^{-\lambda t_3} - L, & t_2 \le t < t_3 \\ I(t) = \frac{1}{2}(\sigma-1)(t-T)\left[2a+\kappa(t+T)\right], & t_3 \le t \le T \end{cases} \quad (19)$$

The inventory holding cost during the period $(0,T)$ is given by

$$C_H = c_1\int_0^{\mu} I(t)dt + c_1\int_{\mu}^{t_1} I(t)dt + c_1\int_{t_1}^{t_2} I(t)dt \quad (20)$$

The costdue to deterioration of units in the period $(0,T)$ is given by

$$C_D = c_2\int_0^{\mu}\theta(t)I(t)dt + c_2\int_{\mu}^{t_1}\theta(t)I(t)dt + c_2\int_{t_1}^{t_2}\theta(t)I(t)dt \quad (21)$$

with $\theta(t) = \alpha\beta(t-\gamma)^{\beta-1}$ .The cost due to shortages in the period $(0,T)$ is given by;

$$C_S = -c_3\int_{t_2}^{t_3} I(t)dt - c_3\int_{t_3}^{T} I(t)dt \quad (22)$$

The opportunity cost due to lost sales in the period $(0,T)$ is given by;

$$C_o = c_4\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt \quad (23)$$

The total cost ψ in the system over the period $(0,T)$ is given by;

$$\psi(t_1,t_2,t_3,T) = c' + C_H + C_D + C_S + C_o \quad (24)$$

c' being a constant. The average cost ϕ per cycle is given by;

$$\phi(t_1,t_2,t_3,T) = \frac{1}{T}\psi(t_1,t_2,t_3,T) \quad (25)$$

Hence

$$\phi(t_1,t_2,t_3,T) = \frac{1}{T}\begin{Bmatrix} c' + c_1\int_0^{\mu} I(t)dt + c_1\int_{\mu}^{t_1} I(t)dt + c_1\int_{t_1}^{t_2} I(t)dt \\ + c_2\int_0^{\mu}\theta(t)I(t)dt + c_2\int_{\mu}^{t_1}\theta(t)I(t)dt + c_2\int_{t_1}^{t_2}\theta(t)I(t)dt \\ - c_3\int_{t_2}^{t_3} I(t)dt - c_3\int_{t_3}^{T} I(t)dt + c_4\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt \end{Bmatrix} \quad (26)$$

The necessary conditions for minimization of $\phi(t_1,t_2,t_3,T)$ are;

$$\partial\phi(t_1,t_2,t_3,T)/\partial t_1 = \partial\phi(t_1,t_2,t_3,T)/\partial t_2 = \partial\phi(t_1,t_2,t_3,T)/\partial t_3 = \partial\phi(t_1,t_2,t_3,T)/\partial T = 0 \quad (27)$$

The above provides us with the optimum values of t_1, t_2, t_3 and T which minimize the average cost ϕ .

Now we have;

$$\frac{\partial \phi(t_1,t_2,t_3,T)}{\partial t_1}=\frac{1}{T}\left\{\begin{array}{l} c_1\frac{\partial}{\partial t_1}\int_0^{\mu} I(t)dt+c_1\frac{\partial}{\partial t_1}\int_{\mu}^{t_1} I(t)dt+c_1\frac{\partial}{\partial t_1}\int_{t_1}^{t_2} I(t)dt \\ +c_2\frac{\partial}{\partial t_1}\int_0^{\mu}\theta(t)I(t)dt+c_2\frac{\partial}{\partial t_1}\int_{\mu}^{t_1}\theta(t)I(t)dt+c_2\frac{\partial}{\partial t_1}\int_{t_1}^{t_2}\theta(t)I(t)dt \\ -c_3\frac{\partial}{\partial t_1}\int_{t_2}^{t_3} I(t)dt-c_3\frac{\partial}{\partial t_1}\int_{t_3}^{T} I(t)dt+c_4\frac{\partial}{\partial t_1}\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt \end{array}\right\} \quad (28)$$

In $0\le t<\mu$, $\frac{\partial}{\partial t_1}\int_0^{\mu} I(t)dt=0$ since the integrand does not involve t_1. Similarly $\frac{\partial}{\partial t_1}\int_0^{\mu}\theta(t)I(t)dt=0$, $\frac{\partial}{\partial t_1}\int_{t_2}^{t_3} I(t)dt=0$, $\frac{\partial}{\partial t_1}\int_{t_3}^{T} I(t)dt=0$ and $\frac{\partial}{\partial t_1}\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)f(t)dt=0$. Hence

$$\frac{\partial \phi(t_1,t_2,t_3,T)}{\partial t_1}=\frac{1}{T}\left\{c_1\frac{\partial}{\partial t_1}\int_{\mu}^{t_1} I(t)dt+c_1\frac{\partial}{\partial t_1}\int_{t_1}^{t_2} I(t)dt+c_2\frac{\partial}{\partial t_1}\int_{\mu}^{t_1}\theta(t)I(t)dt+c_2\frac{\partial}{\partial t_1}\int_{t_1}^{t_2}\theta(t)I(t)dt\right\} \quad (29)$$

Employingin what follows, the Lebnitz rule for differentiating the integral $I(\alpha)=\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx$ given by

$$\frac{dI(\alpha)}{d\alpha}=f(b,\alpha)\frac{db}{d\alpha}-f(b,\alpha)\frac{da}{d\alpha}+\int_a^b\frac{\partial f(x,\alpha)}{\partial\alpha}dx \quad (30)$$

Thus

$$\frac{\partial}{\partial t_1}\int_{\mu}^{t_1} I(t)dt=I(t_1)+\int_{\mu}^{t_1}\frac{\partial}{\partial t_1}I(t)dt \text{, and}$$

$$I(t)=Se^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}}-(\sigma-1)e^{-\alpha(t-\gamma)^{\beta}}\int_t^{t_1}(a+\kappa t)e^{\alpha(t-\gamma)^{\beta}}dt$$

$$\begin{aligned}\therefore\quad \frac{\partial}{\partial t_1}I(t)&=\frac{\partial}{\partial t_1}\left[Se^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}}\right]-(\sigma-1)e^{-\alpha(t-\gamma)^{\beta}}\frac{\partial}{\partial t_1}\int_t^{t_1}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}}ds\\ &=S\alpha\beta(t_1-\gamma)^{\beta-1}e^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}}-(\sigma-1)e^{-\alpha(t-\gamma)^{\beta}}\left[(a+\kappa t_1)e^{\alpha(t_1-\gamma)^{\beta}}+\int_t^{t_1}\frac{\partial}{\partial t_1}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}}ds\right]\\ &=S\alpha\beta(t_1-\gamma)^{\beta-1}e^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}}-(\sigma-1)(a+\kappa t_1)e^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}}\\ &=\left[S\alpha\beta(t_1-\gamma)^{\beta-1}-(\sigma-1)(a+\kappa t_1)\right]e^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}}\end{aligned}$$

(31)

Similarly

$\frac{\partial}{\partial t_1}\int_{t_1}^{t_2} I(t)dt=-I(t_1)+\int_{t_1}^{t_2}\frac{\partial}{\partial t_1}I(t)dt$, with $I(t)=e^{-\alpha(t-\gamma)^{\beta}}\int_t^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}}ds$ in $t_1\le t<t_2$.

$$\therefore\quad \frac{\partial}{\partial t_1}\int_{t_1}^{t_2} I(t)dt=-I(t_1)=-e^{-\alpha(t_1-\gamma)^{\beta}}\int_{t_1}^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}}ds \quad (32)$$

Next we have;

$\frac{\partial}{\partial t_1}\int_{\mu}^{t_1}\theta(t)I(t)dt=\theta(t_1)I(t_1)+\int_{\mu}^{t_1}\frac{\partial}{\partial t_1}\theta(t)I(t)dt=\theta(t_1)I(t_1)$ with $\theta(t)=\alpha\beta(t-\gamma)^{\beta-1}$ and

$I(t)=Se^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}}-(\sigma-1)e^{-\alpha(t-\gamma)^{\beta}}\int_t^{t_1}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}}ds$ in $\mu\le t<t_1$

For the last term in equation (28), we have

$$\frac{\partial}{\partial t_1}\int_{t_1}^{t_2}\theta(t)I(t)dt=-\theta(t_1)I(t_1)+\int_{t_1}^{t_2}\frac{\partial}{\partial t_1}\theta(t)I(t)dt=-\theta(t_1)I(t_1) \quad (33)$$

with $I(t) = e^{-\alpha(t-\gamma)^{\beta}} \int_{t}^{t_2} (a+\kappa s) e^{\alpha(s-\gamma)^{\beta}} ds$

in $t_1 \le t < t_2$.

$$\therefore \quad \frac{\partial \phi(t_1,t_2,t_3,T)}{\partial t_1} = \frac{1}{T}\left\{ \begin{array}{l} c_1\left[S\alpha\beta(t_1-\gamma)^{\beta-1} - (\sigma-1)(a+\kappa t_1)\right] e^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}} \\ +c_1\left[-e^{-\alpha(t_1-\gamma)^{\beta}} \int_{t_1}^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}} ds\right] + c_2\left[\theta(t_1) I(t_1)\right] + c_2\left[-\theta(t_1) I(t_1)\right] \end{array} \right\}$$

$$= \frac{c_1}{T}\left\{ \left[S\alpha\beta(t_1-\gamma)^{\beta-1} - (\sigma-1)(a+\kappa t_1)\right] e^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}} - e^{-\alpha(t_1-\gamma)^{\beta}} \int_{t_1}^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}} ds \right\}$$

(34)

We now consider

$$\frac{\partial \phi(t_1,t_2,t_3,T)}{\partial t_2} = \frac{1}{T}\left\{ \begin{array}{l} c_1 \frac{\partial}{\partial t_2}\int_0^{\mu} I(t)dt + c_1 \frac{\partial}{\partial t_2}\int_{\mu}^{t_1} I(t)dt + c_1 \frac{\partial}{\partial t_2}\int_{t_1}^{t_2} I(t)dt \\ \quad + c_2 \frac{\partial}{\partial t_2}\int_0^{\mu} \theta(t) I(t)dt + c_2 \frac{\partial}{\partial t_2}\int_{\mu}^{t_1} \theta(t) I(t)dt + c_2 \frac{\partial}{\partial t_2}\int_{t_1}^{t_2} \theta(t) I(t)dt \\ \quad - c_3 \frac{\partial}{\partial t_2}\int_{t_2}^{t_3} I(t)dt - c_3 \frac{\partial}{\partial t_2}\int_{t_3}^{T} I(t)dt + c_4 \frac{\partial}{\partial t_2}\int_{t_2}^{t_3} \left(1-e^{-\lambda t}\right) D(t)dt \end{array} \right\} \quad (35)$$

As before, in $0 \le t < \mu$, $\frac{\partial}{\partial t_2}\int_0^{\mu} I(t)dt = 0$ since the integrand does not involve t_2.

Similarly $\frac{\partial}{\partial t_2}\int_{\mu}^{t_1} I(t)dt = 0$, $\frac{\partial}{\partial t_2}\int_0^{\mu} \theta(t) I(t)dt = 0$, $\frac{\partial}{\partial t_2}\int_{\mu}^{t_1} \theta(t) I(t)dt = 0$, and $\frac{\partial}{\partial t_2}\int_{t_3}^{T} I(t)dt = 0$.

$$\therefore \quad \frac{\partial \phi(t_1,t_2,t_3,T)}{\partial t_2} = \frac{1}{T}\left\{ c_1 \frac{\partial}{\partial t_2}\int_{t_1}^{t_2} I(t)dt + c_2 \frac{\partial}{\partial t_2}\int_{t_1}^{t_2} \theta(t) I(t)dt - c_3 \frac{\partial}{\partial t_2}\int_{t_2}^{t_3} I(t)dt + c_4 \frac{\partial}{\partial t_2}\int_{t_2}^{t_3} \left(1-e^{-\lambda t}\right) D(t)dt \right\}$$

(36)

We have;

$\frac{\partial}{\partial t_2}\int_{t_1}^{t_2} I(t)dt = I(t_2) + \int_{t_1}^{t_2} \frac{\partial}{\partial t_2} I(t)dt$ with $I(t) = e^{-\alpha(t-\gamma)^{\beta}} \int_{t}^{t_2} (a+\kappa s) e^{\alpha(s-\gamma)^{\beta}} ds$ in $t_1 \le t < t_2$.

$$\frac{\partial}{\partial t_2} I(t) = \frac{\partial}{\partial t_2}\left[e^{-\alpha(t-\gamma)^{\beta}} \int_{t}^{t_2} (a+\kappa s) e^{\alpha(s-\gamma)^{\beta}} ds \right]$$

$$= e^{-\alpha(t-\gamma)^{\beta}} \frac{\partial}{\partial t_2}\int_{t}^{t_2} (a+\kappa s) e^{\alpha(s-\gamma)^{\beta}} ds = (a+\kappa t_2) e^{\alpha(t_2-\gamma)^{\beta}} + \int_{t}^{t_2} \frac{\partial}{\partial t_2}(a+\kappa s) e^{\alpha(s-\gamma)^{\beta}} ds \quad (37)$$

$$= (a+\kappa t_2) e^{\alpha(t_2-\gamma)^{\beta}}$$

Hence

$$\frac{\partial}{\partial t_2}\int_{t_1}^{t_2} I(t)dt = I(t_2) + \int_{t_1}^{t_2} \frac{\partial}{\partial t_2} I(t)dt = e^{-\alpha(t_2-\gamma)^{\beta}} \int_{t_2}^{t_2} (a+\kappa s) e^{\alpha(s-\gamma)^{\beta}} ds + (a+\kappa t_2) e^{\alpha(t_2-\gamma)^{\beta}} \quad (38)$$

$$= (a+\kappa t_2) e^{\alpha(t_2-\gamma)^{\beta}}$$

Next we consider;

$$\frac{\partial}{\partial t_2}\int_{t_1}^{t_2} \theta(t) I(t)dt = \theta(t_2) I(t_2) + \int_{t_1}^{t_2} \frac{\partial}{\partial t_2}\theta(t) I(t)dt \quad (39)$$

with $I(t) = e^{-\alpha(t-\gamma)^{\beta}} \int_{t}^{t_2} (a+\kappa s) e^{\alpha(s-\gamma)^{\beta}} ds$ in $t_1 \le t < t_2$.

$$\frac{\partial}{\partial t_2}\theta(t)I(t)=\theta(t)e^{-\alpha(t-\gamma)^\beta}\frac{\partial}{\partial t_2}\int_t^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^\beta}ds$$

$$=\theta(t)e^{-\alpha(t-\gamma)^\beta}\left[(a+\kappa t_2)e^{\alpha(t_2-\gamma)^\beta}+\int_t^{t_2}\frac{\partial}{\partial t_2}(a+\kappa s)e^{\alpha(s-\gamma)^\beta}ds\right] \quad (40)$$

$$=\theta(t)(a+\kappa t_2)e^{\alpha(t_2-\gamma)^\beta-\alpha(t-\gamma)^\beta}$$

Therefore

$$\frac{\partial}{\partial t_2}\int_{t_1}^{t_2}\theta(t)I(t)dt=\theta(t_2)I(t_2)+\int_{t_1}^{t_2}\theta(t)(a+\kappa t_2)e^{\alpha(t_2-\gamma)^\beta-\alpha(t-\gamma)^\beta}dt$$

$$=\theta(t_2)e^{-\alpha(t_2-\gamma)^\beta}\int_{t_2}^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^\beta}ds+\int_{t_1}^{t_2}\theta(t)(a+\kappa t_2)e^{\alpha(t_2-\gamma)^\beta-\alpha(t-\gamma)^\beta}dt \quad (41)$$

$$=\int_{t_1}^{t_2}\theta(t)(a+\kappa t_2)e^{\alpha(t_2-\gamma)^\beta-\alpha(t-\gamma)^\beta}dt$$

Next we have

$$\frac{\partial}{\partial t_2}\int_{t_2}^{t_3}I(t)dt=-I(t_2)+\int_{t_2}^{t_3}\frac{\partial}{\partial t_2}I(t)dt \quad (42)$$

with $I(t)=\left(\frac{a\lambda+\kappa\lambda t+\kappa}{\lambda^2}\right)e^{-\lambda t}+\left(\frac{a\lambda+\kappa\lambda t_3+\kappa}{\lambda^2}\right)e^{-\lambda t_3}-L$ in $t_2\le t<t_3$.

$$\therefore \quad \frac{\partial}{\partial t_2}\int_{t_2}^{t_3}I(t)dt=-I(t_2)=-\left(\frac{a\lambda+\kappa\lambda t_2+\kappa}{\lambda^2}\right)e^{-\lambda t_2}-\left(\frac{a\lambda+\kappa\lambda t_3+\kappa}{\lambda^2}\right)e^{-\lambda t_3}+L \quad (43)$$

For the last term in ( ) we have;

$$\frac{\partial}{\partial t_2}\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt=-\left(1-e^{-\lambda t_2}\right)D(t_2)+\int_{t_2}^{t_3}\frac{\partial}{\partial t_2}\left(1-e^{-\lambda t}\right)D(t)dt \quad (44)$$

with $D(t)=a+\kappa t$ for $t\ge\mu$. Hence

$$\frac{\partial}{\partial t_2}\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt=-\left(1-e^{-\lambda t_2}\right)D(t_2)=-\left(1-e^{-\lambda t_2}\right)(a+\kappa t_2) \quad (45)$$

$$\therefore \quad \frac{\partial\phi(t_1,t_2,t_3,T)}{\partial t_2}=\frac{1}{T}\left\{\begin{array}{l}c_1(a+\kappa t_2)e^{\alpha(t_2-\gamma)^\beta}+c_2(a+\kappa t_2)e^{\alpha(t_2-\gamma)^\beta}\int_{t_1}^{t_2}\theta(t)e^{-\alpha(t-\gamma)^\beta}dt-c_4\left(1-e^{-\lambda t_2}\right)(a+\kappa t_2)\\ -c_3\left[-\left(\frac{a\lambda+\kappa\lambda t_2+\kappa}{\lambda^2}\right)e^{-\lambda t_2}-\left(\frac{a\lambda+\kappa\lambda t_3+\kappa}{\lambda^2}\right)e^{-\lambda t_3}+L\right]\end{array}\right\}$$

(46)

Next we consider;

$$\frac{\partial\phi(t_1,t_2,t_3,T)}{\partial t_3}=\frac{1}{T}\left\{\begin{array}{l}c_1\underbrace{\frac{\partial}{\partial t_3}\int_0^{\mu}I(t)dt}_{0}+c_1\underbrace{\frac{\partial}{\partial t_3}\int_{\mu}^{t_1}I(t)dt}_{0}+c_1\underbrace{\frac{\partial}{\partial t_3}\int_{t_1}^{t_2}I(t)dt}_{0}\\ +c_2\underbrace{\frac{\partial}{\partial t_3}\int_0^{\mu}\theta(t)I(t)dt}_{0}+c_2\underbrace{\frac{\partial}{\partial t_3}\int_{\mu}^{t_1}\theta(t)I(t)dt}_{0}+c_2\underbrace{\frac{\partial}{\partial t_3}\int_{t_1}^{t_2}\theta(t)I(t)dt}_{0}\\ -c_3\frac{\partial}{\partial t_3}\int_{t_2}^{t_3}I(t)dt-c_3\frac{\partial}{\partial t_3}\int_{t_3}^{T}I(t)dt+c_4\frac{\partial}{\partial t_3}\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt\end{array}\right\} \quad (47)$$

$$=\frac{1}{T}\left\{-c_3\frac{\partial}{\partial t_3}\int_{t_2}^{t_3}I(t)dt-c_3\frac{\partial}{\partial t_3}\int_{t_3}^{T}I(t)dt+c_4\frac{\partial}{\partial t_3}\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt\right\}$$

We have

$\frac{\partial}{\partial t_3}\int_{t_2}^{t_3} I(t)dt = I(t_3) + \int_{t_2}^{t_3}\frac{\partial}{\partial t_3} I(t)dt$ with $I(t) = \left(\frac{a\lambda + \kappa\lambda t + \kappa}{\lambda^2}\right)e^{-\lambda t} + \left(\frac{a\lambda + \kappa\lambda t_3 + \kappa}{\lambda^2}\right)e^{-\lambda t_3} - L$

in $t_2 \le t < t_3$.

Now

$$\frac{\partial}{\partial t_3} I(t) = \frac{\partial}{\partial t_3}\left[\left(\frac{a\lambda + \kappa\lambda t + \kappa}{\lambda^2}\right)e^{-\lambda t} + \left(\frac{a\lambda + \kappa\lambda t_3 + \kappa}{\lambda^2}\right)e^{-\lambda t_3} - L\right]$$

$$= \frac{\partial}{\partial t_3}\left[\left(\frac{a\lambda + \kappa\lambda t_3 + \kappa}{\lambda^2}\right)e^{-\lambda t_3}\right] = \frac{\kappa}{\lambda}e^{-\lambda t_3} - \left(\frac{a\lambda + \kappa\lambda t_3 + \kappa}{\lambda}\right)e^{-\lambda t_3} = -(a + \kappa t_3)e^{-\lambda t_3}$$

$$\therefore \quad \frac{\partial}{\partial t_3}\int_{t_2}^{t_3} I(t)dt = \left[\left(\frac{a\lambda + \kappa\lambda t_3 + \kappa}{\lambda^2}\right)e^{-\lambda t_3} + \left(\frac{a\lambda + \kappa\lambda t_3 + \kappa}{\lambda^2}\right)e^{-\lambda t_3} - L\right] - \int_{t_2}^{t_3}(a + \kappa t_3)e^{-\lambda t_3}dt$$
$$= \frac{2}{\lambda^2}(a\lambda + \kappa\lambda t_3 + \kappa)e^{-\lambda t_3} - (t_3 - t_2)(a + \kappa t_3)e^{-\lambda t_3} - L \qquad (48)$$

The next term is

$\frac{\partial}{\partial t_3}\int_{t_3}^{T} I(t)dt = -I(t_3) + \int_{t_3}^{T}\frac{\partial}{\partial t_3} I(t)dt$ with $I(t) = \frac{1}{2}(\sigma - 1)(t - T)\left[2a + \kappa(t + T)\right]$ in $t_3 \le t \le T$.

But $\frac{\partial}{\partial t_3} I(t) = 0$.

$$\therefore \quad \frac{\partial}{\partial t_3}\int_{t_3}^{T} I(t)dt = -\frac{1}{2}(\sigma - 1)(t_3 - T)\left[2a + \kappa(t_3 + T)\right] \qquad (49)$$

For the last term in (48) we have

$$\frac{\partial}{\partial t_3}\int_{t_2}^{t_3}\left(1 - e^{-\lambda t}\right)D(t)dt = \left(1 - e^{-\lambda t_3}\right)D(t_3) + \int_{t_2}^{t_3}\frac{\partial}{\partial t_3}\left(1 - e^{-\lambda t}\right)D(t)dt$$
$$= \left(1 - e^{-\lambda t_3}\right)D(t_3) \qquad (50)$$

$$\therefore \quad \frac{\partial \phi(t_1, t_2, t_3, T)}{\partial t_3} = \frac{1}{T}\left\{\begin{array}{l} -\frac{2c_3}{\lambda^2}(a\lambda + \kappa\lambda t_3 + \kappa)e^{-\lambda t_3} + c_3(t_3 - t_2)(a + \kappa t_3)e^{-\lambda t_3} + c_3 L \\ + \frac{1}{2}c_3(\sigma - 1)(t_3 - T)\left[2a + \kappa(t_3 + T)\right] + c_4\left(1 - e^{-\lambda t_3}\right)D(t_3) \end{array}\right\} \qquad (51)$$

Finally we have;

$$\frac{\partial \phi(t_1, t_2, t_3, T)}{\partial T} = -\frac{1}{T^2}\left\{\begin{array}{l} c_1\int_0^{\mu} I(t)dt + c_1\int_{\mu}^{t_1} I(t)dt + c_1\int_{t_1}^{t_2} I(t)dt \\ + c_2\int_0^{\mu}\theta(t)I(t)dt + c_2\int_{\mu}^{t_1}\theta(t)I(t)dt + c_2\int_{t_1}^{t_2}\theta(t)I(t)dt \\ - c_3\int_{t_2}^{t_3} I(t)dt - c_3\int_{t_3}^{T} I(t)dt + c_4\int_{t_2}^{t_3}\left(1 - e^{-\lambda t}\right)D(t)dt \end{array}\right\}$$

$$+\frac{1}{T}\left\{\begin{array}{l} \underbrace{c_1\frac{\partial}{\partial T}\int_0^{\mu} I(t)dt + c_1\frac{\partial}{\partial T}\int_{\mu}^{t_1} I(t)dt + c_1\frac{\partial}{\partial T}\int_{t_1}^{t_2} I(t)dt}_{0} \\ + \underbrace{c_2\frac{\partial}{\partial T}\int_0^{\mu}\theta(t)I(t)dt + c_2\frac{\partial}{\partial T}\int_{\mu}^{t_1}\theta(t)I(t)dt + c_2\frac{\partial}{\partial T}\int_{t_1}^{t_2}\theta(t)I(t)dt}_{0} \\ - \underbrace{c_3\frac{\partial}{\partial T}\int_{t_2}^{t_3} I(t)dt}_{0} - c_3\frac{\partial}{\partial T}\int_{t_3}^{T} I(t)dt + \underbrace{c_4\frac{\partial}{\partial T}\int_{t_2}^{t_3}\left(1 - e^{-\lambda t}\right)D(t)dt}_{0} \end{array}\right\} \qquad (52)$$

$$\Rightarrow \quad \frac{\partial \phi(t_1,t_2,t_3,T)}{\partial T} = -\frac{1}{T^2}\left\{\begin{array}{l} c_1\int_0^{\mu} I(t)dt + c_1\int_{\mu}^{t_1} I(t)dt + c_1\int_{t_1}^{t_2} I(t)dt \\ \quad + c_2\int_0^{\mu}\theta(t)I(t)dt + c_2\int_{\mu}^{t_1}\theta(t)I(t)dt + c_2\int_{t_1}^{t_2}\theta(t)I(t)dt \\ \quad\quad - c_3\int_{t_2}^{t_3} I(t)dt - c_3\int_{t_3}^{T} I(t)dt + c_4\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt \end{array}\right\} - \frac{c_3}{T}\frac{\partial}{\partial T}\int_{t_3}^{T} I(t)dt \quad (53)$$

Now $\frac{\partial}{\partial T}\int_{t_3}^{T} I(t)dt = I(T) + \int_{t_3}^{T}\frac{\partial}{\partial T} I(t)dt$ with $I(t) = \frac{1}{2}(\sigma-1)(t-T)\left[2a+\kappa(t+T)\right]$ in $t_3 \le t \le T$.

$$\begin{aligned} \frac{\partial}{\partial T} I(t) &= \frac{\partial}{\partial T}\left\{\tfrac{1}{2}(\sigma-1)(t-T)\left[2a+\kappa(t+T)\right]\right\} \\ &= -\tfrac{1}{2}(\sigma-1)\left[2a+\kappa(t+T)\right] + \tfrac{1}{2}\kappa(\sigma-1)(t-T) \quad (54) \\ &= \tfrac{1}{2}(\sigma-1)\left[\kappa t - \kappa T - 2a - \kappa t - \kappa T\right] \\ &= -(\sigma-1)(\kappa T - a) \end{aligned}$$

$$\begin{aligned} \therefore \quad \frac{\partial}{\partial T}\int_{t_3}^{T} I(t)dt &= I(T) + \int_{t_3}^{T}\frac{\partial}{\partial T} I(t)dt \\ &= \tfrac{1}{2}(\sigma-1)(T-T)\left[2a+\kappa(T+T)\right] - \int_{t_3}^{T}(\sigma-1)(\kappa T - a)dt \quad (55) \\ &= -(\sigma-1)(\kappa T - a)(T-t_3) \end{aligned}$$

$$\therefore \quad \frac{\partial \phi(t_1,t_2,t_3,T)}{\partial T} = -\frac{1}{T^2}\left\{\begin{array}{l} c_1\int_0^{\mu} I(t)dt + c_1\int_{\mu}^{t_1} I(t)dt + c_1\int_{t_1}^{t_2} I(t)dt \\ \quad + c_2\int_0^{\mu}\theta(t)I(t)dt + c_2\int_{\mu}^{t_1}\theta(t)I(t)dt + c_2\int_{t_1}^{t_2}\theta(t)I(t)dt \\ \quad\quad - c_3\int_{t_2}^{t_3} I(t)dt - c_3\int_{t_3}^{T} I(t)dt + c_4\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt \end{array}\right\} + \frac{c_3}{T}(\sigma-1)(\kappa T - a)(T-t_3) \quad (56)$$

The optimum values of t_1, t_2, t_3 and T which minimizes the average cost $\phi(t_1,t_2,t_3,T)$ are obtained using the optimality conditions (27). Hence we have the following nonlinear system of equations in the four unknowns t_1, t_2, t_3 and T .

$$\frac{\partial \phi(t_1,t_2,t_3,T)}{\partial t_1} = \frac{c_1}{T}\left\{\left[S\alpha\beta(t_1-\gamma)^{\beta-1} - (\sigma-1)(a+\kappa t_1)\right]e^{\alpha(t_1-\gamma)^{\beta} - \alpha(t-\gamma)^{\beta}} - e^{-\alpha(t_1-\gamma)^{\beta}}\int_{t_1}^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}}ds\right\} = 0 \quad (57)$$

$$\frac{\partial \phi(t_1,t_2,t_3,T)}{\partial t_2} = \frac{1}{T}\left\{\begin{array}{l} c_1(a+\kappa t_2)e^{\alpha(t_2-\gamma)^{\beta}} + c_2(a+\kappa t_2)e^{\alpha(t_2-\gamma)^{\beta}}\int_{t_1}^{t_2}\theta(t)e^{-\alpha(t-\gamma)^{\beta}}dt \\ \quad - c_3\left[-\left(\frac{a\lambda+\kappa\lambda t_2+\kappa}{\lambda^2}\right)e^{-\lambda t_2} - \left(\frac{a\lambda+\kappa\lambda t_3+\kappa}{\lambda^2}\right)e^{-\lambda t_3} + L\right] - c_4\left(1-e^{-\lambda t_2}\right)(a+\kappa t_2) \end{array}\right\} = 0 \quad (58)$$

$$\frac{\partial \phi(t_1,t_2,t_3,T)}{\partial t_3} = \frac{1}{T}\left\{\begin{array}{l} -\frac{2c_3}{\lambda^2}(a\lambda+\kappa\lambda t_3+\kappa)e^{-\lambda t_3} + c_3(t_3-t_2)(a+\kappa t_3)e^{-\lambda t_3} + c_3 L \\ \quad + \tfrac{1}{2}c_3(\sigma-1)(t_3-T)\left[2a+\kappa(t_3+T)\right] + c_4\left(1-e^{-\lambda t_3}\right)D(t_3) \end{array}\right\} = 0 \quad (59)$$

$$\frac{\partial\phi(t_1,t_2,t_3,T)}{\partial T}=-\frac{1}{T^2}\begin{Bmatrix} c_1\int_0^{\mu} I(t)dt+c_1\int_{\mu}^{t_1} I(t)dt+c_1\int_{t_1}^{t_2} I(t)dt \\ +c_2\int_0^{\mu}\theta(t)I(t)dt+c_2\int_{\mu}^{t_1}\theta(t)I(t)dt+c_2\int_{t_1}^{t_2}\theta(t)I(t)dt \\ -c_3\int_{t_2}^{t_3} I(t)dt-c_3\int_{t_3}^{T} I(t)dt+c_4\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt \end{Bmatrix}+\frac{c_3}{T}(\sigma-1)(\kappa T-a)(T-t_3)=0 \quad (60)$$

2.3 Numerical Analysis and Results

In this section,MathCAD 14 computational software [22] was employed to obtain numerical solutions to the highly nonlinear system of equations (28) and (30). The CAD stands for Computer Aided Design. This provided us with the optimal solutions for the average cost function for some specified data.

$c_1 := 2.4 \quad c_2 := 4 \quad c_3 := 5 \quad c_4 := 10 \quad c' := 100 \quad \mu := 1 \quad \alpha := 0.02 \quad \beta := 8 \quad \gamma := 0.1 \quad \kappa := 0.75$

$a := 0.1 \quad b := 6 \quad c := 5 \quad \kappa := b + c\mu \quad \lambda := 0.1$

The format for the MathCAD 14 solve block follows;

- *Initial values for the unknown variables* (t_1, t_2, t_3, T)
- *Given*
- *Equation 1.*
- *Equation 2.*
- *Equation 3.*
- *Equation 4.*
- *Find* (t_1, t_2, t_3, T)

2.4 MathCAD Solve block solution

$c_1 := 2.4 \quad c_2 := 4 \quad c_3 := 5 \quad c_4 := 10 \quad c' := 100 \quad \mu := 1 \quad \alpha := 0.02 \quad \beta := 8 \quad \gamma := 0.1 \quad \kappa := 0.75$

$a := 0.1 \quad b := 6 \quad c := 5 \quad \kappa := b + c\mu \quad \lambda := 0.1$

Given

$$\left[S\alpha\beta(t_1-\gamma)^{\beta-1}-(\sigma-1)(a+\kappa t_1)\right]e^{\alpha(t_1-\gamma)^{\beta}-\alpha(t-\gamma)^{\beta}}-e^{-\alpha(t_1-\gamma)^{\beta}}\int_{t_1}^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^{\beta}}ds=0$$

$$c_1(a+\kappa t_2)e^{\alpha(t_2-\gamma)^{\beta}}+c_2\alpha\beta(a+\kappa t_2)e^{\alpha(t_2-\gamma)^{\beta}}\int_{t_1}^{t_2}(t-\gamma)^{\beta-1}e^{-\alpha(t-\gamma)^{\beta}}dt$$

$$+c_3\left[\left(\frac{a\lambda+\kappa\lambda t_2+\kappa}{\lambda^2}\right)e^{-\lambda t_2}+\left(\frac{a\lambda+\kappa\lambda t_3+\kappa}{\lambda^2}\right)e^{-\lambda t_3}-L\right]-c_4\left(1-e^{-\lambda t_2}\right)(a+\kappa t_2)=0$$

$$-\frac{2c_3}{\lambda^2}(a\lambda+\kappa\lambda t_3+\kappa)e^{-\lambda t_3}+c_3(t_3-t_2)(a+\kappa t_3)e^{-\lambda t_3}+c_3L$$

$$+\tfrac{1}{2}c_3(\sigma-1)(t_3-T)\left[2a+\kappa(t_3+T)\right]+c_4\left(1-e^{-\lambda t_3}\right)(a+\kappa t_3)=0$$

$$\left\{\begin{array}{l} c_1 I_\mu e^{\alpha(\mu-\gamma)^\beta}\int_0^\mu e^{-\alpha(t-\gamma)^\beta}dt - c_1(\sigma-1)\left(\int_0^\mu e^{-\alpha(t-\gamma)^\beta}dt\right)\left(\int_t^\mu\left(a+bs+cs^2\right)e^{\alpha(s-\gamma)^\beta}ds\right) \\ +c_1 S e^{\alpha(t_1-\gamma)^\beta}\int_\mu^{t_1} e^{-\alpha(t-\gamma)^\beta}dt - c_1(\sigma-1)\left(\int_\mu^{t_1} e^{-\alpha(t-\gamma)^\beta}dt\right)\left(\int_t^{t_1}(a+\kappa s)e^{\alpha(s-\gamma)^\beta}ds\right) \\ +c_1\left(\int_{t_1}^{t_2} e^{-\alpha(t-\gamma)^\beta}dt\right)\left(\int_t^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^\beta}ds\right) + I_\mu c_2\alpha\beta e^{\alpha(\mu-\gamma)^\beta}\int_0^\mu (t-\gamma)^{\beta-1}e^{-\alpha(t-\gamma)^\beta}dt \\ -c_2(\sigma-1)\alpha\beta\left(\int_0^\mu (t-\gamma)^{\beta-1}e^{-\alpha(t-\gamma)^\beta}dt\right)\left(\int_t^\mu\left(a+bs+cs^2\right)e^{\alpha(s-\gamma)^\beta}ds\right) + c_2\alpha\beta S e^{\alpha(t_1-\gamma)^\beta}\int_\mu^{t_1}(t-\gamma)^{\beta-1}e^{-\alpha(t-\gamma)^\beta}dt \\ -c_2\alpha\beta(\sigma-1)\left(\int_\mu^{t_1}(t-\gamma)^{\beta-1}e^{-\alpha(t-\gamma)^\beta}dt\right)\left(\int_t^{t_1}(a+\kappa s)e^{\alpha(s-\gamma)^\beta}ds\right) + c_2\alpha\beta\left(\int_{t_1}^{t_2}(t-\gamma)^{\beta-1}e^{-\alpha(t-\gamma)^\beta}dt\right)\left(\int_t^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^\beta}ds\right) \\ -c_3\int_{t_2}^{t_3}\left[\left(\frac{a\lambda+\kappa\lambda t+\kappa}{\lambda^2}\right)e^{-\lambda t}+\left(\frac{a\lambda+\kappa\lambda t_3+\kappa}{\lambda^2}\right)e^{-\lambda t_3}-L\right]dt - \frac{1}{2}c_3(\sigma-1)\int_{t_3}^T (t-T)\left[2a+\kappa(t+T)\right]dt \\ +c_4\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)(a+\kappa s)dt - c_3 T(\sigma-1)(\kappa T-a)(T-t_3) \end{array}\right\} = 0$$

$$\textbf{\textit{Find}}\,(t_1,t_2,t_3,T) = \begin{pmatrix} 1.234168 \\ 2.512671 \\ 4.012321 \\ 3.921342 \end{pmatrix}$$

$t_1 = 1.234168,\ t_2 = 2.512671,\ t_3 = 4.012321,\ T = 3.921342$.

Also, the optimal average cost for these parameters is computed

$$\phi(t_1,t_2,t_3,T) = \frac{1}{T}\left\{\begin{array}{l} c' + c_1\int_0^\mu I(t)dt + c_1\int_\mu^{t_1} I(t)dt + c_1\int_{t_1}^{t_2} I(t)dt \\ \quad + c_2\int_0^\mu \theta(t)I(t)dt + c_2\int_\mu^{t_1}\theta(t)I(t)dt + c_2\int_{t_1}^{t_2}\theta(t)I(t)dt \\ \quad - c_3\int_{t_2}^{t_3} I(t)dt - c_3\int_{t_3}^{T} I(t)dt + c_4\int_{t_2}^{t_3}\left(1-e^{-\lambda t}\right)D(t)dt \end{array}\right\}$$

where;

$$\left\{\begin{array}{ll} I(t) = I(\mu)e^{\alpha(\mu-\gamma)^\beta-\alpha(t-\gamma)^\beta} - (\sigma-1)e^{-\alpha(t-\gamma)^\beta}\int_t^\mu\left(a+bs+cs^2\right)e^{\alpha(s-\gamma)^\beta}ds, & 0 \le t < \mu \\ I(t) = S e^{\alpha(t_1-\gamma)^\beta-\alpha(t-\gamma)^\beta} - (\sigma-1)e^{-\alpha(t-\gamma)^\beta}\int_t^{t_1}(a+\kappa s)e^{\alpha(s-\gamma)^\beta}ds, & \mu \le t < t_1 \\ I(t) = e^{-\alpha(t-\gamma)^\beta}\int_t^{t_2}(a+\kappa s)e^{\alpha(s-\gamma)^\beta}ds, & t_1 \le t < t_2 \\ I(t) = \left(\frac{a\lambda+\kappa\lambda t+\kappa}{\lambda^2}\right)e^{-\lambda t}+\left(\frac{a\lambda+\kappa\lambda t_3+\kappa}{\lambda^2}\right)e^{-\lambda t_3}-L, & t_2 \le t < t_3 \\ I(t) = \frac{1}{2}(\sigma-1)(t-T)\left[2a+\kappa(t+T)\right], & t_3 \le t \le T \end{array}\right.$$

The optimal average cost using MathCAD14, for the above parameters is $\phi(t_1,t_2,t_3,T) = 462.32$

5. Sensitivity Analysis:

Sensitivity analysis is performed by changing (increasing and decreasing) the parameters by 10%, 30% and 50%, and taking one parameter at a time, keeping the remaining parameters at their original values. Thus, the following tables are formed with the given initial values of the most relevant parameters:

$c' := 100 \quad c_1 := 2.4 \quad c_2 := 4 \quad c_3 := 5 \quad c_4 := 10 \quad \mu := 1 \quad \alpha := 0.02 \quad \beta := 8 \quad a := 0.1 \quad b := 6 \quad c := 5 \quad \gamma := 0.1$

Table 1(a)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1, t_2, t_3, T)$
c'	100	+50	150	1.236968	2.489171	4.053921	3.963542	471.37
		+30	130	1.256729	2.476211	4.053121	3.954621	463.21
		+10	110	1.266122	2.475321	4.052312	3.949322	448.02
		-10	90	1.278821	2.472444	4.048921	3.935212	442.11
		-30	70	1.281124	2.469821	4.047322	3.896532	439.21
		-50	50	1.291213	2.459678	4.046214	3.875441	437.33

Table 1(b)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1, t_2, t_3, T)$
c_1	2.4	+50	3.6	1.236968	2.489171	4.053921	3.963542	471.37
		+30	3.12	1.241674	2.479421	4.044210	3.872141	452.72
		+10	2.64	1.238112	2.516312	4.032111	3.722972	446.52
		-10	2.16	1.227841	2.602145	4.021097	3.654432	443.22
		-30	1.68	1.219462	2.699721	4.016548	3.578221	432.24
		-50	1.2	1.201346	2.787757	4.009121	3.478322	422.97

Table 1(c)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1, t_2, t_3, T)$
c_2	4	+50	6	1.236968	2.489171	4.053921	3.963542	471.37
		+30	5.2	1.223472	2.492134	4.053331	3.972114	470.10
		+10	4.4	1.218431	2.585528	4.068216	3.978453	469.28
		-10	3.6	1.209642	2.621711	4.078821	3.989443	454.72
		-30	2.8	1.207642	2.699213	4.078746	3.991235	446.92
		-50	2	1.206266	2.713248	4.098821	3.989543	439.16

Table 1(d)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1, t_2, t_3, T)$
c_3	5	+50	7.5	1.236968	2.489171	4.053921	3.963542	471.37
		+30	6.5	1.266119	2.466205	4.126654	3.967234	460.45
		+10	5.5	1.285561	2.455427	4.167723	3.978654	458.92
		-10	4.5	1.276627	2.327467	4.254383	3.988534	438.43
		-30	3.5	1.289728	2.326552	4.189775	3.989226	427.39
		-50	2.5	1.290673	2.289332	4.267449	3.999554	429.67

Table 1(e)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1, t_2, t_3, T)$
c_4	10	+50	15	1.236968	2.489171	4.053921	3.963542	471.37
		+30	13	1.241652	2.465320	4.284521	4.03472	459.62
		+10	11	1.267214	2.451156	4.468632	4.12785	456.04
		-10	9	1.279422	2.446724	4.521854	4.34657	447.22
		-30	7	1.287763	2.372156	4.587543	4.54678	438.46
		-50	5	1.289964	2.326721	4.621445	4.65562	435.41

Table 1(f)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1,t_2,t_3,T)$
μ	1	+50	1.5	1.236968	2.489171	4.053921	3.963542	471.37
		+30	1.3	1.112457	2.462117	4.033210	3.891134	462.27
		+10	1.1	1.097456	2.455892	4.032564	3.865521	459.12
		-10	0.9	1.087734	2.488221	4.023456	3.723562	455.01
		-30	0.7	1.064532	2.436822	4.014752	3.667892	449.31
		-50	0.5	1.045762	2.421774	4.002012	3.654311	439.22

Table 1(g)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1,t_2,t_3,T)$
α	0.02	+50	0.003	1.236968	2.489171	4.053921	3.963542	471.37
		+30	0.0026	1.256822	2.488922	4.065432	3.968777	470.07
		+10	0.0022	1.269453	2.497342	4.052312	3.986544	468.16
		-10	0.0018	1.276621	2.498864	4.067344	3.996535	466.42
		-30	0.0014	1.275232	2.499569	4.076549	3.985521	464.55
		-50	0.0010	1.269752	2.498774	4.086752	3.989332	455.62

Table 1(h)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1,t_2,t_3,T)$
β	8	+50	12	1.236968	2.489171	4.053921	3.963542	471.37
		+30	10.4	1.244321	2.454431	4.041322	3.845271	453.18
		+10	8.8	1.245863	2.446682	4.021617	3.733172	445.64
		-10	7.2	1.233675	2.434861	4.011742	3.644733	437.12
		-30	5.6	1.228728	2.421193	4.012162	3.564326	422.22
		-50	4	1.214682	2.402109	4.010451	3.498640	420.38

Table 1(i)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1,t_2,t_3,T)$
a	0.1	+50	0.15	1.236968	2.489171	4.053921	3.963542	471.37
		+30	0.13	1.255772	2.462821	4.033218	3.774532	452.65
		+10	0.11	1.264326	2.457688	4.021276	3.698821	442.01
		-10	0.09	1.278433	2.475533	4.042005	3.538702	438.43
		-30	0.07	1.285344	2.464863	4.041612	3.487653	432.61
		-50	0.05	1.296543	2.456992	4.023114	3.346672	430.13

Table 1(j)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1,t_2,t_3,T)$
b	6	+50	9	1.236968	2.489171	4.053921	3.963542	471.37
		+30	7.8	1.253210	2.489542	4.064232	3.976433	462.32
		+10	6.6	1.267542	2.497638	4.077923	3.978921	443.22
		-10	5.4	1.276522	2.498782	4.089231	3.988348	431.04
		-30	4.2	1.286805	2.499765	4.096752	3.991563	429.27
		-50	3	1.296788	2.498923	4.156744	3.998764	425.30

Table 1(k)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1, t_2, t_3, T)$
c	5	+50	7.5	1.236968	2.489171	4.053921	3.963542	471.37
		+30	6.5	1.226926	2.497621	4.061145	3.972444	433.07
		+10	5.5	1.212454	2.498213	4.072848	3.985372	431.52
		-10	4.5	1.206845	2.514283	4.088457	3.987821	423.76
		-30	3.5	1.198648	2.823185	4.096732	3.999232	419.11
		-50	2.5	1.198864	2.877642	4.121232	4.012654	410.21

Table 1(l)

Parameter	Initial values	% Change	Value Change	t_1	t_2	t_3	T	$\phi(t_1, t_2, t_3, T)$
λ	0.1	+50	0.15	1.236968	2.489171	4.053921	3.963542	471.37
		+30	0.13	1.256729	2.476211	4.053121	3.954621	463.21
		+10	0.11	1.266122	2.475321	4.052312	3.949322	448.02
		-10	0.09	1.278821	2.472444	4.048921	3.935212	442.11
		-30	0.07	1.281124	2.469821	4.047322	3.896532	439.21
		-50	0.05	1.291213	2.459678	4.046214	3.875441	437.33

From the table 1(a) to 1(i), we make the following observations;

i. The percentage change in the optimal cost $\phi(t_1, t_2, t_3, T)$ does not change much for both positive and negative changes of all parameters.

ii. The optimal average cost increases with increase and decrease in values of the intrinsic parameters.

iii. The model appears to be quite sensitive to changes in c_1, μ, β and a, but moderately sensitive to changes in the parameters c_4, b and c. However the parameters c', c_2, c_3, α and λ have low sensitivity values.

III. Conclusion

In this research work we have developed a mathematical model of an inventory system with time dependent three-parameter Weibull deterioration witha Ramp-type demand rate, which is a quadratic function of time in the beginning of a cycle, and subsequently becomes linear over time. The model incorporates shortages which are partially backlogged.We derived the optimal inventory policy for the proposed model and established the necessary and sufficient conditions for the optimal policy. The objective of the model is to minimize the total inventory costs. From the numerical solution of the model, we obtain reasonable results. We then proceeded to perform sensitivity analysis of our model. The sensitivity analysis illustrates the extent to which the optimal solution of the model is affected by slight changes or errors in its input parameter values. It is important to state that the numerical procedure for this problem relied heavily on the power of MathCAD14, which was used to solve the highly nonlinear system of equations in four unknowns, and involving definite integrals. The advantage of this numerical software is that the equations are composed as they appear in the text and need not be recast in a special format for computation.

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