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A Simple Characterization of the Fatou Set of a Class of Polynomials

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Abstract: *In the course of the iteration of a function of a complex variable, the resulting family of iterates provide a surprising occurrence. This paper carefully investigates this occurrence and provides a general characterization for obtaining the Fatou sets for a class of polynomials. The contrast in the nature of these sets for different classes is then illustrated using computer programming techniques and graphics and the results are consistent with what obtains in the literature.*

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I. Introduction.

Coming from an informed background of several problems on iteration of function of a complex variable arising from investigations from the local and the global theories, we shall exploit the recent trends used in the global theory by Jian Hua et al (2002), Qiu and Wu (2006) as well as Braverman and Yampolsky (2008). The example which is the subject of the paper is meant to illustrate interesting patterns which abound as we search for new directions in mathematical research that involve mathematical analysis and computer programming, Rippon and Stallard (2005) provide an excellent introduction to the current interaction between mathematical analysis and computer programming.

The Global Theory

Under the global theory of iteration initiated by the works of Pierre Fatou (1919, 1920) and Gaston Julia (1918), it considers the iteration of rational and entire transcendental functions and is usually described as the Fatou – Julia theory. Indeed it is well known that unless $f(z)$ is a rational function of order 0 or 1, $\mathcal{F}(f(z))$, $the\ Faou\ set\ of\ f$, is always a non- empty perfect set with no interior points unless it's the whole plane. Other results established by the founding fathers include:

1.2.1 $\mathcal{F}\{f_n(z)\} = \mathcal{F}\{f(z)\}$ for every integer $n \geq 1$

1.2.2 $\mathcal{F}\{f(z)\}$ is completely invariant under the mapping $z\colon\to f(z)$ ie if $\alpha \in \mathcal{F}(f(z))$, then so will $f(\alpha)$ and every solution $\beta\ of\ f(\beta) = \alpha$

1.2.3 For every $\alpha \in \mathcal{F}\{f(z)\}$ and for every complex number β (excluding at most two exceptional β −values) there exists a sequence of positive integers $\{n_k\}, k = 1,2, \ldots$ and a sequence of complex numbers $\{\alpha_k\}$ such that $f_{n_k}(\alpha_k) = \rho$ and $\lim_{k\to\infty} \alpha_k = \alpha$

1.2.4 $\mathcal{F}\{f(z)\}$, contains every repulsive fixed point of $f(z)$.

1.2.5 $\mathcal{F}\{f(z)\}$ is the derivative of the set of fixed points of $f(z)$

A central role is played by the Fatou set $\mathcal{F}\{f(z)\}$ of those points in the complex plane, in the neighbourhood of which no sequence of iterates,$\{f_n(z)\}$, n = 0,1,2,… , form a normal family in the sense of Montel.

II. Characterization of the Fatou Set of a Class of Polynomials

We begin our consideration by quoting a result of Brohn, (1965), and use the following techniques inspired by the works of Jian Hua et al. (2002); Qin and Wa (2006), Braverman and Yampolski(2008) as well as Liverpool and Egahi (2012).

We shall need the following lemma which we proceed to give;

Lemma 2.1.1

Let $f(z) = z^3 + pz$, be a polynomial with p real. Then, $\mathcal{F}\{z^3 + pz\}$ is totally disconnected. If $p < -3$, then $\mathcal{F}\{z^3 + pz\}$ is real and $\mathcal{F}\{z^3 + pz\} \subset [-q_2, q_2]\}$, where$q_2 = (1-p)^{1/2}$.

We now investigate $\mathcal{F}\{(z^3 - 8z)\}$, with $p = -8$, then clearly ± 3 are repulsive fixed points and therefore belong to $\mathcal{F}$.

Let $E_o = [-3,\ 3\,]$, then $\mathcal{F}\{z^3 - 8z\} \subset E_o$ by lemma 2.1.1. We then map $E_o$by the three inverse branches of $f(x)$ and denote $f_{-1}(E_o)$ by E_1.

By solving $x^3 - 8x + 3 = 0$, we get the solutions -3, $\frac{1}{2} - \frac{1}{2}\sqrt{5}$, and $\frac{1}{2}\sqrt{5} + \frac{3}{2}$.
Similarly, considering $x^3 - 8x - 3 = 0$, we get the solutions $3, -\frac{1}{2}\sqrt{5} - \frac{1}{2}$ and $\frac{1}{2}\sqrt{5} + \frac{3}{2}$.
From these, we see that $E_1 = \left[-3,\ \frac{1}{2}\sqrt{5} - \frac{3}{2}\right] \cup \left[\frac{1}{2}\sqrt{5} - \frac{3}{2},\ \frac{3}{2} - \frac{1}{2}\sqrt{5}\right] \cup \left[\frac{3}{2} + \frac{1}{2}\sqrt{5},\ 3\right]$
We then map E_1 by $f_{-1}(x)$ and so on. Then $\mathcal{F} \subset \cdots \subset E_{n+1} \subset E_n \subset \cdots \subset E_o$.
Where $f_{-1}(E_n) = E_{n+1}, n = 0,1.2 \cdots$
Now we see that E_n is the union of 3^n disjoint intervals, whose end points belong to $\mathcal{F}$.
We denote the intervals by ω_{n_v} where v varies from 1 to 3^n
Thus, $E_n = \cup_{v=1}^{3^n} \omega_{n_v}$ and $\mathcal{F} = \cap_{n=1}^{\infty} E_n$.
By this analysis, we have constructed $\mathcal{F}\{(z^3 - 8z)\}$.
It is a generalized cantor set on the real line, symmetric with respect to the origin.
We now seek a generalization of this argument as follows:
We consider $\mathcal{F}\{z^3 + pz\}$ with p real and p< -3. Then $q = \sqrt{(1-p)}$ is a repulsive fixed point, for p < 3 , and therefore belong to $\mathcal{F}$.
Let $E_o = [-q, q]$. then, clearly $\mathcal{F}\{z^3 + pz\} \subset E_o$ by lemma 2.1.1
We now map E_o by the inverse branches of $f(x)$ and denote $f_{-1}(E_o)$ by E_1.
Then E_1 is clearly mapped by $f_{-1}(x)$ and so on. It is clear that $\mathcal{F} \subset \cdots \subset E_{n+1} \subset E_n \subset \cdots \subset E_o$ where $f_{-1}(E_n) = E_{n+1}, n = 0,1,2 \cdots$.
It follows that E_n is the union of 3^n disjoint intervals whose end points belong to $\mathcal{F}$. Denote same by w_{n_v}, where v varies from 1 to 3^n,
Thus $E_n = \bigcap_{v=1}^{3^n} \omega_{n_v}$, and $\mathcal{F} = \bigcap_{n=1}^{\infty} E_n$.
We now seek the solution of the equations $x^2 + px = q$ and $x^2 + px = -q$ and get
$E_1 = \left[-q, \frac{-q-q'}{2}\right] \cup \left[\frac{q'-q}{2},\ \frac{q-q'}{2}\right] \cup \left[\frac{q+q'}{2}, q\right]$, where $q' = \sqrt{(-3-p)}$.
We have thus constructed $\mathcal{F}\{(z^3 + pz)\}$, with p real and p$< -3$. It is a generalized cantor set on the real line, symmetric with respect to the origin. In this general case, we can prove that $\frac{q-q'}{2} < \sqrt{\frac{-p}{3}}$.
Indeed $\frac{q-q'}{2} < \sqrt{\frac{-p}{3}}$, if $\frac{\sqrt{(1-p)} - \sqrt{(-3-p)}}{2} < \sqrt{\frac{-p}{3}}$
That is, if $1 - p - 3 - p - 2\frac{\sqrt{(1-p)}.\sqrt{-3-p}}{4} < \frac{-p}{3}$
i.e. if $9(p^2 + 2p - 3) > (p+3)^2$
i.e. if $(p+3)(2p-3) > 0$ and the result is established.
We have therefore established theorem 2.1.1
Theorem 2.1.1 Let $f(z) = z^3 + pz$ with p real and p < -3, then $\mathcal{F}\{(z^3 + pz)\} = \cap_{n=1}^{\infty} E_n$ where $q = \sqrt{(1-p)}$, $E_o = [-q, q]$, $f_{-1}(E_o) = E_1$ and inductively $f_{-1}(E_n) = E_{n+1}, n = 0,1,2 \cdots$ with $E_n = \cup_{n=1}^{3^n} \omega_{n_v}$ is the union of 3^n disjoint intervals, ω_{n_v}, whose end points belong to $\mathcal{F}$ and $\mathcal{F} \subset \cdots \subset E_{n+1} \subset E_n \subset \cdots \subset E_o$. It follows that E_n is the union of 3^n disjoint intervals whose end points belong to $\mathcal{F}$. It is a generalized cantor set on the real line, symmetric with respect to the origin with $\frac{q-q'}{2} < \sqrt{\frac{-p}{3}}$ where $q' = \sqrt{(-3-p)}$ and where q' defines E_1.

3.1 The Examples. We present here the plots of Fatou sets $F\{(z^3 + pz)\}$ for specific values of p. These sets are connected for $p \geq -3$ and disconnected for $p < -3$. The computer generated plots of the connected and disconnected Fatou sets confirm that the analytic exploration, sometimes produces more profound results. The change in the properties of the Fatou set as p changes value is of current interest and could be explored using a combination of computer programming and mathematical analysis techniques. Diagram 3a contains Fig 3.1a, 3.2a and 3.3a of connected Julia sets and their corresponding Fatou setsand Diagram 3b which contains Fig 3.1b, 3.2b, and 3.3b of disconnected Julia sets and their corresponding Fatou sets respectively for chosen values of a class of polynomials.

Diagram 3a is a plot of connected Julia sets and their corresponding Fatou sets for a class of polynomials.

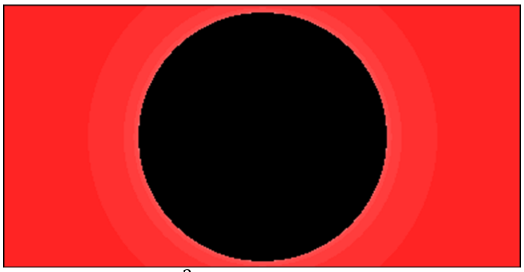
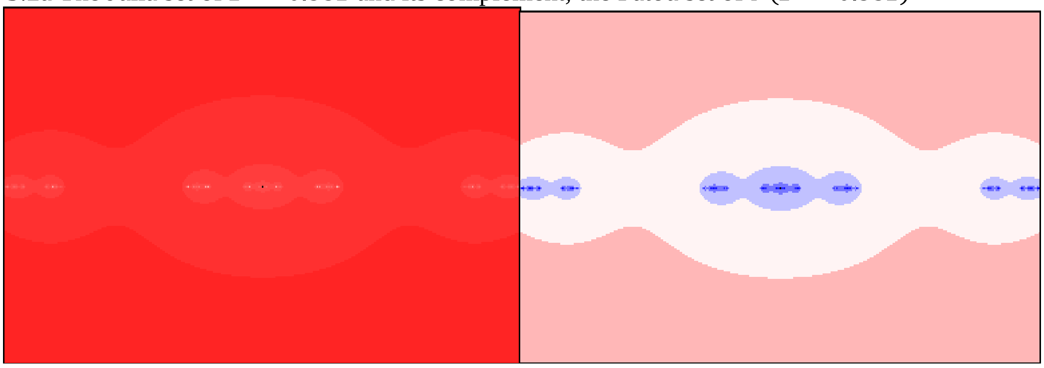


Figure 3.1a: The Julia set of z^3 and its complement, the Fatou set of $F(z^3)$

Fig 3.2a The Julia set of $z^3 - 0.55z$ and its complement, the Fatou set of $F(z^3 - 0.55z)$



(a) (b)

Fig 3.3a The Julia set of $z^3 - 4z$ and its corresponding complement, the Fatou set of $F(z^3 - 4z)$

Figure (a) and (b) are the same only that the iteration for (a) is 100 while that of (b) is 20.

Diagram 3b–is a plot of disconnected Julia sets and their corresponding Fatou sets.

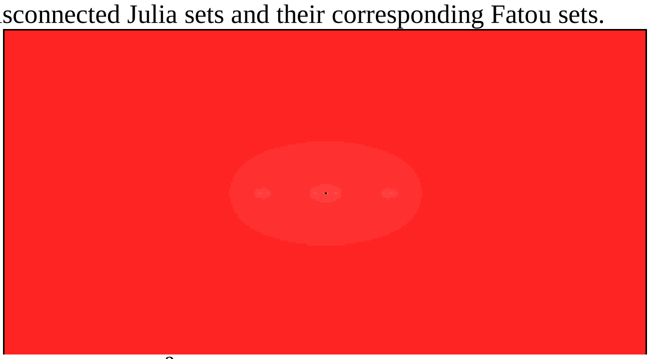


Fig 3.1b The Julia set of $z^3 - 6z$ and its complement, the Fatou set of $F(z^3 - 6z)$

Fig 3.2b The Julia set of $z^3 - 3z$ and its complement, the Fatou set of $F(z^3 - 3z)$
The Julia set is the interval [-2, 2], the black line.

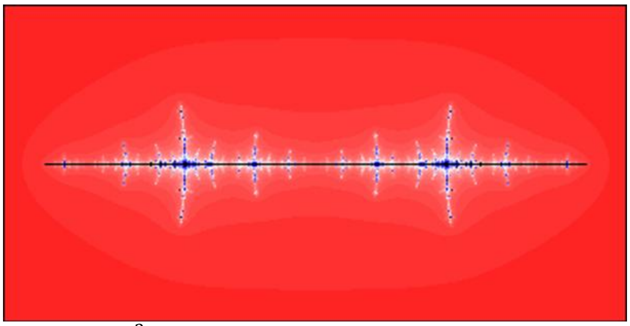


Fig 3.3b The Julia set of $z^3 - 2.3z$ and its complement, the Fatou set of $F(z^3 - 2.3z)$

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