IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 18, Issue 6 Ser. II (Nov. – Dec. 2022), PP 61-66
www.iosrjournals.org

Transitivity and sensitivity of non-autonomous discrete product systems

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Abstract:*This paper studies the chaotic properties between product non-autonomous discrete systems $(X\times Y, f_{1,\infty}\times g_{1,\infty})$ and its factor non-autonomous discrete systems $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$. The following results are obtained. (1) $f_{1,\infty}\times g_{1,\infty}$ is chain transitive if and only if $f_{1,\infty}$ and $g_{1,\infty}$ is chain transitive. (2) If $f_{1,\infty}\times g_{1,\infty}$ is* F *-poin-transitive, then $f_{1,\infty}$ and $g_{1,\infty}$ is* F *-point-transitive. (3) If $f_{1,\infty}\times g_{1,\infty}$ is* F *-transitive (resp. topologically exact), then $f_{1,\infty}$ and $g_{1,\infty}$ is* F *-transitive (resp. topologically exact).(4) $f_{1,\infty}\times g_{1,\infty}$ iscofinitely sensitive if and only if $f_{1,\infty}$ or $g_{1,\infty}$ is cofinitely sensitive. (5) If $f_{1,\infty}$ or $g_{1,\infty}$ is* F *-sensitive, then $f_{1,\infty}\times g_{1,\infty}$ is* F *-sensitive.*

Background*: The question concerning stability and chaoticityof dynamical systems is of significant importance not only in theoretical mathematics, but also in physics, technology etc. Many of them are still unresolved, for example Navier-Stokes equation describing motion of fluids, thus any information addressing them is of use. Among those, number of problems can be reduced to nonautonomous discrete mappings*.

Materials and Methods*: Theoretical analysis, proof and deduction.*

Results*: (1) $f_{1,\infty}\times g_{1,\infty}$ is chain transitive if and only if $f_{1,\infty}$ and $g_{1,\infty}$ is chain transitive. (2) If $f_{1,\infty}\times g_{1,\infty}$ is* F *-poin-transitive, then $f_{1,\infty}$ and $g_{1,\infty}$ is* F *-point-transitive. (3) If $f_{1,\infty}\times g_{1,\infty}$ is* F *-transitive (resp. topologically exact), then $f_{1,\infty}$ and $g_{1,\infty}$ is* F *-transitive (resp. topologically exact).(4) $f_{1,\infty}\times g_{1,\infty}$ iscofinitely sensitive if and only if $f_{1,\infty}$ or $g_{1,\infty}$ is cofinitely sensitive. (5) If $f_{1,\infty}$ or $g_{1,\infty}$ is* F *-sensitive, then $f_{1,\infty}\times g_{1,\infty}$ is* F *-sensitive.*

Conclusion:*According to the results of this paper, for the chaotic properties such aschain transitive,* F *-point-transitive,* F *-transitive, cofinitely sensitive,* F *-sensitive,etc., the discussion of the product system can be simplified to the discussion of the factor systems. This is conducive to dimension reduction, thus simplifying the research process in practical problems.*

Key Word*: transitivity; sensitivity; non-autonomous discrete systems; product system.*

Date of Submission: 07-12-2022 Date of Acceptance: 21-12-2022

I. Introduction

Chaos is an inherent characteristic of nonlinear dynamical systems and a common phenomenon in nonlinear systems. The study of non-autonomous discrete systems (NDDSs) is an enduring hot spot in the field of chaos research. In 1996, the concept of NDDSs was first proposed by Kolyada and Snoha[1]. They also defined topological entropy in NDDSs. Since then,many scholars have studied in this direction (for example [2-9]).

Let Z^+ denotes the set of positive integers. (X,d) is a compact metric space with the metric d, $f_n: X\to X \quad (n\in Z^+)$ is a continuous mapping sequence. Denote $f_{1,\infty}=(f_1, f_2,\cdots)=(f_n)_{n=1}^{\infty}$, under this mapping sequence, the orbit of the point $x\in X$ is

$$Orb_{f_{1,\infty}}(x)=(x, f_1(x), f_2\circ f_1(x),\cdots, f_1^n(x),\cdots),$$

where $f_1^n(x)=f_n\circ f_{n-1}\circ\cdots\circ f_1(x)$ .Especially, f_1^0 represents identity mapping.

In NDDSs, Tian and Chen ([4], 2006) studied chaotic properties in the sense of Devaney. Shi and Chen ([5], 2009) introduce several concepts of chaotic characteristics, such as transitive, sensitive, Li-Yorke chaos,and so on. Balibrea and Oprocha[6], 2012) studied Li-Yorke chaos, topological weak mixing, and topological entropy in a compact metric space. Vasisht and Das ([7], 2018) obtained necessary and sufficient conditions for a non-autonomous system to be $\mathcal{F}$ -transitive and $\mathcal{F}$ -mixing.Then, they discussed the stronger forms of sensitivity via Furstenberg families ([8], 2019). Recently, [9] define the concepts of collectively accessible, collectively sensitive, collectively infinitely sensitive and collectively Li-Yorke sensitive in NDDSs. And it is proved that, if the mapping sequence $f_{1,\infty}=(f_1, f_2,\cdots)$ is $\mathcal{P}$ -chaotic, then $f_{n,\infty}=(f_n, f_{n+1},\cdots)$ would also be $\mathcal{P}$ -chaotic. Where $\mathcal{P}$ -chaotic represents one of the following five properties: sensitive, collectively accessible, collectively sensitive, collectively infinitely sensitive, and collectively Li–Yorke sensitive. More studies on the chaotic properties of non-autonomous discrete systems see [10-16] and others.

Inspired by [9], the present paper study transitive properties between product non-autonomous discrete systems $(X\times Y, f_{1,\infty}\times g_{1,\infty})$ and factor non-autonomous discrete systems $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$, such as chain transitive, $\mathcal{F}$ -point-transitive, $\mathcal{F}$ -transitive, and topologically exact. In addition, sensitive properties between product systems and factor systems, such as cofinitely sensitive, and $\mathcal{F}$ -sensitive, are discussed.

This paper is organized as follows. In Section 2, some basic definitions are proposed, and relationships between these concepts are given. In Section 3, under the metric $d((x_1,y_1),(x_2,y_2))=max\{d_1(x_1,x_2),d_2(y_1,y_2)\}$, some necessary or sufficient conditions for transitive properties and sensitive properties of non-autonomous product systems are given.

II. Definitions and relationships between them

For $M,N\in Z^+$ with $M\le N$, denote

$$[M,N]=\{M,M+1,M+2,\cdots,N\} \text{ and } [M,\infty)=\{M,M+1,M+2,\cdots\}.$$

A δ -chain from x to y of length n is a finite sequence $x_0=x, x_1,\cdots, x_n=y$ such that $d(f_{i+1}(x_i),x_{i+1})<\delta$ for any $i=0,1\cdots,n-1$. $S\subset Z^+$ is said to be thick if S contains arbitrarily large blocks of consecutive numbers. S is called syndeticif $Z^+\setminus S$ is not thick, i.e., for any $m\in Z^+$, there is an $M\in Z^+$ such that $[m,M]\cap S\ne\varnothing$. $S\subset Z^+$, we say that S is cofinite if $Z^+\setminus S$ is finite, i.e., there is an $M\in Z^+$ such that $[M,\infty)\subset S$. $S\subset Z^+$ is called with thepositive upper density if

$$\limsup_{n\to+\infty}\frac{\#(S\cap\{1,2,\cdots n\})}{n}>0,$$

where $\#(\cdot)$ be the cardinality of the set.

For $A\subset X$ and $\delta>0$, let

$$N_{f_{1,\infty}}(A,\delta)=\{n\in Z^+ : \text{there exist } x,y\in A \text{ satisfying } d(f_1^n(x), f_1^n(y))>\delta\}.$$

For any point $x\in X$ and pair of nonempty open sets $U,V\subset X$, let

$$N_{f_{1,\infty}}(x,U)=\{n\in Z^+ : f_1^n(x)\in U\} \text{ and } N_{f_{1,\infty}}(U,V)=\{n\in Z^+ : f_1^n(U)\cap V\ne\varnothing\}.$$

The following give some definitions relate to transitivity and sensitivity.

Definition 2.1 Let $(X, f_{1,\infty})$ be a NDDS. $\mathcal{F}$ is a Furstenberg family.

(1) $f_{1,\infty}$ is topologically transitive if the set $N_{f_{1,\infty}}(U,V)$ is nonempty for every pair of nonempty open sets $U,V\subset X$;

(2) $f_{1,\infty}$ issyndetically transitive if the set $N_{f_{1,\infty}}(U,V)$ is syndetic for every pair of nonempty open sets $U,V\subset X$;

(3) $f_{1,\infty}$ is topologically mixing if the set $N_{f_{1,\infty}}(U,V)$ is cofinite for every pair of nonempty open sets $U,V\subset X$;

(4) $f_{1,\infty}$ is topologically ergodic if the set $N_{f_{1,\infty}}(U,V)$ has positive upper density for every pair of nonempty open sets $U,V\subset X$;

(5) $f_{1,\infty}$ is chain transitive if for any $\delta>0$ and any two points $x, y\in X$, there exists an $n\in Z^+$ such that a δ -chain from x to y of length n can be found;

(6) $f_{1,\infty}$ is $\mathcal{F}$ -point-transitive if there exists a point $x\in X$ such that $N_{f_{1,\infty}}(U,V)\in\mathcal{F}$ for every nonempty open set $U,V\subset X$, where x is called a $\mathcal{F}$ -transitive point of $f_{1,\infty}$;

(7) $f_{1,\infty}$ is $\mathcal{F}$ -transitive if $N_{f_{1,\infty}}(U,V)\in\mathcal{F}$ for every pair of nonempty open sets $U,V\subset X$;

(8) $f_{1,\infty}$ is topologically exact if there exists some $n\in Z^{+}$ such that $f_1^{n}(U)=X$ for every nonempty open set $U\subset X$.

Remark 2.1 According to the above Definition 2.1 and the literature [10], one can get that, topological mixing implies syndetic transitivity, and syndetic transitivity implies topological ergodicity, etc. That is,

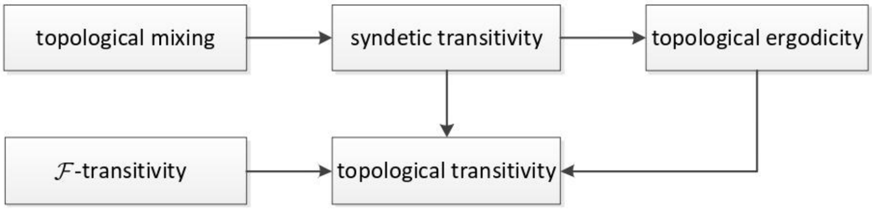

Figure 1:Relationships of some concepts relate to transitivity

Definition 2.2 Let $(X, f_{1,\infty})$ be a NDDS. $\mathcal{F}$ is a Furstenberg family.

(1) $f_{1,\infty}$ is sensitive if there exists a $\delta>0$ such that $N_{f_{1,\infty}}(A,\delta)\neq\varnothing$ for any nonempty open set $A\subset X$, where δ is called sensitivity constant of $f_{1,\infty}$;

(2) $f_{1,\infty}$ issyndetically sensitive if there is a $\delta>0$ such that $N_{f_{1,\infty}}(A,\delta)$ is syndetic for any nonempty open set $A\subset X$;

(3) $f_{1,\infty}$ is multi-sensitive if there is a $\delta>0$ such that

$$\bigcap_{i=1}^{k} N_{f_{1,\infty}}(A_i,\delta)\neq\varnothing$$

for any integer $k>0$ and for any nonempty open sets $A_1, A_2,\cdots A_k$;

(4) $f_{1,\infty}$ iscofinitely sensitive if there is a $\delta>0$ such that $N_{f_{1,\infty}}(A,\delta)$ is cofinite for any nonempty open set $A\subset X$;

(5) $f_{1,\infty}$ isergodically sensitive if there is a $\delta>0$ such that $N_{f_{1,\infty}}(A,\delta)$ has positive upper density for any nonempty open set $A\subset X$;

(6) $f_{1,\infty}$ is $\mathcal{F}$ -sensitiveif there is a $\delta>0$ such that $N_{f_{1,\infty}}(U,V)\in\mathcal{F}$ for any nonempty open set $A\subset X$;

(7) $f_{1,\infty}$ is double multi-sensitive if $f_{1,\infty}\times f_{1,\infty}$ is multi-sensitive;

(8) $f_{1,\infty}$ is double ergodically sensitive if $f_{1,\infty}\times f_{1,\infty}$ is ergodically sensitive;

(9) $f_{1,\infty}$ is double $\mathcal{F}$ -sensitive if $f_{1,\infty}\times f_{1,\infty}$ is $\mathcal{F}$ -sensitive.

Remark 2.2 According to Definition 2.2 and literatures [11-13], it is not difficult to obtain the relationships of these concepts as follow.

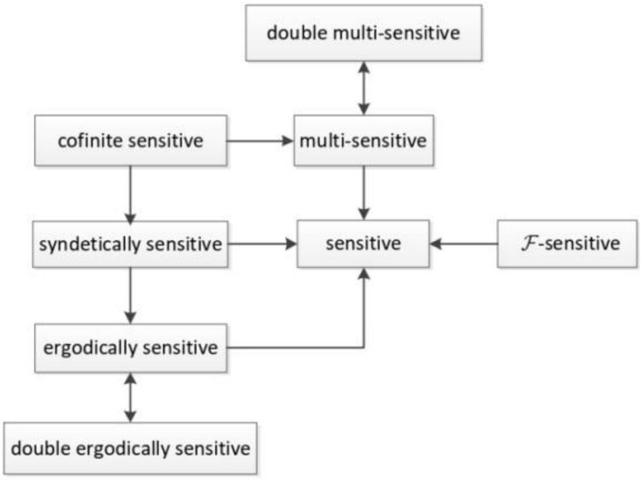

Figure 2: Relationships of concepts relate to sensitivity

III. Transitivity and sensitivity of product systems

Let $f_{1,\infty}=(f_n)_{n=1}^{\infty}$ and $g_{1,\infty}=(g_n)_{n=1}^{\infty}$ are two continuous mapping sequences on compact metric spaces (X,d_1) and (Y,d_2). The product system of $(X,f_{1,\infty})$ and $(Y,g_{1,\infty})$ is defined as $(X\times Y, f_{1,\infty}\times g_{1,\infty})$.

For any $(x_1,y_1),(x_2,y_2)\in X\times Y$. This paper defined

$$d((x_1,y_1),(x_2,y_2))=max\{d_1(x_1,x_2),d_2(y_1,y_2)\}$$

is the metric on $X\times Y$. The following conclusions are based on the metric d on compact metric space $X\times Y$.

Theorem 3.1 $f_{1,\infty}\times g_{1,\infty}$ is chain transitive if and only if $f_{1,\infty}$ and $g_{1,\infty}$ are chain transitive.

Proof. (Necessity) If $f_{1,\infty}\times g_{1,\infty}$ is chain transitive, then for any $\delta>0$ and $(x,y),(\bar{x},\bar{y})\in X\times Y$, there exist $(x_0,y_0)=(x,y),(x_1,y_1),\cdots,(x_n,y_n)=(\bar{x},\bar{y})$ such that $d((f_{i+1}\times g_{i+1})(x_i,y_i),(x_{i+1},y_{i+1}))<\delta$. So

$$\begin{aligned}d((f_{i+1}\times g_{i+1})(x_i,y_i),(x_{i+1},y_{i+1}))&=d((f_{i+1}(x_i),g_{i+1}(y_i)),(x_{i+1},y_{i+1}))\\&=max\{d_1(f_{i+1}(x_i),x_{i+1}),d_2(g_{i+1}(y_i),y_{i+1})\}<\delta.\end{aligned}$$

Thus, $d_1(f_{i+1}(x_i),x_{i+1})<\delta$ and $d_2(g_{i+1}(y_i),y_{i+1})<\delta$. Therefore, $f_{1,\infty}$ and $g_{1,\infty}$ are chain transitive.

(Sufficiency) Assume that $f_{1,\infty}$ and $g_{1,\infty}$ are chain transitive. Then for any $\delta>0$ and $x,\bar{x}\in X$, $y,\bar{y}\in Y$, there exist $x_0=x,x_1,,\cdots,x_n=\bar{x}$ and $y_0=y,y_1,,\cdots,\ y_n=\bar{y}$ such that

$$d_1(f_{i+1}(x_i),x_{i+1})<\delta \text{ and } d_2(g_{i+1}(y_i),y_{i+1})<\delta.$$

So

$$max\{d_1(f_{i+1}(x_i),x_{i+1}),d_2(g_{i+1}(y_i),y_{i+1})\}<\delta,$$

i.e.,

$$d((f_{i+1}\times g_{i+1})(x_i,y_i),(x_{i+1},y_{i+1}))<\delta.$$

Thus, $f_{1,\infty}\times g_{1,\infty}$ is chain transitive.

Theorem 3.2 If $f_{1,\infty}\times g_{1,\infty}$ is $\mathcal{F}$ -point-transitive, then $f_{1,\infty}$ and $g_{1,\infty}$ are $\mathcal{F}$ -point-transitive.

Proof. Since $f_{1,\infty}\times g_{1,\infty}$ is $\mathcal{F}$ -point-transitive, then there is a $\mathcal{F}$ -transitive point $(x,y)\in X\times Y$ such that the set $N_{f_{1,\infty}\times g_{1,\infty}}((x,y),U\times V)\in\mathcal{F}$ for any nonempty open set $U\times V\subset X\times Y$. For a given positive integer $n\in N_{f_{1,\infty}\times g_{1,\infty}}((x,y),U\times V)$, one has $(f_1^n\times g_1^n)((x,y))=(f_1^n(x),g_1^n(y))\in U\times V$. So $f_1^n(x)\in U$ and $g_1^n(x)\in V$. That is to say, $n\in N_{f_{1,\infty}}(x,U)$ and $n\in N_{g_{1,\infty}}(y,V)$. So

$$N_{f_{1,\infty}\times g_{1,\infty}}((x,y),U\times V)\subset N_{f_{1,\infty}}(x,U) \text{ and } N_{f_{1,\infty}\times g_{1,\infty}}((x,y),U\times V)\subset N_{g_{1,\infty}}(y,V).$$

By the upward heritability of $\mathcal{F}$, $N_{f_{1,\infty}}(x,U)\in\mathcal{F}$ and $N_{g_{1,\infty}}(y,V)\in\mathcal{F}$.

Then $f_{1,\infty}$ and $g_{1,\infty}$ are $\mathcal{F}$ -point-transitive.

Theorem 3.3 If $f_{1,\infty}\times g_{1,\infty}$ is $\mathcal{F}$ -transitive, then $f_{1,\infty}$ and $g_{1,\infty}$ are $\mathcal{F}$ -transitive.

Proof. Since $f_{1,\infty}\times g_{1,\infty}$ is $\mathcal{F}$ -transitive, then for any nonempty open sets $U_1\times V_1, U_2\times V_2\subset X\times Y$, the set $N_{f_{1,\infty}\times g_{1,\infty}}(U_1\times V_1, U_2\times V_2)\in\mathcal{F}$. For a given positive integer $n\in N_{f_{1,\infty}\times g_{1,\infty}}(U_1\times V_1, U_2\times V_2)$,since

$$(f_1^n\times g_1^n)(U_1\times V_1)\cap(U_2\times V_2)\neq\varnothing,$$

then $f_1^n(U_1)\cap U_2\neq\varnothing$ and $g_1^n(V_1)\cap V_2\neq\varnothing$. That is to say, $n\in N_{f_{1,\infty}}(U_1\times U_2)$ and $n\in N_{g_{1,\infty}}(V_1\times V_2)$. So

$$N_{f_{1,\infty}\times g_{1,\infty}}(U_1\times V_1, U_2\times V_2)\subset N_{f_{1,\infty}}(U_1\times U_2)$$

and

$$N_{f_{1,\infty}\times g_{1,\infty}}(U_1\times V_1, U_2\times V_2)\subset N_{g_{1,\infty}}(V_1\times V_2).$$

According to the upward heritability of $\mathcal{F}$, $N_{f_{1,\infty}}(U_1\times U_2)\in\mathcal{F}$ and $N_{g_{1,\infty}}(V_1\times V_2)\in\mathcal{F}$.

Thus, $f_{1,\infty}$ and $g_{1,\infty}$ is $\mathcal{F}$ -transitive.

Theorem 3.4 If $f_{1,\infty}\times g_{1,\infty}$ is topologically exact, then $f_{1,\infty}$ and $g_{1,\infty}$ are topologically exact.

Proof. Since $f_{1,\infty}\times g_{1,\infty}$ is topologically exact, then for any nonempty open set $U\times V\subset X\times Y$, there exist some $n\in Z^+$, such that

$$(f_1^n\times g_1^n)(U\times V)=f_1^n(U)\times g_1^n(V)=X\times Y.$$

So, $f_1^n(U)=X$, $g_1^n(V)=Y$. Thus, $f_{1,\infty}$ and $g_{1,\infty}$ are topologically exact.

Theorem 3.5 $f_{1,\infty}\times g_{1,\infty}$ is cofinitely sensitive if and only if $f_{1,\infty}$ or $g_{1,\infty}$ is cofinitely sensitive.

Proof. (Necessity) Assume that $f_{1,\infty}\times g_{1,\infty}$ is cofinitely sensitive. Then there is a $\delta>0$ such that $N_{f_{1,\infty}\times g_{1,\infty}}(A\times B,\delta)$ is cofinite for any nonempty open set $A\times B\subset X\times Y$. That is to say, there is an $M>0$, for any $n\geq M$, there are two points $(a_1,b_1),(a_2,b_2)\in A\times B$ such that

$$\begin{aligned}
&d((f_1^n\times g_1^n)(a_1,b_1),(f_1^n\times g_1^n)(a_2,b_2))\\
&=d((f_1^n(a_1),g_1^n(b_1)),(f_1^n(a_2),g_1^n(b_2))\\
&=max\{d_1(f_1^n(a_1),f_1^n(a_2)),d_2(g_1^n(b_1),g_1^n(b_2))\}\\
&>\delta.
\end{aligned}$$

Then, there at least one of the inequality $d_1(f_1^n(a_1),f_1^n(a_2))>\delta$ or $d_2(g_1^n(b_1),g_1^n(b_2))>\delta$ is held. Without loss of generality, assume that $d_1(f_1^n(a_1),f_1^n(a_2))>\delta$. Then there is a $\delta>0$ such that $N_{f_{1,\infty}}(A,\delta)$ is cofinite for any nonempty open set $A\subset X$ .So, $f_{1,\infty}$ is cofinitely sensitive.

(Sufficiency) Assume that $f_{1,\infty}$ is cofinitely sensitive. Then there is a $\delta>0$ such that $N_{f_{1,\infty}}(A,\delta)$ is cofinite for any nonempty open set $A\subset X$, i.e., there is an $M>0$, for any $n\geq M$, there are two points $a_1,a_2\in A$ such that $d_1(f_1^n(a_1),f_1^n(a_2))>\delta$. Thus, for a nonempty open set $B\subset Y$ and any $b_1,b_2\in B$,

$$\begin{aligned}
&d((f_1^n\times g_1^n)(a_1,b_1),(f_1^n\times g_1^n)(a_2,b_2))\\
&=d((f_1^n(a_1),g_1^n(b_1)),(f_1^n(a_2),g_1^n(b_2))\\
&=max\{d_1(f_1^n(a_1),f_1^n(a_2)),d_2(g_1^n(b_1),g_1^n(b_2))\}\\
&\geq d_1(f_1^n(a_1),f_1^n(a_2))\\
&>\delta
\end{aligned}$$

Obviously, $A\times B$ is a nonempty open set in $X\times Y$. Therefore, $f_{1,\infty}\times g_{1,\infty}$ is cofinitelysensitive.

Theorem 3.6 If $f_{1,\infty}$ or $g_{1,\infty}$ is $\mathcal{F}$ -sensitive, then $f_{1,\infty}\times g_{1,\infty}$ is $\mathcal{F}$ -sensitive.

Proof. Assume that $f_{1,\infty}$ is $\mathcal{F}$ -sensitive. Then there is a $\delta > 0$ such that $N_{f_{1,\infty}}(A,\delta) \in \mathcal{F}$ for any nonempty open set $A \subset X$. For a given positive integer $n \in N_{f_{1,\infty}}(A,\delta)$, there are two points $a_1, a_2 \in A$ such that $d_1(f_1^n(a_1), f_1^n(a_2)) > \delta$. Then for a nonempty open set $B \subset Y$ and any $b_1, b_2 \in B$, one has

$$\begin{aligned} & d((f_1^n \times g_1^n)(a_1,b_1),(f_1^n \times g_1^n)(a_2,b_2)) \\ & = d((f_1^n(a_1), g_1^n(b_1)),(f_1^n(a_2), g_1^n(b_2)) \\ & = max\{d_1(f_1^n(a_1), f_1^n(a_2)), d_2(g_1^n(b_1), g_1^n(b_2))\} \\ & \geq d_1(f_1^n(a_1), f_1^n(a_2)) \\ & > \delta . \end{aligned}$$

So, $n \in N_{f_{1,\infty}\times g_{1,\infty}}(A\times B,\delta)$. And because $N_{f_{1,\infty}}(A,\delta) \in \mathcal{F}$ and $N_{f_{1,\infty}}(A,\delta) \subset N_{f_{1,\infty}\times g_{1,\infty}}(A\times B,\delta)$ .By the upward heritability of $\mathcal{F}$, $N_{f_{1,\infty}\times g_{1,\infty}}(A\times B,\delta) \in \mathcal{F}$ .Thus, $f_{1,\infty} \times g_{1,\infty}$ is $\mathcal{F}$ -sensitive.

Remark 3.1 In fact, many scholars have studied the chaotic properties between product non-autonomous discrete systems and factor non-autonomous discrete systems, and obtained the following results. If $f_{1,\infty} \times g_{1,\infty}$ is transitive, then $f_{1,\infty}$ and $g_{1,\infty}$ is transitive. $f_{1,\infty} \times g_{1,\infty}$ is topologically mixing if and only if $f_{1,\infty}$ and $g_{1,\infty}$ is topologically mixing. If $f_{1,\infty} \times g_{1,\infty}$ is sensitive, then $f_{1,\infty}$ or $g_{1,\infty}$ is sensitive. Moreover, [14] discussed the sensitivity of set-valued discrete systems, and obtained that, if $(\mathrm{K}(X\times Y), \overline{f_{1,\infty} \times g_{1,\infty}})$ is ergodically sensitive, then $(X, f_{1,\infty})$ or $(Y, g_{1,\infty})$ is ergodically sensitive, where the system $(\mathrm{K}(X\times Y), \overline{f_{1,\infty} \times g_{1,\infty}})$ is the product dynamical system of the two set-valued non-autonomous discrete systems. [15]studied the sensitivity of iterated function systems under the product operation. It is obtained that $\mathcal{F} \times \mathcal{G}$ is syndeticallysensitive if and only if $f_\lambda \times g_\gamma$ is syndetically sensitive, where $\mathcal{F}$ and $\mathcal{G}$ are twofeebly open iterated function systems, the maps $f_\lambda \in \mathcal{F}$ and $g_\gamma \in \mathcal{G}$ are surjective.

Remark 3.2 The conclusions of Theorems 3.2, 3.3, 3.5 and 3.6 are not sufficient and necessary conditions. Are the inverse propositions true? These are next work to think about.

IV. Conclusion

For the chaotic properties such aschain transitive, $\mathcal{F}$ -point-transitive, $\mathcal{F}$ -transitive, cofinitely sensitive, $\mathcal{F}$ -sensitive, etc., the discussion of the product system can be simplified to the discussion of the factor systems. This is conducive to dimension reduction, thus simplifying the research process in practical problems.

Acknowledgement

This work was funded by the Project of the Department of Science and Technology of Sichuan Provincial (No. 2021ZYD0005), the Scientific Research Project of Sichuan University of Science and Engineering (No. 2020RC24), and the Graduate Student Innovation Fundings (No. y2022189).

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