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Some Topological Properties of Bicomplex Space and Its Subsets

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Abstract
Bicomplex space is one of the most recent generalizations of the complex space which have found applications in some of the most recent branches of science, e.g., electromagnetics, quantum mechanics, telecommunications and medical sciences.

Recently, in 2008, Srivastava has systematically put forward the topological structure of the bicomplex space C_2. He has defined three types of topologies on C_2. Certain best possible inclusion relations between the basis elements of these topologies have been established

In this paper, we have defined the concept of dense subsets in C_2 equipped with the idempotent topology. It has been shown that the metric space C_2 with Euclidean metric is separable. We have given two counter examples which show that, the openness (closedness) of a subset S in C_2 is not necessary for the idempotent parts ${}^1S$ and ${}^2S$ to be open (closed) in A_1 and A_2, respectively.

Keywords: Bicomplex idempotent topology, open idempotent Discus, Idempotent parts of a set, denseness, open and closed sets.

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Introduction: Bicomplex numbers were discovered by Corrado Segre [S1] in 1892. Throughout this paper the set of bicomplex numbers is denoted by C_2 and the set of real and complex numbers are denoted by C_0 and C_1 respectively.

A bicomplex numbers is defined as follows:

$$\xi = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in C_0$$

where $i_1^2 = i_2^2 = -1;\ i_1i_2 = i_2i_1$.

The set of bicomplex numbers is defined as follows:

$$C_2 = \{ x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in C_0 \}$$

or equivalently as

$$C_2 = \{ z_1 + i_2z_2 : z_1, z_2 \in C_1 \} .$$

Idempotent elements:
Beside 0 and 1, there are exactly two nontrivial idempotent elements in C_2 denoted as e_1 and e_2 and defined as $e_1 = (1 + i_1 i_2)/2$, $e_2 = (1 - i_1 i_2)/2$.
Obviously, $e_1 + e_2 = 1$ and $e_1.e_2 = e_2.e_1 = 0$.

Every bicomplex number $\xi = z_1 + i_2 z_2$ has a unique idempotent representation as a complex combination of the non trivial idempotent elements, viz. $\xi = {}^1\xi .e_1 + {}^2\xi .e_2$. The complex numbers ${}^1\xi = z_1 - i_1 z_2$, ${}^2\xi = z_1 + i_1 z_2$ are called the **idempotent components** of ξ (cf. [S2]).
The **idempotent parts** of a set S in C_2 are denoted as ${}^1S$ and ${}^2S$ and defined as
${}^1S = \{{}^1\xi : \xi \in S\}$ and ${}^2S = \{{}^2\xi : \xi \in S\}$.
Norm in C_2 is defined as

$$\|\xi\| = \left\{|z_1|^2 + |z_2|^2\right\}^{1/2} = \left\{\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}\right\}^{1/2}.$$

Under this norm, C_2 is a modified Banach algebra.
The Auxiliary Complex Spaces:
Define the complex spaces A_1, A_2 as follows: $A_1 = \{(z_1 - i_1 z_2) : z_1, z_2 \in C_1\}$
$A_2 = \{(z_1 + i_1 z_2) : z_1, z_2 \in C_1\}$.
Cartesian Idempotent set [S2]: The Cartesian (idempotent) set determined by the set P in A_1 and Q in A_2 is denoted as $P \times_e Q$ and is defined as

$$X = P \times_e Q = \left\{ w_1 e_1 + w_2 e_2 : w_1 \in P,\ w_2 \in Q \right\}.$$

Open Ball: An open ball with centre ξ and radius r is denoted as $B(\xi; r)$ and is defined as $B(\xi; r) = \{\eta : \eta \in C_2, \|\xi - \eta\| < r\}$.

Open idempotent Discus: An open idempotent discus with centre ξ and associated radii r_1 and r_2 is denoted as $D(\xi; r_1, r_2)$ and is defined as the Cartesian (idempotent) set determined by open circular discs $S({}^1\xi; r_1)$ in A_1 and $S({}^2\xi; r_2)$ in A_2. Thus,

$$D(\xi; r_1, r_2) = \left\{ \eta : \eta \in C_2, |{}^1\xi - {}^1\eta| < r_1, |{}^2\xi - {}^2\eta| < r_2 \right\}.$$

Srivastava **[S2]** has defined three types of topologies on C_2 as follows:

- **(i)** The Norm Topology τ_1 on C_2, a basis for which is the collection of all open balls.
- **(ii)** The Idempotent Topology τ_2 on C_2, a basis for which is the collection of all open (idempotent) discuses.
- **(iii)** The Complex Topology τ_3 on C_2, a basis for which is the collection of all open (complex) discuses.

§1: In this section, we explore some basic sets in the Bicomplex Topological space (C_2, τ_2). For definitions of the topological terms in this section, we refer to [M1].
The results in bicomplex analysis which are required are as follows (cf. [P1]):

Functions $h_1 : C_2 \to A_1$ and $h_2 : C_2 \to A_2$ are defined as $h_1(\xi) = {}^1\xi$ and $h_2(\xi) = {}^2\xi$, respectively.

(i) The functions h_1 and h_2, restricted to a set X in C_2, map X into sets ${}^1X$ and ${}^2X$ in A_1 and A_2 respectively. Thus

$$h_1(X) = {}^1X, \qquad X \subset C_2, {}^1X \subset A_1;$$
$$h_2(X) = {}^2X, \qquad X \subset C_2, {}^2X \subset A_2.$$

(ii) If X is an open set in C_2, then both ${}^1X$ and ${}^2X$ are open sets in A_1 and A_2, respectively.

(iii) If X is an closed set in C_2, then both ${}^1X$ and ${}^2X$ are closed sets in A_1 and A_2, respectively.
we have obtained following results:

Theorem 1.1: If a subset X is dense in C_2, then ${}^1X$ and ${}^2X$ are dense in A_1 and A_2 respectively.

Proof: Let z be an arbitrary point of $A_1 \sim {}^1X$ then there is a point ξ in C_2 such that $h_1(\xi) = {}^1\xi = z$. Also, $h_2(\xi) = {}^2\xi = u$ (say) is a point in A_2. Obviously, $\xi \notin X$. Since X is dense in C_2, ξ is a limit point of X. Given $\varepsilon > 0$, there exist a point say $\eta \in X$ such that $\eta \in B(\xi; \varepsilon)$, $\eta \neq \xi$.

Now, $\eta \in B(\xi; \varepsilon) \Rightarrow \|\eta - \xi\| < \varepsilon$

$$\Rightarrow \left\{ \frac{\left|{}^1\eta - {}^1\xi\right|^2 + \left|{}^2\eta - {}^2\xi\right|^2}{2} \right\}^{1/2} < \varepsilon$$

$$\Rightarrow \left|{}^1\eta - {}^1\xi\right| < \sqrt{2}\varepsilon \text{ and } \left|{}^2\eta - {}^2\xi\right| < \sqrt{2}\varepsilon$$

Since, $\eta \in X \Rightarrow h_1(\eta) = {}^1\eta = w \in {}^1X$

$\Rightarrow |w - z| < \sqrt{2}\varepsilon$

$\Rightarrow$ z is a limit point of ${}^1X$

$\Rightarrow$ each point in $A_1 \sim {}^1X$ is a limit point of ${}^1X$

$\Rightarrow {}^1X$ is dense in A_1.

Similarly we can show that ${}^2X$ is dense in A_2.

The converse of this theorem does not seem to be true, in general.

We are working on a counter example for this. However we have the following result for the cartesian idempotent set.

Theorem 1.2: Let X be the cartesian idempotent set determined by ${}^1X$ in A_1 and ${}^2X$ in A_2. If ${}^1X$ and ${}^2X$ are dense in A_1 and A_2, respectively, then X is dense in C_2 .

Proof: Let ξ be an arbitrary point of $C_2 \sim X$, $D(\xi; r_1, r_2)$ be an arbitrary open neighborhood of ξ, in the idempotent topology.

Since, $\xi \in C_2 \sim X$ three cases arise:

(i) ${}^1\xi \in A_1 \sim {}^1X$, ${}^2\xi \in {}^2X$

(ii) ${}^2\xi \in A_2 \sim {}^2X$, ${}^2\xi \in {}^1X$

(iii) ${}^1\xi \in A_1 \sim {}^1X$, ${}^2\xi \in A_2 \sim {}^2X$

We take up case (iii).

Case (iii): Suppose that ${}^1\xi \in A_1 \sim {}^1X$ and ${}^2\xi \in A_2 \sim {}^2X$. Since ${}^1X$ is dense in A_1, there exist a point $z \in {}^1X$ such that $z \in S({}^1\xi; r_1)$. Also, since ${}^2X$ is dense in A_2, $\exists$ $w \in {}^2X$ such that $w \in S({}^2\xi; r_2)$.

The discus generated by open circular disc $S({}^1\xi; r_1)$ in A_1 and $S({}^2\xi; r_2)$ in A_2 is $D(\xi; r_1, r_2)$ such that $\eta = ze_1 + we_2 \in D(\xi; r_1, r_2)$. Also since $z \in {}^1X$, $w \in {}^2X$ and X is the Cartesian idempotent set determined by ${}^1X$ and ${}^2X$, $\eta = ze_1 + we_2 \in X$.

$\Rightarrow \xi \in C_2 \sim X$ is a limit point of X

$\Rightarrow X$ is dense in C_2. Similarly we can prove other cases.

Theorem 1.3: C_2 is a separable space.

Proof: Let $X = \{ \xi = x_0 + i_1x_1 + i_2x_2 + i_1i_2x_3 : x_0, x_1, x_2, x_3 \in Q \}$. Then the one to one correspondence between $Q \times Q \times Q \times Q$ and X is given by the map

$$(x_0, x_1, x_2, x_3) \mapsto \xi = x_0 + i_1x_1 + i_2x_2 + i_1i_2x_3$$

and, since $Q \times Q \times Q \times Q$ is countable, X is countable. Let $\xi = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \in C_2 \sim X$ and $\varepsilon > 0$ be given. Since Q is dense in C_0, there exist rationales b_0, b_1, b_2, b_3 such that

$$|a_0 - b_0| < \frac{\varepsilon}{4}, |a_1 - b_1| < \frac{\varepsilon}{4}, |a_2 - b_2| < \frac{\varepsilon}{4} \text{ and } |a_3 - b_3| < \frac{\varepsilon}{4}.$$

Then, since $\|(a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3) - (b_0 + i_1 b_1 + i_2 b_2 + i_1 i_2 b_3)\|$ =

$$\left\{|a_0 - b_0|^2 + |a_1 - b_1|^2 + |a_2 - b_2|^2 + |a_3 - b_3|^2\right\}^{\frac{1}{2}} < \left\{\frac{\varepsilon^2}{16} + \frac{\varepsilon^2}{16} + \frac{\varepsilon^2}{16} + \frac{\varepsilon^2}{16}\right\}^{\frac{1}{2}}$$

$$= \frac{\varepsilon}{2}, \quad \text{i.e. } X \text{ is dense,}$$

so that X is countable dense subset of C_2.

The metric space C_2 with Euclidean metric is separable.

§2: In this section we have constructed two counter examples. The first counter example asserts that the converse of the result **(ii)** in **§2** is not true in general. The next counter example proves similar behavior of closed sets.

(i) Define a set $S = P \cup Q$ in C_2, where P is the set consisting of all bicomplex numbers for which $\|\xi\| < 2$ and Q is the singleton set $\{\eta\}$ where $\eta = 2e_1 + 2i_1 e_2$.

Note that $\xi \in P \Rightarrow \|\xi\| < 2$

$$\Rightarrow \left\{\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}\right\}^{1/2} < 2$$

$$\Rightarrow |{}^1\xi| < 2\sqrt{2} \text{ and } |{}^2\xi| < 2\sqrt{2}$$

$$\Rightarrow {}^1P = \left\{{}^1\xi : |{}^1\xi| < 2\sqrt{2}\right\} \text{ and } {}^2P = \left\{{}^2\xi : |{}^2\xi| < 2\sqrt{2}\right\}$$

Similarly, $\eta \in Q \Rightarrow \eta = 2e_1 + 2i_1 e_2 \Rightarrow {}^1\eta = 2$, ${}^2\eta = 2i_1$

$$\Rightarrow {}^1Q = \left\{{}^1\eta : {}^1\eta = 2\right\}, {}^2Q = \left\{{}^2\eta : {}^2\eta = 2i_1\right\}$$

We claim that

(i) $S = P \cup Q$ is not open in C_2 .

(ii) ${}^1S = {}^1P \cup {}^1Q$ and ${}^2S = {}^2P \cup {}^2Q$ are open sets in A_1 and A_2 respectively.

Since $\|\eta\| = 2 \Rightarrow \eta = 2e_1 + 2i_1e_2$ is a boundary point of the open ball $B(0; 2)$.

Thus S is not open set in C_2.

$${}^1S = {}^1P \cup {}^1Q = \left\{{}^1\xi : \left|{}^1\xi\right| < 2\sqrt{2}\right\} \cup \left\{{}^1\eta : {}^1\eta = 2\right\} = \left\{{}^1\xi : \left|{}^1\xi\right| < 2\sqrt{2}\right\}$$

Since ${}^1S$ is an open circular disc in A_1, this implies that ${}^1S$ is an open set in A_1.

Similarly, it can be shown that ${}^2S$ is open set in A_2.

(ii) Define a set $S = P \cup Q$ in C_2, where P is the set consisting of all bicomplex numbers for which $\|\xi\| \le r$ and Q is the consisting of all bicomplex numbers for which $\|\xi - \alpha\| < \frac{r}{4}$, where $\alpha = re_1 + ri_1e_2$.

$$\xi \in P \Rightarrow \|\xi\| \le r$$

$$\Rightarrow \left\{\frac{\left|{}^1\xi\right|^2 + \left|{}^2\xi\right|^2}{2}\right\}^{1/2} \le r$$

$$\Rightarrow \left|{}^1\xi\right| \le \sqrt{2}r \text{ and } \left|{}^2\xi\right| \le \sqrt{2}r$$

$$\Rightarrow {}^1P = \left\{{}^1\xi : \left|{}^1\xi\right| \le \sqrt{2}r\right\} \text{ and } {}^2P = \left\{{}^2\xi : \left|{}^2\xi\right| \le \sqrt{2}r\right\}.$$

Similarly, $\xi \in Q \Rightarrow \|\xi - \alpha\| < \frac{r}{4}$

$$\Rightarrow \left\{\frac{\left|{}^1\xi - {}^1\alpha\right|^2 + \left|{}^2\xi - {}^2\alpha\right|^2}{2}\right\}^{1/2} < r/4$$

$$\Rightarrow \left|{}^1\xi - {}^1\alpha\right| < \frac{r}{2\sqrt{2}} \text{ and } \left|{}^2\xi - {}^2\alpha\right| < \frac{r}{2\sqrt{2}}$$

$$\Rightarrow \left|{}^1\xi - {}^1\alpha\right| < \frac{r}{2\sqrt{2}} \text{ and } \left|{}^2\xi - {}^2\alpha\right| < \frac{r}{2\sqrt{2}}$$

$$\Rightarrow {}^1Q = \left\{{}^1\xi : \left|{}^1\xi - r\right| < \frac{r}{2\sqrt{2}}\right\} \text{ and } {}^2Q = \left\{{}^2\xi : \left|{}^2\xi - ri_1\right| < \frac{r}{2\sqrt{2}}\right\}.$$

We claim that

(i) $S = P \cup Q$ is not closed

(ii) ${}^1S = {}^1P \cup {}^1Q$ and ${}^2S = {}^2P \cup {}^2Q$ are closed sets in A_1 and A_2 respectively.

Let $\xi = \frac{5r}{4}e_1 + \frac{5r}{4}i_1e_2$ Note that $\|\xi\| = \frac{5r}{4} > r$, so that ξ lies outside the ball P. Further, $\|\alpha - \xi\| = \frac{r}{4}$ implies that ξ does not belong to the open ball Q. Hence, $\xi \notin S$. To show that ξ is a limit point of S, let R = $B(\xi;\varepsilon)$ be an arbitrary neighborhood around ξ. Since $\|\alpha - \xi\| = \frac{r}{4}$, which is less than $\frac{r}{4} + \varepsilon$. This means that distance between centers of the open balls R and Q is less than sum of its radii.

Hence ξ is a limit point of $B\left(\alpha; \frac{r}{4}\right)$, and hence limit point of S which does not belongs to S. Thus $S = P \cup Q$ is not closed.

(ii) ${}^1S = {}^1P \cup {}^1Q = \left\{{}^1\xi : \left|{}^1\xi\right| \le \sqrt{2}r\right\} \cup \left\{{}^1\xi : \left|{}^1\xi - r\right| < \frac{r}{2\sqrt{2}}\right\}$

Let z be an arbitrary point of ${}^1Q \Rightarrow |z - r| < \frac{r}{2\sqrt{2}}$.

$$\Rightarrow |z| \le |z - r| + |r| < \frac{r}{2\sqrt{2}} + r$$

$$\Rightarrow |z| < \sqrt{2}r$$

$$\Rightarrow z \in {}^1P$$

$$\Rightarrow {}^1S = {}^1P, \text{ which is closed disc in } A_1,$$

Thus ${}^1S$ is closed in A_1.

Similarly it can be shown that ${}^2S$ is closed set in A_2.

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