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Odd Sum Labeling of Some Grid Graphs

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Abstract:
In this paper we have discussed the odd sum labeling of grid graph, path union of grid graphs with different size, graph obtained by joining vertex of a grid graph and a complete bipartite graph $K_{2,t}$ by a path, step grid graph and the graph obtained by joining step grid graphs of different size by arbitrary paths.
***Key Word**: Odd sum labeling, odd sum graph, grid graph, step grid graph, path union.*

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I. Introduction

Throughout this paper by a graph we mean a finite, simple undirected graph. We use the notation p for number of vertices and q for number of edges in a graph. Graph labeling was initiated by Rosa[1]. Since then many researchers have contributed in the field of graph labeling. A detailed survey on graph labeling is updated every year by Gallian[2]. The concept of odd sum labeling was given by Arockiaraj and Mahalakshmi[3] with odd sum labeling of path, cycle, balloon graph, ladder graph, quadrilateral snake graph, bistar graph and cyclic ladder graph. Arockiaraj et al.[4,5] discussed the odd sum property of some subdivision graphs and graphs obtained by duplicating any edge of some graphs. Gopi[6] investigated odd sum labeling of some tree related graphs such as the H graph of path, twig graph, the graph $P(m,n)$ and the graph (P_m, S_n). Gopi and Iraudaya Mary[7] studied the odd sum labeling of slanting ladder graph, the shadow graph of a star graph and bistar graph, the mirror graph of a path and the graph obtained by duplicating a vertex in a path. Odd sum labeling and odd sum graph is defined[3] as, "An injective function $f: V(G) \to \{0, 1, 2, \dots\dots, |E(G)|\}$ is said to be an odd sum labeling if the induced edge labeling f^* defined by $f^*(uv) = f(u) + f(v),\ \forall\, uv \in E(G)$ is a bijective and $f^*: E(G) \to \{1, 3, 5, \dots\dots, 2|E(G)| - 1\}$. A graph is said to be an odd sum graph if it admits an odd sum labeling".

This paper deals with odd sum labeling of grid graph $P_n \times P_m$, path union of grid graphs $P_{n_1} \times P_{m_1}, P_{n_2} \times P_{m_2}, \dots\dots\dots, P_{n_t} \times P_{m_t}$, graph obtained by joining vertex of a grid graph $P_n \times P_m$ and a complete bipartite graph $K_{2,t}$ by a path P_r, step grid graph St_n and the graph obtained by joining step grid graphs $St_{n_1}, St_{n_2}, \dots\dots\dots, St_{n_t}$ by paths $P_{r_1}, P_{r_2}, \dots\dots\dots, P_{r_{t-1}}$.

Definition 1: The Cartesian product of two paths P_n and P_m is known as a grid graph and it is denoted by $P_n \times P_m$. It is obvious that $|V(P_n \times P_m)| = nm$ and $|E(P_n \times P_m)| = 2nm - (n + m)$.

Definition 2: For a graph G, if $G_1, G_2, \dots\dots, G_t$ $(t \geq 2)$ are t copies of G then a graph obtained by adding an edge from G_i to G_{i+1} $(1 \leq i \leq t-1)$ is said to be a path union of graph G which is denoted by $P(t \cdot G)$.

Let $G_1, G_2, \dots\dots, G_t$ $(t \geq 2)$ be connected graphs. Consider paths $P_{n_1}, P_{n_2}, \dots\dots, P_{n_{t-1}}$. Then the graph obtained by joining each pair of graphs (G_i, G_{i+1}) by the path P_{n_i} $(1 \leq i \leq t-1)$ is denoted by $\langle G_1, P_{n_1}, G_2, P_{n_2}, G_3, \dots\dots, G_{t-1}, P_{n_{t-1}}, G_t \rangle$. If $P_{n_1} = P_{n_2} = \dots\dots, = P_{n_{t-1}} = P_n$ then such a path union is denoted by $P_n(G_1, G_2, \dots\dots, G_t)$. A graph $P_2(G_1, G_2, \dots\dots, G_t)$ can also be simply denoted as $P(G_1, G_2, \dots\dots, G_t)$.

Definition 3: Consider paths $P_n, P_n, P_{n-1}, \dots\dots, P_3, P_2$ on $n, n, n-1, n-2, \dots\dots, 3, 2$ vertices and arrange them vertically. A graph obtained by joining horizontal vertices of given successive paths is known as a step grid graph[8] of size n, where $n \geq 3$. It is denoted by St_n. Clearly, $|V(St_n)| = \frac{n^2+3n-2}{2}$ and $|E(St_n)| = n^2 + n - 2$.

II. Main Results

Theorem 1: Every grid graph $P_n \times P_m$ admits odd sum labeling.

Proof: Consider a grid graph $P_n \times P_m$ as shown in Figure 1.

The vertex set $V(P_n \times P_m) = \{v_{i,j} :\ i = 1,2,\dots,n;\ j = 1,2,\dots,m\}$ and

the edge set $E(P_n \times P_m) = \{v_{i,j}\ v_{i+1,j} :\ i = 1,2,\dots,n-1;\ j = 1,2,\dots,m\}$
$\cup \{v_{i,j}\ v_{i,j+1} \mid i = 1,2,\dots,n;\ j = 1,2,\dots,m-1\}$.

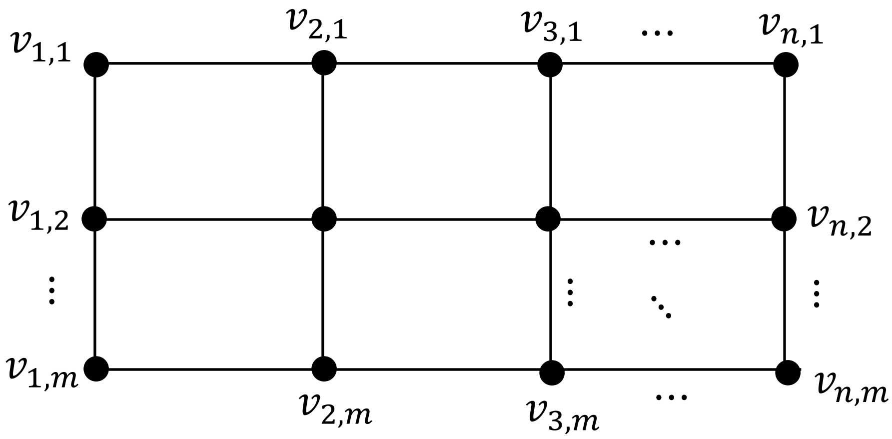

Figure – 1: Ordinary labeling of $P_n \times P_m$

Clearly, $q = |E(P_n \times P_m)| = 2mn - (m + n)$.
Now, define $f: V(P_n \times P_m) \to \{0,1,2, \dots, q\}$ as
$f(v_{i,1}) = (i - 1)(2m - 1), \forall\, i = 1,2, \dots, n;$
$f(v_{i,j}) = f(v_{i,j-1}) + 1, \forall\, i = 1,2, \dots, n, \forall\, j = 2,3, \dots, m.$
The induced edge labeling function $f^*: E(P_n \times P_m) \to \{1,3,5, \dots, 2q - 1\}$ is given by

$$f^*(uv) = f(u) + f(v), \forall\, uv \in E(P_n \times P_m).$$

The above labeling pattern yields odd sum labeling of $P_n \times P_m$. Hence, $P_n \times P_m$ admits odd sum labeling.

Illustration 1: Odd sum labeling of grid graph $P_4 \times P_3$ is shown in Figure 2.

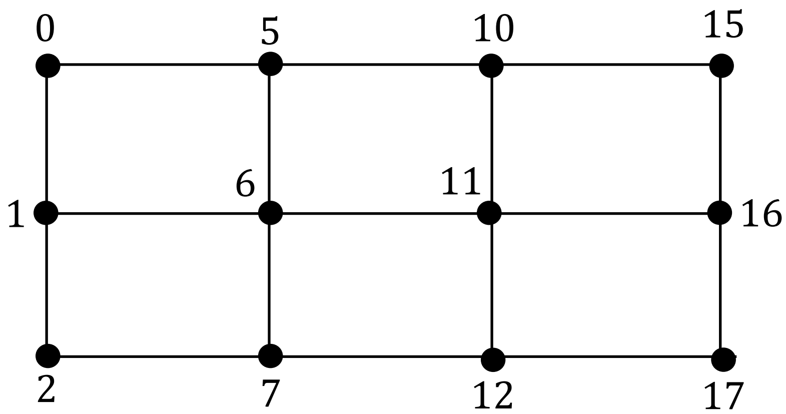

Figure – 2: Odd sum labeling of a grid graph $P_4 \times P_3$

Theorem 2: A graph $P(P_{n_1} \times P_{m_1}, P_{n_2} \times P_{m_2}, \dots\dots\dots, P_{n_t} \times P_{m_t})$ is an odd sum graph.
Proof: Let G be a graph $P(P_{n_1} \times P_{m_1}, P_{n_2} \times P_{m_2}, \dots\dots\dots, P_{n_t} \times P_{m_t})$ in which the vertex of i^{th} column and j^{th} row of $P_{n_k} \times P_{m_k}$ is denoted by $v_{k,i,j}$ and the vertex $v_{k,1,1}$ be joined with $v_{k-1,n_{k-1},m_{k-1}}$ by an edge for each $k = 2,3, \dots\dots, t$ as shown in Figure 3.
Clearly, the number of edges in $P_{n_k} \times P_{m_k}$ is $q_k = 2m_k n_k - (m_k + n_k)$, $\forall\, k = 1,2, \dots\dots, t$.
Hence the number of edges in G is

$$q = (t - 1) + \sum_{k=1}^{t} q_k \,.$$

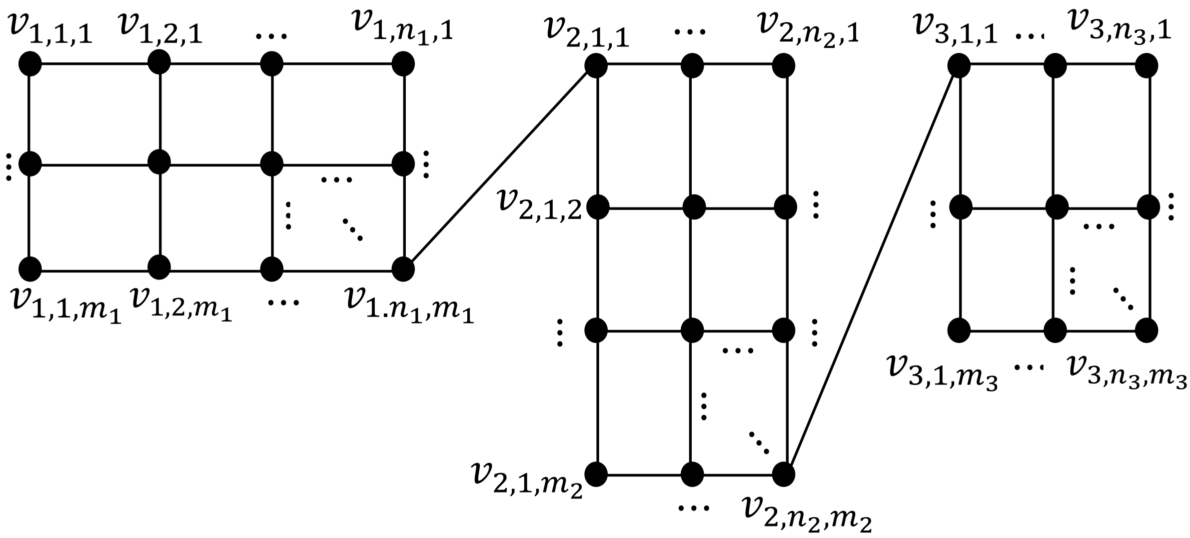

Figure – 3: Ordinary vertex labeling of path union of grid graphs

We define vertex labeling function $f: V(G) \rightarrow \{0,1,2, \ldots \ldots, q\}$ as follow:
$f(v_{1,i,1}) = (i-1)(2m_1 - 1)$, $\forall\, i = 1, 2, \ldots \ldots, n_1$;
$f(v_{k,i,1}) = (i-1)(2m_k - 1) + k - 1 + \sum_{j=1}^{k-1} q_j$, $\forall\, i = 1, 2, \ldots \ldots, n_k, \forall\, k = 2, 3, \ldots \ldots, t$;
$f(v_{k,i,j}) = f(v_{k,i,j-1}) + 1$, $\forall\, i = 1,2, \ldots \ldots, n_k$, $\forall\, j = 2,3, \ldots \ldots, m_k$, $\forall\, k = 1,2, \ldots \ldots, t$.
The induced edge labeling function $f^*: E(G) \rightarrow \{1, 3, 5, \ldots, 2q-1\}$ is given by

$$f^*(uv) = f(u) + f(v), \qquad \forall\, uv \in E(G).$$

The above labeling pattern shows the odd sum labeling of the graph G.
Hence, the graph $P(P_{n_1} \times P_{m_1}, P_{n_2} \times P_{m_2}, \ldots \ldots \ldots, P_{n_t} \times P_{m_t})$ is an odd sum graph.

Illustration 2: The graph $P(P_4 \times P_3, P_3 \times P_4, P_3 \times P_3)$ is an odd sum graph as shown in Figure 4.

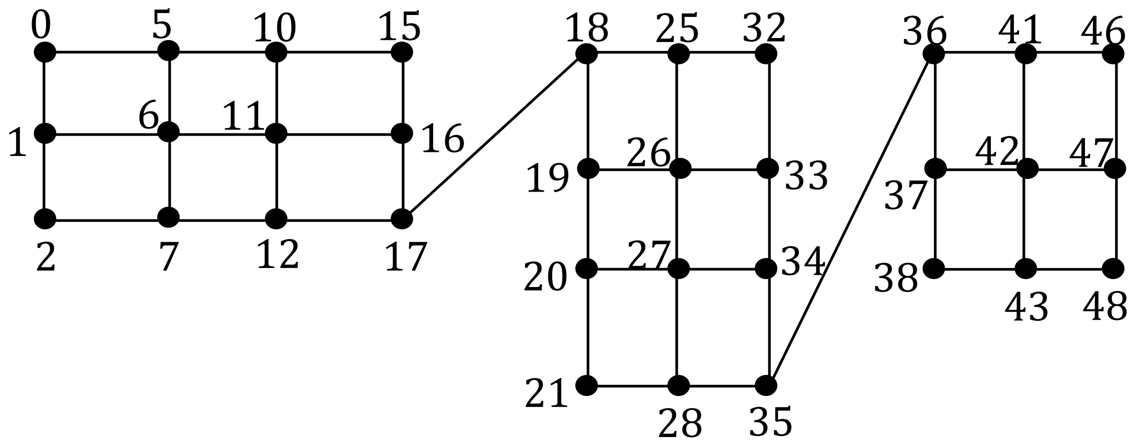

Figure – 4: Odd sum labeling of the graph $P(P_4 \times P_3, P_3 \times P_4, P_3 \times P_3)$

Theorem 3: A graph obtained by joining vertex of a grid graph $P_n \times P_m$ and a complete bipartite graph $K_{2,t}$ by a path P_r i.e. a graph $\langle P_n \times P_m, P_r, K_{2,t} \rangle$ is an odd sum graph.
Proof: Let G be a graph obtained by joining vertex $v_{n,m}$ of a grid graph $P_n \times P_m$ and a vertex u_1 of a complete bipartite graph $K_{2,t}$ by a path P_r as shown in Figure 5. Thus, $G = \langle P_n \times P_m, P_r, K_{2,t} \rangle$.
Here, $V(G) = \{v_{i,j} : i = 1,2, \ldots, n; j = 1,2, \ldots, m\} \cup \{u_1, u_2, u'_1, u'_2, \ldots, u'_t\} \cup \{w_1, w_2, \ldots, w_r\}$ and

$$E(G) = \{v_{i,j}\, v_{i+1,j} : i = 1,2, \ldots, n-1; j = 1,2, \ldots, m\} \cup \{v_{i,j}\, v_{i,j+1} : i = 1,2, \ldots n; j = 1,2, \ldots, m-1\}$$
$$\cup \{w_i w_{i+1} : i = 1,2, \ldots, r-1\} \cup \{u_1 u'_i : i = 1,2, \ldots, t\} \cup \{u_2 u'_i : i = 1,2, \ldots, t\}$$

where $w_1 = v_{n,m}$ and $u_1 = w_r$.

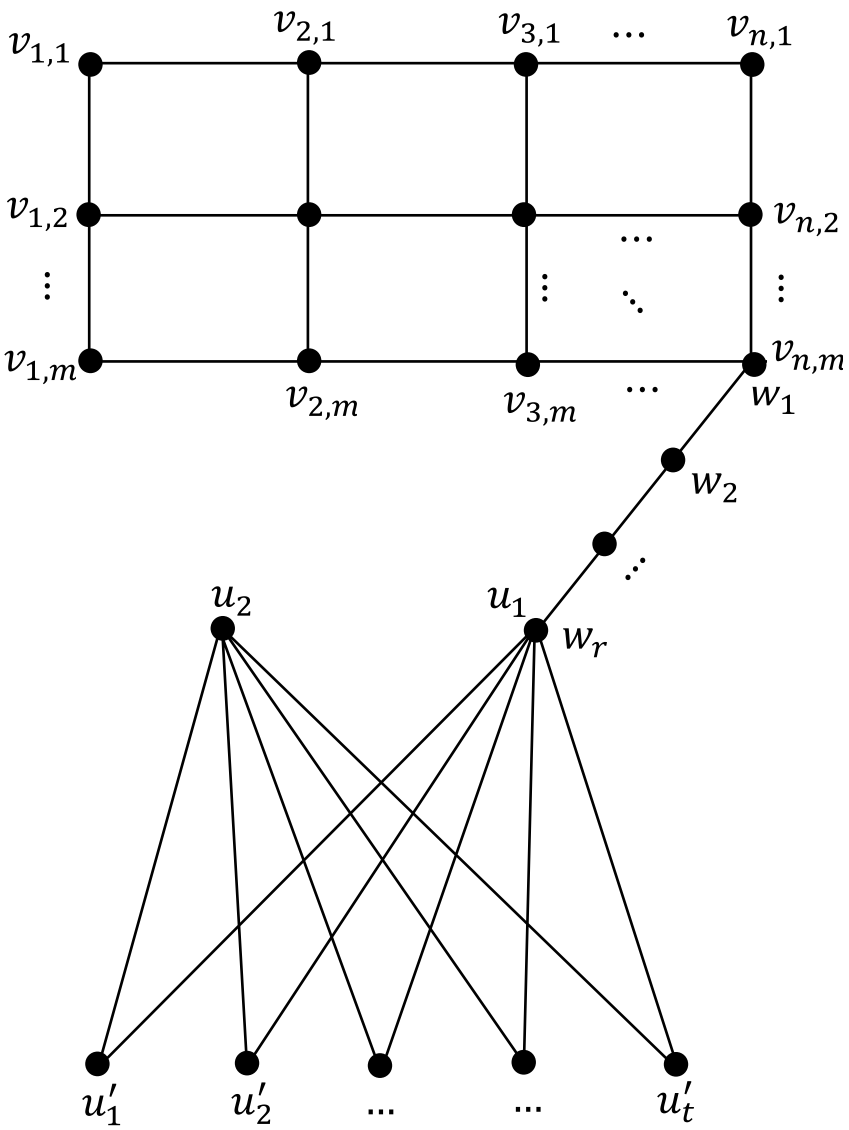

Figure – 5: Ordinary labeling of the graph $\langle P_n \times P_m,\ P_r,\ K_{2,t}\rangle$

Clearly, $|E(G)| = q = 2mn - (m+n) + 2t + r - 1$.
Now, define $f: V(G) \to \{0,1,2,3,\dots,q\}$ as follow:
$f(v_{i,1}) = (i-1)(2m-1), \forall\ i = 1,2,\dots,n;$
$f(v_{i,j}) = f(v_{i,j-1}) + 1, \forall\ i = 1,2,\dots,n, \forall\ j = 2,3,\dots,m;$
$f(w_1) = f(v_{n,m});$
$f(w_i) = f(v_{n,m}) + i - 1, \forall\ i = 1,2,\dots,r;$
$f(u_1) = f(w_r);$
$f(u_2) = q;$
$f(u_i^{'}) = f(u_1) + 2i - 1, \forall\ i = 1,2,\dots,t.$
The induced edge labeling function $f^*: E(G) \to \{1,3,5,\dots,2q-1\}$ is given by $f^*(uv) = f(u) + f(v), \forall\ uv \in E(G)$.
The above labeling pattern tends to give odd sum labeling pattern of a graph G. Hence, the graph $\langle P_n \times P_m,\ P_r,$ K2,t is an odd sum graph.

Illustration 3: Odd sum labeling of a graph $\langle P_4 \times P_3,\ P_4,\ K_{2,5} \rangle$ is shown in Figure 6.

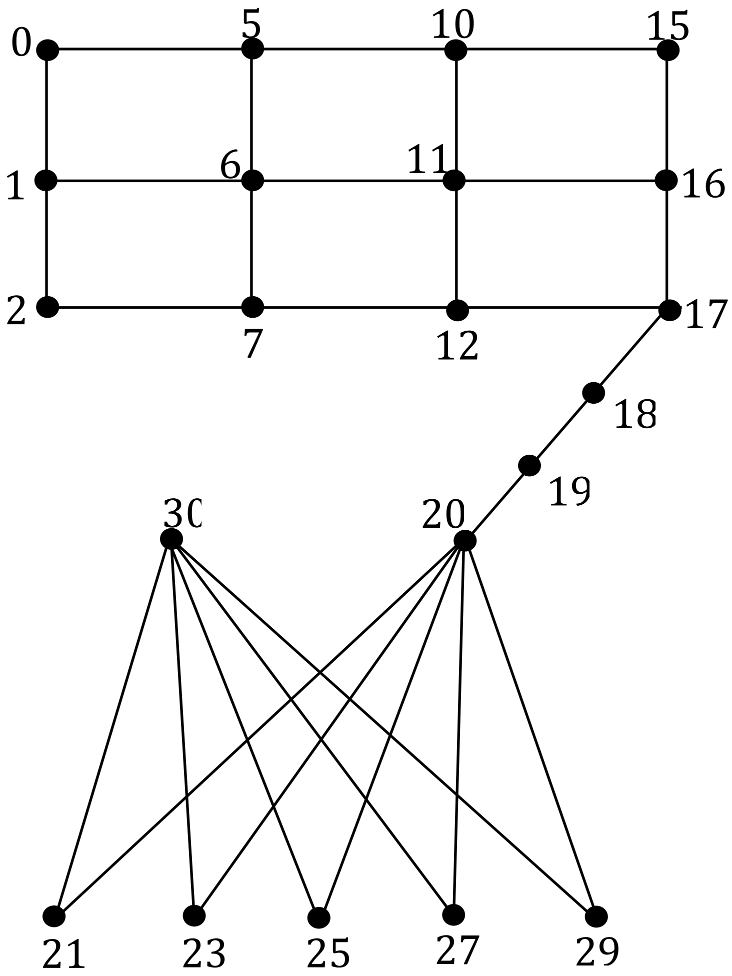

Figure – 6: Odd sum labeling of a graph $\langle P_4 \times P_3,\ P_4,\ K_{2,5} \rangle$

Theorem 4: Every step grid graph St_n $(n \geq 3)$ is an odd sum graph.
Proof: Consider a step grid graph St_n of size n which is a graph obtained by joining horizontal vertices of successive paths $P_n, P_n, P_{n-1}, P_{n-2}, \dots \dots \dots, P_2$ as shown in Figure 7.
Here, $V(St_n) = \{u_{1,j} : 1 \leq j \leq n\} \cup \{u_{i,j} : 2 \leq i \leq n;\ 1 \leq j \leq n+2-i\}$ and
$E(St_n) = \{u_{1,j}u_{1,j+1} : 1 \leq j \leq n-1\} \cup \{u_{i,j}\ u_{i,j+1} : 2 \leq i \leq n;\ 1 \leq j \leq n+1-i\} \cup$
$\{u_{1,j}\ u_{2,j} : 1 \leq j \leq n\} \cup \{u_{i,j}\ u_{i+1,j-1} : 2 \leq i \leq n-1;\ 2 \leq j \leq n+2-i\}$.
Clearly, $q = |E(St_n)| = n^2 + n - 2$.
We define vertex labeling function $f: V(St_n) \rightarrow \{0, 1, 2, \dots, q\}$ as follow:
$f(u_{1,1}) = 0;$
$f(u_{i,1}) = (i-1)(2n+2-i) - 1, \forall\ i = 2, 3, \dots, n;$
$f(u_{1,j}) = f(u_{1,j-1}) + 1, \forall\ j = 2, 3, \dots, n;$
$f(u_{i,j}) = f(u_{i,j-1}) + 1,\ \forall\ i = 1, 2, \dots, n, \forall\ j = 2, 3, \dots, n+2-i.$

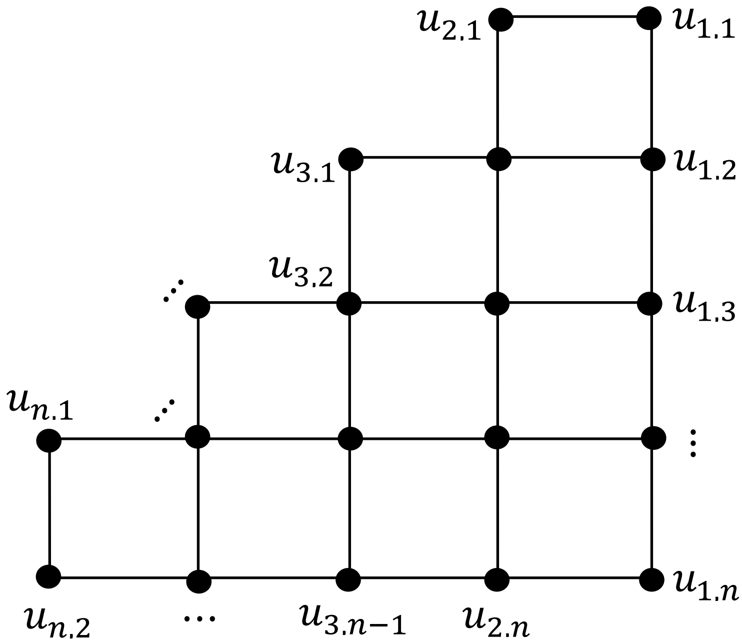

Figure – 7: Ordinary labeling of a step grid graph St_n

The induced edge labeling function $f^*: E(St_n) \rightarrow \{1, 3, 5, \ldots, 2q-1\}$ given by $f^*(uv) = f(u) + f(v)$, $\forall\, uv \in E(St_n)$ with the above vertex labeling pattern shows that the graph St_n is an odd sum graph.

Illustration 4: Odd sum labeling of step grid graph St_5 is shown in Figure 8.

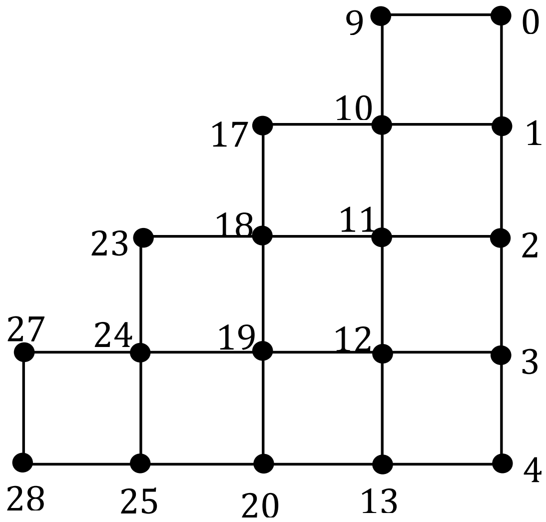

Figure – 8: Odd sum labeling of a step grid graph St_5

Theorem 5: A graph $\langle St_{n_1}, P_{r_1}, St_{n_2}, P_{r_2}, St_{n_3}, \ldots \ldots, St_{n_{t-1}}, P_{r_{t-1}}, St_{n_t} \rangle$ is an odd sum graph.
Proof: Consider step grid graphs $St_{n_1}, St_{n_2}, \ldots \ldots \ldots, St_{n_t}$ of size $n_1, n_2, \ldots \ldots \ldots, n_t$ respectively.
For $k = 1, 2, \ldots \ldots \ldots, t$, we have $V(St_{n_k}) = \{v_{k,1,j} : 1 \leq j \leq n_k\} \cup \{v_{k,i,j} : 2 \leq i \leq n_k;\ 1 \leq j \leq n_k + 2 - i\}$ and
$E(St_{n_k}) = \{v_{k,1,j}v_{k,1,j+1} : 1 \leq j \leq n_k - 1\} \cup \{v_{k,i,j}\, v_{k,i,j+1} : 2 \leq i \leq n_k;\ 1 \leq j \leq n_k + 1 - i\}$
$\cup\ \{v_{k,1,j}\, v_{k,2,j} : 1 \leq j \leq n_k\} \cup \{v_{k,i,j}\, v_{k,i+1,j-1} : 2 \leq i \leq n_k - 1;\ 2 \leq j \leq n_k + 2 - i\}$.
Clearly, $p_k = |V(St_{n_k})| = \frac{{n_k}^2 + 3n_k - 2}{2}$ and $q_k = |E(St_{n_k})| = {n_k}^2 + n_k - 2$, $\forall\, k = 1, 2, \ldots \ldots \ldots, t$.
Let G be a graph as shown in Figure 9 which is obtained by joining each vertex $v_{k,n_k,2}$ of St_{n_k} and a vertex $v_{k+1,1,1}$ of $St_{n_{k+1}}$ by a path P_{r_k} of arbitrary size r_k with $V(P_{r_k}) = \{w_{k,1}, w_{k,2}, \ldots \ldots \ldots, w_{k,r_k}\}$ and $E(P_{r_k}) = \{w_{k,i}\, w_{k,i+1} : 1 \leq i \leq r_k - 1\}$ where $k = 1, 2, \ldots \ldots \ldots, t-1$. Thus,

$G = \langle St_{n_1}, P_{r_1}, St_{n_2}, P_{r_2}, St_{n_3}, \ldots\ldots, St_{n_{t-1}}, P_{r_{t-1}}, St_{n_t} \rangle$ where each $n_k \geq 3$ and each $r_k \geq 2$. Note that $v_{k,n_k,2} = w_{k,1}$ and $w_{k,r_k} = v_{k+1,1,1}$, $\forall\, k = 1, 2, \ldots\ldots, t-1$.

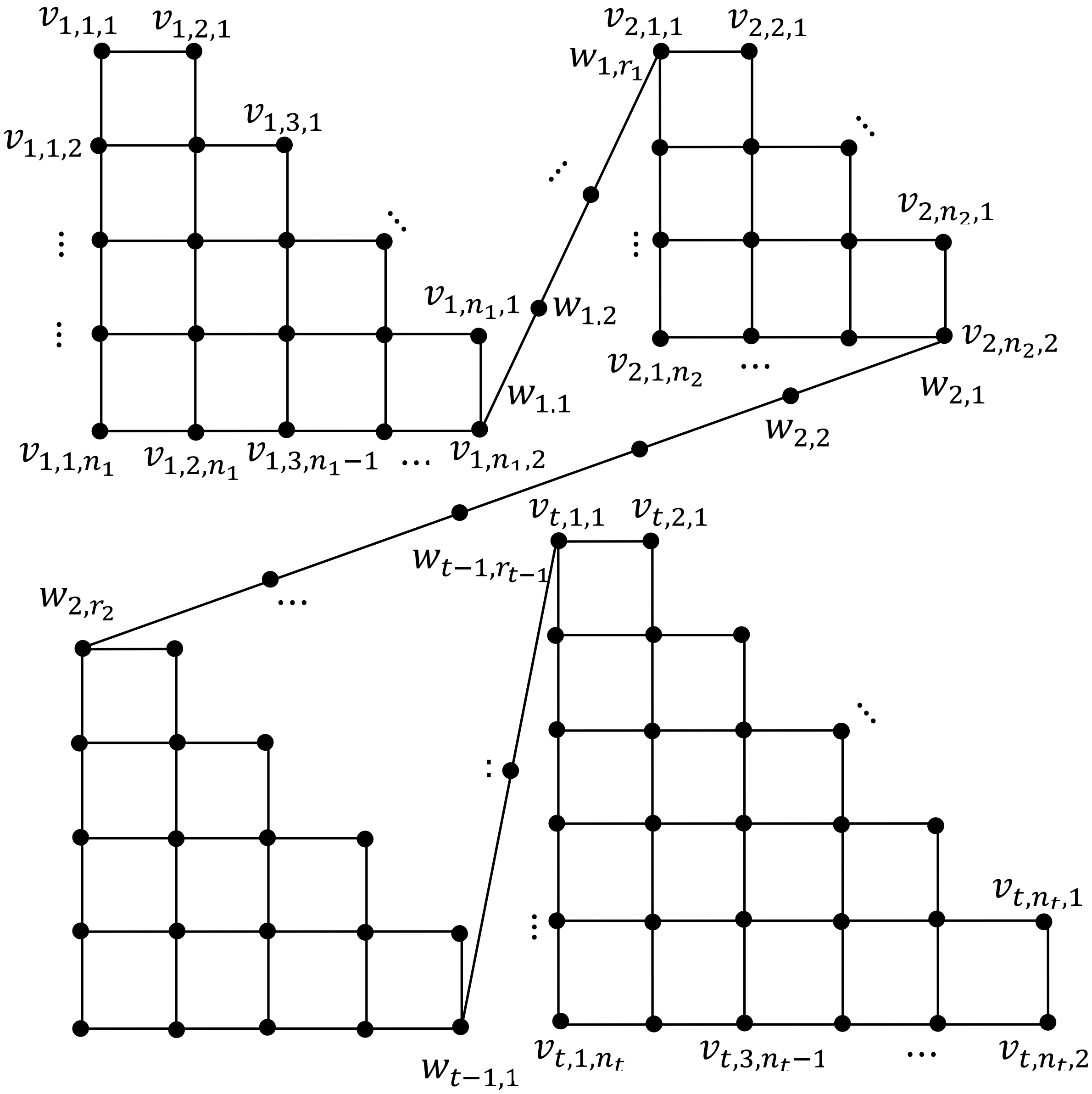

Figure – 9: Ordinary labeling of the graph $\langle St_{n_1}, P_{r_1}, St_{n_2}, P_{r_2}, St_{n_3}, \ldots\ldots, St_{n_{t-1}}, P_{r_{t-1}}, St_{n_t} \rangle$

Thus, we have $V(G) = \left(\bigcup_{k=1}^{t} V(St_{n_k})\right) \cup \left(\bigcup_{k=1}^{t-1} V(P_{r_k})\right)$, $E(G) = \left(\bigcup_{k=1}^{t} E(St_{n_k})\right) \cup \left(\bigcup_{k=1}^{t-1} E(P_{r_k})\right)$,

$p = |V(G)| = 2(1-t) + \sum_{k=1}^{t} p_k + \sum_{k=1}^{t-1} r_k$ and $q = |E(G)| = (1-t) + \sum_{k=1}^{t} q_k + \sum_{k=1}^{t-1} r_k$.

Now, we define the vertex labeling function $f: V(G) \to \{0, 1, 2, \ldots, q\}$ as follow:

$f(v_{1,1,1}) = 0$;

$f(v_{1,i,1}) = (i-1)(2n_1 + 2 - i) - 1, \forall\, i = 2, 3, \ldots\ldots, n_1$;

$f(v_{k,1,1}) = (1-k) + \sum_{j=1}^{k-1} q_j + \sum_{j=1}^{k-1} r_j, \forall\, k = 2, 3, \ldots\ldots, t$;

$f(v_{k,i,1}) = (i-1)(2n_k + 2 - i) - k + \sum_{j=1}^{k-1} q_j + \sum_{j=1}^{k-1} r_j, \forall\, k = 2, 3, \ldots\ldots, t, \forall\, i = 2, 3, \ldots\ldots, n_k$;

$f(v_{k,1,j}) = f(v_{k,1,j-1}) + 1, \forall\, k = 1, 2, \ldots\ldots, t, \forall\, j = 2, 3, \ldots\ldots, n_k$;

$f(v_{k,i,j}) = f(v_{k,i,j-1}) + 1, \forall\ k = 1, 2, \dots\dots, t, \forall\ i = 1, 2, \dots\dots, n_k$ and $\forall\ j = 2, 3, \dots\dots, n_k + 2 - I$;
$f(w_{k,i}) = f(v_{k,n_k,2}) + (i-1), \forall\ k = 1, 2, \dots, t-1, \ \forall\ i = 2,3, \dots\dots, r_k - 1$.
The induced edge labeling function $f^*: E(G) \to \{1, 3, 5, \dots, 2q-1\}$ is given by $f^*(uv) = f(u) + f(v)$, $\forall$ uv $\in E(G)$.
The above labeling pattern shows the odd sum labeling of the graph G. Thus, G is an odd sum graph. Hence, the graph $\langle St_{n_1}, P_{r_1}, St_{n_2}, P_{r_2}, St_{n_3}, \dots\dots, St_{n_{t-1}}, P_{r_{t-1}}, St_{n_t}\rangle$ is an odd sum graph.

Illustration 5: A graph $\langle St_5, P_4, St_4, P_6, St_5, P_3, St_6\rangle$ with its odd sum labeling is shown in Figure 10.

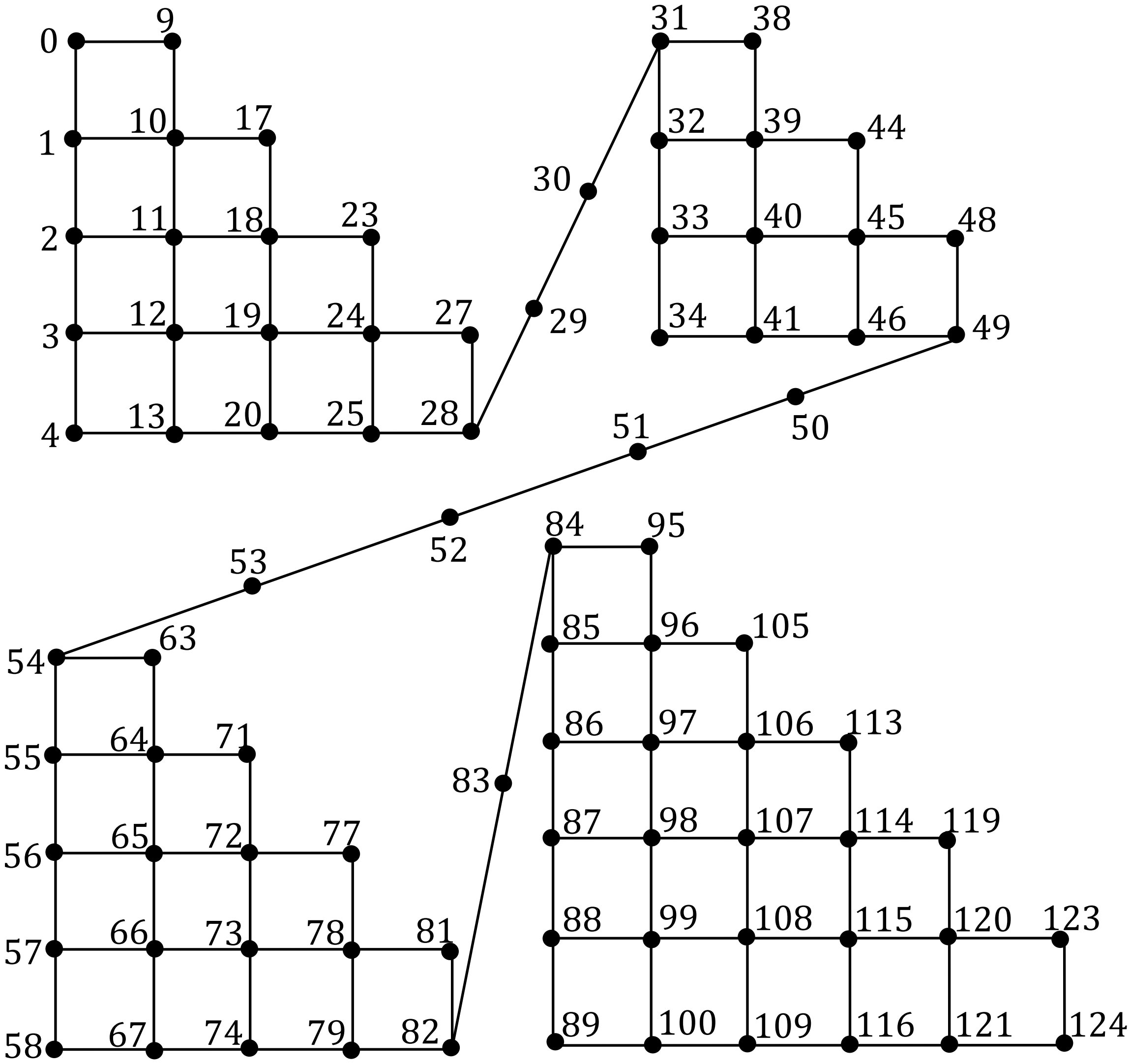

Figure – 10: Odd sum labeling of a graph $\langle St_5, P_4, St_4, P_6, St_5, P_3, St_6\rangle$

III. Conclusion

In this paper, we have discussed odd sum labeling of grid graph, path union of grid graphs with different size, graph obtained by joining vertex of a grid graph and a complete bipartite graph $K_{2,t}$ by a path, step grid graph and the graph obtained by joining step grid graphs of different size by arbitrary paths.

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