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On Hardy's Inequality for Several variables: Skew-Symmetric Functions

Esameldin Abdalla Sidahmed Yahiya [1], Zakieldeen Aboabuda Mohamed Alhassn Ali[2] and Mutaz Mohammed Elbagir Eltigani Abdelsalam [3].
[1]*Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.*
[2]*Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.*
[3]*Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia.*

Abstract
In this paper we assume that Hardy's inequality on a Skew-symmetric function. In general, it has much better constants. This enables us they depend on the lowest degree of spherical harmonics Skew-symmetric polynomial. We prove the existence of Hardy inequalities on the class of a Skew-symmetric function. In addition, we find some conditions for specific Schrödinger operators in Skew-symmetric functions that do not have nonpositive eigenvalues. Then we discuss some cases of Caffarelli-Kohn-Nirenberg inequalities and apply our results to spectral properties of Schrödinger operators.

Key Words: *Skew-symmetric function, Holder's inequality, Hardy inequality, Caffarelli-Kohn-Nirenberg inequality.*

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I. Introduction

We consider the different forms of Hardy's inequality and their applications are extensive, which cannot be covered in this paper. We only review paper [4] and Books [8], [9], [10] and [11].

In [14] and [15], Hardy's inequalities and their applications are studied in the spectral theory of Schrödinger's operators when $N = 2$.

The inferred inequality in (proposition (2.4) and Corollary (2.7) for $N = 2$ are finite cases of Caffarelli-Kohn-Nirenberg inequalities [1], which cannot be adopted without Skew-symmetric conditions.

Additional applications of Hardy's inequalities for Skew-symmetric functions are used to prove the spectral properties of the Schrödinger's operators with decay strength and corresponding estimates are given in [2].

There is a nontrivial inequality [3] because the zero point of the spectrum is not a resonant state of the Schrödinger magnetic operator with the Aharonoff-Bohm magnetic field in the 2-Dimention state in [5]. Also in [5],[6] some spectral inequality of parameters of voltage functions $in\ L^1\ (\mathbb{R}_+, L^\infty\ (S), r\ dr)$.

In this paper, we use 2-Dimention Hardy's inequality for Skew-symmetric functions, which allows us to show nonpositve eigenvalues for Schrödinger's operators in [Theorem 3. 2] .These classes were taken to prove the Lieb-Thirring inequality for Schrödinger's operators on Hardy in [23]. In [7; Proposition 4.1] the constant $C_A(N)$ depends on the lowest Dirichlet eigenvalue of the Laplace–Beltrami operator, and then we calculated this eigenvalue directly due to the special structure of the Skew-symmetric functions.

Finally, in [12],[13] we show the absence of bound cases in the triplet S-sector for Schrödinger's operators and Some properties of fermionic wave functions.

$(\mathbf{1.1})$ **Definition** $(classical\ Hardy\ inequality)$

$$\int_{\mathbb{R}^N} |\nabla v|^2\, dx \ \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2}\ dx. \qquad (1)$$

Where $v \in H^1(\mathbb{R}^N)$ and $N \geq 3$.

We consider Definition (1.1) on the class of Skew-symmetric functions $H^1(\mathbb{R}^N)$, which we denote by $H_A^1(\mathbb{R}^N)$. These functions are supposed satisfy the following Skew-symmetric conditions:

$$v(\dots, x_i, \dots, x_j, \dots) = -v(\dots, x_j, \dots, x_i, \dots) \qquad (2)$$

Where $x = (x_1, \dots, x_N) \in \mathbb{R}^N$.

Obviously $H_A^1(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$, the constant in (1.1) is expected to be much larger. To show that, for $v \in H_A(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, dx \ \geq \ C_A(N) \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} \ dx, \quad N \geq 2 \tag{3}$$

Where

$$C_A(N) = \frac{(N-2)^2}{4}$$

In [16] and [17] some sharp inequalities were obtained, and it was proved that for $v(x) = -v(-x) \in H^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, dx \ \geq \frac{N^2}{4} \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} \ dx \tag{4}$$

If $N = 2$, the constant corresponds with $C_A(N) = 1$.

(1.2) Definition $\left(Laplace\text{–}Beltrami\ operator\ \Delta_\theta\ on\ S^{N-1}\right)$

The Laplacian in polar coordinates (r, θ) is

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{N-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Delta_\theta$$

where $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$, $\theta = \frac{x}{r}$, $r = |x|, and\ x \in \mathbb{R}^N$.

$P_M(x)$ The harmonic homogeneous polynomial of degree M , and its spherical harmonic $Y_M = P_M/r^M$. The Y_M spherical harmonics are eigenfunctions of the $-\Delta_\theta$:

$$-\Delta_\theta Y_M = M(M + N - 2)Y_M = \lambda_{M,N}\, Y_M.$$

And the polymorphism of the eigenvalue $\lambda_{M,N}$ is

$$h(M,N) = \binom{N + M - 1}{N - 1} - \binom{N + M - 3}{N - 1}.$$

(1.3) proposition

Assume Ψ be an analytic Skew-symmetric function on $\mathbb{R}^N$ and $\mathcal{V}_N$ is Vandermonde determinant

$$\mathcal{V}_N = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_N \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_N^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & \dots & x_N^{N-1} \end{vmatrix} \tag{5}$$

Satisfying condition (1.2), Then ϕ the a symmetric analytic function such that $\Psi = \phi\, \mathcal{V}_N$.

Proof

Since Ψ is Skew-symmetric for $x_k = x_j$. We have $\Psi(x) = 0$. Hence Ψ has the factors $x_k - x_j$ for all $k = j$. From Properties of analytic Skew-symmetric functions we conclude $\phi(x)$ is symmetric and analytic, Such that:

$$\phi(x) = \frac{\Psi(x)}{\Pi_{k<j}(x_k - x_j)} = \Psi(x)(\mathcal{V}_N)^{-1}.$$

(1.4) Definition

Let $P_{M(N)} \neq 0$ of degree $M(N)$ such that there is a Skew-symmetric harmonic homogeneous polynomial by $M(N)$ and $N > 1$.The $\mathcal{V}_N$ defined in (1.5) is such a harmonic polynomial, then

$$M(N) = \frac{N(N-1)}{2}. \tag{6}$$

(1.5)Corollary

The Δ_θ defined on Skew-symmetric functions in $L^2(S^{N-1})$ satisfies the inequality in the quadratic form

$$-\Delta_\theta \geq M(N) = N(N - 1)/2$$

II. Main results

(2.1)Theorem

Let $N \geq 2$ and $v \in H_A^1(\mathbb{R}^N)$. Then

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, dx \ \geq \ C_A(N) \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} \ dx, \tag{7}$$

Where

$$C_A(N) = \frac{(N^2-2)^2}{4} \tag{8}$$

Proof
Let $x = (r, \theta), r \in (0, \infty), and\ \theta \in S^{N-1}$.Then

$$\int_{\mathbb{R}^N} |\nabla v|^2\, dx = \int_0^\infty \int_{S^{N-1}} \left(\left|\frac{\partial v}{\partial r}\right|^2 + \frac{1}{r^2} |\Delta_\theta v|^2 \right) r^{N-1} d\theta\, dr \qquad (9)$$

Suppose $\mathfrak{U} \equiv$ orthogonal of spherical harmonic functions and $\mathfrak{U}_A \subset \mathfrak{U}$, for any $v \in H_A^1(\mathbb{R}^N)$, then

$$v(r, \theta) = \sum_{k: Y_k \in \mathfrak{U}_A} v_k(r) Y_k(\theta). \qquad (10)$$

Where
$\mathfrak{U}_A$ be the orthogonal subset of $\mathfrak{U}$ skew-symmetric functions. We use $M(N) = min\{k: Y_k \in \mathfrak{U}_A\}$ and

$$\lambda_{M,N} = M(N)(M(N) + N - 2) = \frac{N(N-1)(N^2 + N - 4)}{4}$$

Then

$$\int_{S^{N-1}} |\nabla_\theta v(r,\theta)|^2\, d\theta = \sum_{k=M(N)}^{\infty} \lambda_k\, |v_k(r)|^2$$

$$\geq \lambda_{M,N} \sum_{k=M(N)}^{\infty} |v_k(r)|^2 = \lambda_{M,N} \int_{S^{N-1}} |v(r,\theta)|^2\, d\theta \qquad (11)$$

From the inequality (1) we find:

$$\int_0^\infty \left(\left|\frac{\partial v}{\partial r}\right|^2 \right) r^{N-1} dr \geq \frac{(N-2)^2}{4} \int_0^\infty \frac{|v|^2}{r^2} r^{N-1} dr. \qquad (12)$$

Substituting (11) and (12) into (9), finally we get

$$\int_{\mathbb{R}^N} |\nabla v|^2\, dx \geq \int_0^\infty \int_{S^{N-1}} \left(\left|\frac{\partial v}{\partial r}\right|^2 + \frac{\lambda_{M,N}}{r^2} |\Delta_\theta v|^2 \right) r^{N-1} d\theta\, dr$$

$$\geq C_A(N) \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2}\, dx.$$

(2.2)Proposition
The Inequality

$$\int_{\mathbb{R}^N} |\nabla v|^2\, dx \geq C_A(N) \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2}\, dx$$

is sharp.
Proof
Let $v_0(x) = \phi(r) Y_{M(N)}(\theta)$.
Replacing this function with formula (10), we get

$$\int_{\mathbb{R}^N} |\nabla v_0|^2\, dx \geq \int_0^\infty \int_{S^{N-1}} \left(\left|\frac{\partial \phi}{\partial r}\right|^2 \left|Y_{M(N)}\right|^2 + \frac{1}{r^2} |\phi|^2 \left|\Delta_\theta Y_{M(N)}\right|^2 \right) r^{N-1} d\theta\, dr$$

It is known that inequality

$$\int_0^\infty \left(\left|\frac{\partial \phi}{\partial r}\right|^2 \right) r^{N-1} dr \geq \frac{(N-2)^2}{4} \int_0^\infty \frac{|\phi|^2}{r^2} r^{N-1} dr. \qquad (13)$$

is sharp. Then

$$\int_{S^{N-1}} \frac{1}{r^2} \left|\Delta_\theta Y_{M(N)}\right|^2 = \lambda_{M,N} \int_{S^{N-1}} \frac{1}{r^2} \left|\Delta_\theta Y_{M(N)}\right|^2 \qquad (14)$$

Combining (13) and (14), we complete the proof.
(**2.3**) **Definition** (*classical Sobolev inequality*)

$$\left(\int_{\mathbb{R}^N} |v|^{2N/(N-2)}\, dx \right)^{(N-2)/N} \leq S_N \int_{\mathbb{R}^N} |\nabla v|^2\, dx \qquad (15)$$

Where $N \geq 3$ and S_N is sharp in [18] ,[19].

$$S_N = \frac{N(N-2)}{4} \, |S_N|^{2/N} = \frac{N(N-2)}{4} \, 2^{2/N} \, \pi^{1+1/N} \, \Gamma\left(\frac{N+1}{2}\right)^{-2/N} . \qquad (16)$$

(2.4)Proposition $(Caffarelli\text{–} Kohn\text{–} Nirenberg \ inequalities)$

Case 1: $N \geq 3$, we let

$$\alpha = \frac{2N}{N-2\vartheta}, \qquad \beta = 2N \, \frac{\vartheta - 1}{N - 2\vartheta}, \qquad 0 \leq \vartheta \leq 1,$$

Then

$$\left(\int_{\mathbb{R}^N} |x|^\beta \, |v|^\alpha dx\right)^{2/\alpha} \leq C_{N,\vartheta} \left(\int_{\mathbb{R}^N} |\nabla v|^2 \, dx\right)^{\vartheta} \left(\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} \, dx\right)^{1-\vartheta} \qquad (17)$$

For any skew-symmetric function $v \in H^1(\mathbb{R}^N)$, where

$$C_{N,\vartheta} \leq S_N^{\vartheta} . \qquad (18)$$

Proof

Applying (15) and Hölder's inequalities, we get

$$\left(\int_{\mathbb{R}^N} |x|^\beta \, |v|^\alpha dx\right)^{2/\alpha} = \left(\int_{\mathbb{R}^N} |v|^{\alpha\vartheta} \left(\frac{|v|^2}{|x|^2}\right)^{\alpha(1-\vartheta)} dx\right)^{2/\alpha}$$

$$\leq \left(\int_{\mathbb{R}^N} |v|^{\frac{2N}{N-2}} \, dx\right)^{\frac{\vartheta(N-2)}{N}} \left(\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} \, dx\right)^{1-\vartheta}$$

$$\leq S_N^{\vartheta} \left(\int_{\mathbb{R}^N} |\nabla v|^2 \, dx\right)^{\vartheta} \left(\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2} \, dx\right)^{1-\vartheta} .$$

We complete the proof.

(2.5)Proposition $(Caffarelli\text{–} Kohn\text{–} Nirenberg \ inequalities)$

Case 2: $N = 2$

Then there exists $C_{2,\vartheta} > 0$, for any $0 \leq \vartheta < 1$ and $v \in H_A^1(\mathbb{R}^2)$,

$$\left(\int_{\mathbb{R}^2} \frac{|v|^{2/(1-\vartheta)}}{|x|^2} \, dx\right)^{1-\vartheta} \leq C_{N,\vartheta} \left(\int_{\mathbb{R}^2} |\nabla v|^2 \, dx\right)^{\vartheta} \left(\int_{\mathbb{R}^2} \frac{|v|^2}{|x|^2} \, dx\right)^{1-\vartheta} \qquad (19)$$

Proof

Let

$$\bar{v} = \int_{B_\rho} v \, dx$$

Where $B_\rho = \{0 \leq |x| \leq \rho \ , \rho > 0\}$, For any skew-symmetric function $v \in H_A^1(\mathbb{R}^2)$,then by using the inequality in ([20], [22], [21], [11]) with $\alpha = \frac{2}{1-\vartheta}$, we get

$$\int_{B_\rho} |v|^\alpha \, dx C_{N,\vartheta} \leq C \left(\int_{B_\rho} |\nabla v|^2 \, dx\right)^{\vartheta\alpha/2} \left(\int_{B_\rho} |v|^2 \, dx\right)^{(1-\vartheta)\alpha/2}$$

$$\leq C \left(\int_{\mathbb{R}^2} |\nabla v|^2 \, dx\right)^{\vartheta\alpha/2} \int_{B_\rho} |v|^2 \, dx \qquad (20)$$

Multiply (20) by ρ^{-3} , since (20) is independent of the disc B_ρ with radius ρ, and using the identities

$$\int_0^\infty \rho^{-3} \int_{|x|\leq\rho} |v|^\alpha dx \, d\rho = \frac{1}{2} \int_{\mathbb{R}^2} \frac{|v|^\rho}{|x|^2} dx,$$

$$\int_0^\infty \rho^{-3} \int_{|x|\leq\rho} |v|^2 dx \, d\rho = \frac{1}{2} \int_{\mathbb{R}^2} \frac{|v|^2}{|x|^2} dx,$$

Finally we integrate with respect to ρ over $(0, \infty)$. We get

$$\int_{\mathbb{R}^2} \frac{|v|^\rho}{|x|^2}\,dx \le C \le C\left(\int_{\mathbb{R}^2} |\nabla v|^2\,dx\right)^{\vartheta\alpha/2} \int_{\mathbb{R}^2} \frac{|v|^2}{|x|^2}\,dx.$$

Combining (2.4) and (2.1), we obtain the following corollary

(2.6) Corollary

$$If\ \alpha = \frac{2N}{N-2\vartheta}, \quad \beta = 2N\,\frac{\vartheta-1}{N-2\vartheta}, \quad 0 \le \vartheta \le 1\ and\ N \ge 3,$$

For any skew-symmetric function $v \in H_A^1(\mathbb{R}^N)$, Then

$$\frac{C_A(N)^{1-\vartheta}}{C_{N,\vartheta}}\left(\int_{\mathbb{R}^N} |x|^\beta\,|v|^\alpha dx\right)^{2/\alpha} \le C_{N,\vartheta}\left(\int_{\mathbb{R}^N} |\nabla v|^2\,dx\right)^{\vartheta}\left(\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^2}\,dx\right)^{1-\vartheta} \quad (21)$$

(2.7) Corollary

$If\ C_A(2) = 1, \alpha = \frac{2}{1-\vartheta}, 0 \le \vartheta < 1,$ and applying Proposition(2.5), we obtain

$$\left(\int_{\mathbb{R}^2} \frac{|v|^{2/(1-\vartheta)}}{|x|^2}\,dx\right)^{1-\vartheta} \le C_{2,\vartheta}\int_{\mathbb{R}^2} |\nabla v|^2\,dx \quad (22)$$

For any skew-symmetric function $v \in H_A^1(\mathbb{R}^2)$.

III. Applications to Spectral Properties of Schrödinger's Operators

(3.1) Definition $(Schrödinger's\ Operator)$

$$H = -\Delta - V, \qquad V \ge 0,$$

On $L^2(\mathbb{R}^N)$ and its quadratic form

$$(Hv, v) = \int_{\mathbb{R}^N} (|\nabla v|^2 - V\,|v|^2)dx. \quad (23)$$

(3.2) Theorem

Assume that α and ϑ satisfy the assumptions of Corollaries (2.6) and (2.7) if $N \ge 3$ and $N = 2$ respectively. Let

$$\frac{C_A(N)^{1-\vartheta}}{C_{N,\vartheta}}\left(\int_{\mathbb{R}^N} V^{N/(2\vartheta)}\,|x|^{(1-\vartheta)N/(2\vartheta)}\,dx\right)^{N/(2\vartheta)} \le 1$$

Then

$$H = -\Delta - V \ge 0, \qquad is\ positive. \quad (24)$$

If $N = 2$, then (24) provided that

$$C_{2,\vartheta}\left(\int_{\mathbb{R}^2} V^{1/\vartheta}\,|x|^{(1-\vartheta)/\vartheta}\,dx\right)^{\vartheta} \le 1.$$

Proof

we use Holder's inequality, we find

$$\int_{\mathbb{R}^N} V\,|v|^2\,dx = \int_{\mathbb{R}^N} V\,|x|^\alpha\,|v|^2 |x|^{-\alpha}\,dx$$

$$\le \left(\int_{\mathbb{R}^N} V^q\,|x|^{\alpha q}\,dx\right)^{1/q}\left(\int_{\mathbb{R}^N} |x|^{-\alpha p}\,V\,|v|^p\,dx\right)^{2/p}$$

Where $1/q + 2/p = 1$, then

$$\frac{1}{q} = 1 - 2\frac{N-2\vartheta}{2N} = \frac{2\vartheta}{N}.$$

Choosing

$$\alpha = -\frac{\gamma}{p} = -2N\frac{\vartheta-1}{N-2\vartheta}\frac{N-2\vartheta}{2N} = 1-\vartheta,$$

we have

$$\alpha q = (1-\vartheta)\frac{N}{2\vartheta}.$$

Using Corollary (2.6), we obtain

$$\int_{\mathbb{R}^N}(|\nabla v|^2 - V\,|v|^2)dx \geq \left(1 - \frac{C_A(N)^{1-\vartheta}}{C_{N,\vartheta}}\left(\int_{\mathbb{R}^N} V^{N/(2\vartheta)}\,|x|^{(1-\vartheta)N/(2\vartheta)}\,dx\right)^{N/(2\vartheta)}\right)\int_{\mathbb{R}^N} V\,|v|^2\;dx \geq 0.$$

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