IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 6 Ser. I (Nov. – Dec. 2022), PP 05-06
www.iosrjournals.org

A Strange Sequence

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Date of Submission: 23-10-2022 Date of Acceptance: 05-11-2022

Let 0 <a<b, and consider the sequence, $\{x_n\}$, defined by

$$x_0 = a \qquad x_1 = b \qquad x_{n+1} = \left(1-\frac{1}{2n}\right)x_n + \left(\frac{1}{2n}\right)x_{n-1}$$

When n = 1, the recursive equation yields $x_2 = \left(1-\frac{1}{2}\right)x_1 + \left(\frac{1}{2}\right)x_0 = \frac{a+b}{2}$.When n = 2, the recursive equation yields

$$x_3 = \left(1-\frac{1}{4}\right)x_2 + \left(\frac{1}{4}\right)x_1 = \left(\frac{3}{4}\right)\left(\frac{a+b}{2}\right) + \left(\frac{1}{4}\right)b = \frac{3a+3b+2b}{8} = \frac{3a+5b}{8}$$

Fact: $\boxed{\lim_{n\to\infty} x_{n+1} = a + \frac{b-a}{\sqrt{e}}}$

Proof: $x_{n+1} = \left(1-\frac{1}{2n}\right)x_n + \left(\frac{1}{2n}\right)x_{n-1} = x_n - \frac{x_n}{2n} + \frac{x_{n-1}}{2n} = x_n + \frac{1}{2n}\left(x_{n-1} - x_n\right) \Rightarrow$

$$x_{n+1} - x_n = \frac{-1}{2n}\left(x_n - x_{n-1}\right) \text{(*)}$$

Replacing n by n – 1, yields $x_n - x_{n-1} = \frac{-1}{2n-2}\left(x_{n-1} - x_{n-2}\right)$

which, combined with (*), yields $x_{n+1} - x_n = \frac{(-1)^2}{2n(2n-2)}\left(x_{n-1} - x_{n-2}\right)$

Continuing in this manner, we get $x_{n+1} - x_n = \frac{(-1)^3}{2n(2n-2)(2n-4)}\left(x_{n-2} - x_{n-3}\right)$

Finally $x_{n+1} - x_n = \frac{(-1)^n}{2n(2n-2)(2n-4)\cdots 4\cdot 2}\left(x_1 - x_0\right) = \frac{(-1)^n}{2^n n!}\left(b-a\right) \Rightarrow$

$$x_{n+1} - x_n = (-1)^n \frac{\left(\frac{1}{2}\right)^n}{n!}\left(b-a\right) \text{(**)}$$

Using (**) repeatedly yields

$$x_{n+1} - x_n = (-1)^n \frac{\left(\frac{1}{2}\right)^n}{n!}\left(b-a\right)$$

$$x_n - x_{n-1} = (-1)^{n-1} \frac{\left(\frac{1}{2}\right)^{n-1}}{(n-1)!}\left(b-a\right)$$

$$x_{n-1} - x_{n-2} = (-1)^{n-2} \frac{\left(\frac{1}{2}\right)^{n-2}}{(n-2)!}\left(b-a\right)$$

$$\vdots$$

$$x_1 - x_0 = (-1)^0 \frac{\left(\frac{1}{2}\right)^0}{0!}\left(b-a\right) = b-a$$

Adding these equations, and noticing that the left sides form a telescoping sequence, we have, letting *n*go to infinity:

$$\lim_{n\to\infty}(x_{n+1} - x_0) = \lim_{n\to\infty}\left(1 - \frac{\left(\frac{1}{2}\right)^1}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} - \frac{\left(\frac{1}{2}\right)^3}{3!} + \ldots\right)\left(b-a\right) = e^{-\frac{1}{2}}(b-a) \Rightarrow$$

$$\lim_{n\to\infty} x_{n+1} = a + e^{-\frac{1}{2}}(b-a) = a + \frac{b-a}{\sqrt{e}} \blacksquare$$