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Properties of Sequence in Extended Partial Rectangular b Metric Spaces

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Abstract: *In this paper, we introduced the class of Extended Partial Rectangular b metric spaces as a generalization of both extended rectangular b metric spaces with rectangular partial b metric spaces. Some properties of sequence in this metric as if cauchysequence, convergence sequence, limit sequence are given in this paper.*
Key Word*: partial b metric spaces, extended rectangular b metric spaces, b metric spaces, metric space.*

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I. INTRODUCTION

The distance between two points is called a metric if it satisfies certain axioms [6].With the introduction of the metric concept,mathematicians since then have found it easy to analyzeapproach process(limit). The concept of these metrics continues to evolve over timethe passage of time from research by Hausdroff in 1914 whichproduces about asymmetric distances or known as quasi metrics, finitein 1994 a theoretical computer expert Matthews found thatthe distance from a point to itself is not always zero and is called spacepartial metric[ ].In 1993 Czerwik introduced another new development regardingmetric space by extending the definition of metric space by addingthe constant b in the triangular inequality axiom known as the space bmetric[2]. Furthermore, a researcher named Satish in 2014 [13] combined 2definition of partial metric and b− metric to space b partial metric and onthe same year he also researched again about rectangular metric spaceswhich was introduced by Branciari and combines the three into bpartial rectangular metric spaces [12]. Developments regarding the metric space continue,so in 2019 Zead and Parvaneh developed another definition, there are additional functionson the triangular inequality axiom so that the definition of the metric space is morewider again and its called extended partial b metric spaces[11]. Therefore the author wants to examine the definition of the sequence as wellproperties of sequences in extended rectangular partial metric b− space.

II. PRELIMINARIES

In this section, we review several definitions of the development of metric spaces. The following definitions will help the writer in researching extended rectangular partial b metric space.

Definition 2.1[13]**:**A partial b metric space on a nonempty set X is a function $p_b: X \times X \to \mathbb{R}^+$, such that for all $x, y, z \in X$:

(PB1) $x = y$ if only if $p_b(x,y) = p_b(x,x) = p_b(y,y)$;

(PB2) $p_b(x,x) = p_b(y,x)$;

(PB3) there exist a real number $s \geq 1$ such that $p_b(x,y) \leq s\left(p_b(x,z) + p_b(z,x)\right) - p_b(z,z)$.

A partial b metric space is a pair (X, p_b) such that X is a nonempty set and p_b is a partial b metric space. The number s is called the coefficient of (X, p_b).

Definition 2.2 [12]**:** A partial rectangular b metric space on a nonempty set X is a function $p_b^r: X \times X \to \mathbb{R}^+$, such that for all $w, x, y, z \in X$:

(PBR1) $x = y$ if only if $p_b^r(x,y) = p_b^r(x,x) = p_b^r(y,y)$;

(PBR2) $p_b^r(x,y) = p_b^r(y,x)$;

(PBR3) there exist a real number $s \geq 1$ such that

$$p_b(x,y) \leq s\left(p_b^r(x,z) + p_b^r(z,w) + p_b^r(w,y) - p_b^r(z,z) - p_b^r(w,w)\right) + p_b(z,z) + \frac{1-s}{2}\left(p_b^r(x,x) + p_b^r(y,y)\right).$$

A partial rectangular b metric space is a pair (X, p_b^r) such that X is a nonempty set and p_{b^r} is a rectangular partial b metric space. The number s is called the coefficient of (X, p_b^r).

Definition 2.3 [11]**:** Let X be a nonempty set, $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $\varphi^{-1}(t) \leq t \leq \varphi(t)$ for all $t > 0$ and $0 = \varphi(0)$. A function $p_p: X \times X \to [0,\infty)$ is called an extended partial b metric if for all $x, y, z \in X$ the following conditions are satisfied:

Pp1. $x = y$ iff $p_p(x,x) = p_p(x,y) = p_p(y,y)$;

Pp2. $p_p(x,x) \leq p_p(x,y)$;

Pp3. $p_p(x,y) = p_p(y,x)$;

Pp4. $p_p(x,y) - p_p(x,x) \leq \varphi\left(p_p(x,z) + p_p(z,y) - p_p(z,z) - p_p(x,x)\right)$.

Then (X, p_p) is called an extended partial b metric space.

Definition 2.4 [9]**:** Let X be a nonempty set, $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $t \leq \varphi(t)$ for all $t > 0$ and $0 = \varphi(0)$ and let $b_{dr}: X \times X \to [0,\infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $a, b \in X$, each distinct from w and x satisfies conditions:

Bdr1. $b_{dr}(x,y) = 0$ iff $x = y$;

Bdr2. $b_{dr}(x,y) = b_{dr}(y,x)$;

Bdr3. $b_{dr}(x,y) \leq \varphi[b_{dr}(x,a) + b_{dr}(a,b) + b_{dr}(b,y)]$.

Then (X, b_{dr}) is called an extended rectangular b metric space.

III. RESULT AND DISCUSSION

In this section will be discussed the definitions, theorems, some properties of Sequences of Extended Rectangular Partial b Metric Spaces.

Definition 3.1: Let X be a nonempty set, $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $t \leq \varphi(t)$ for all $t > 0$and $0 = \varphi(0)$. A function $l: X \times X \to [0,\infty)$ for all $x, y, w, z \in X$ the following conditions are satisfied:

L1. $x = y$iff$l(x,x) = l(x,y) = l(y,y)$;

L2. $l(x,x) \leq l(y,x)$;

L3. $l(x,y) = l(y,x)$;

L4. $l(x,y) - l(x,x) \leq \varphi[l(x,z) + l(w,y) - l(z,z) - l(w,w) - l(x,x)]$.

Then (X, l) is called an extended rectangular partial b metric space.

Example 3.2: Let (X, r) is a rectangular metric space and $l(x,y) = 1 + \varphi(r(x,y))$ and $\theta: [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $t \leq \theta(t)$ for all $t > 0$and $0 = \theta(0)$. We will show that $(X, l)$is an extended rectangular partial b metric space with $\varphi(t) = \theta(t)$.

Obviously, conditions L1 – L3 of definition 3.1 are satisfied. On the other hand, for each $x, y, w, z \in X$ then

$$\begin{aligned} l(x,y) - l(x,x) &= 1 + \theta(r(x,y)) - 1 \\ &\leq \theta(r(x,w) + r(w,z) + r(z,y)) \\ &\leq \theta(r(x,w)) + \theta(r(w,z)) + \theta(r(z,y)) \\ &\leq \theta(r(x,w) + 1 + \theta(r(w,z)) + 1 + \theta(r(z,y)) + 1 - 1 - 1 \\ &= \theta((l(x,y) + l(w,z) + l(z,y) - l(z,z) - l(x,x)) \\ &= \varphi(l(x,y) + l(w,z) + l(z,y) - l(z,z) - l(x,x)). \end{aligned}$$

Conditions (L4) ofdefinition 3.1isfulfilledand$l$is a extended rectangular partial b metric space on$X$.

Lemma 3.3: Let$(X, l)$is an extended rectangular partial b metric space then

i. If$l(x,y) = 0$then$x = y$
ii. If$x \neq y$then$l(x,y) > 0$

Proof. For each$x, y \in X$and from definition 3.1 we obtain

i. If$l(x,y) = 0$from (L3) $l(x,x) \leq l(x,y) = 0$. Because for each$x, y \in X$then$l(x,y) \geq 0$and$l(x,x) = 0$.So, $l(x,y) = l(x,x)$from (L1) we get$x = y$.
ii. If$x \neq y$then from (i) we get$l(x,y) \neq 0$. So it was clear that $l(x,y)$must be positive.■

Lemma 3.4: Let$(X, l)$is extended rectangular partial b metric space. Function$l^s: X \times X \to [0,\infty)$with a super-additive function $\varphi$and for each$x, y \in X$, where

$$l^s(x,y) = 2l(x,y) - l(x,x) - l(y,y)$$

Then$l^s$is a extended rectangular b metric.

Proof. If $l^s(x,y) = 0$ and $l^s(x,y)$ is an extended rectangular partial b metric space from Lemma 3.3 we obtain that $x = y$ and the other hand if $x = y$ then

$$\begin{aligned} l^s(x,y) &= 2l(x,y) - l(x,x) - l(y,y) \\ &= 2l(x,x) - l(x,x) - l(x,x) \\ &= 0. \end{aligned}$$

Next,

$$\begin{aligned} l^s(x,y) &= 2l(x,y) - l(x,x) - l(y,y) \\ &= 2l(y,x) - l(y,y) - l(x,x) \\ &= l^s(y,x). \end{aligned}$$

And

$$\begin{aligned} l^s(x,y) &= 2l(x,y) - l(x,x) - l(y,y) \\ &= l(x,y) - l(x,x) + l(x,y) - l(y,y) \\ &\leq \varphi(l(x,z) + l(z,w) + l(w,y) - l(z,z) - l(w,w) - l(x,x)) + \varphi(l(x,z) \\ &\quad + l(z,w) + l(w,y) - l(z,z) - l(w,w) - l(y,y) \\ &\leq \varphi(2l(x,z) + 2l(z,w) + 2l(w,y) - 2l(z,z) - l(x,x) - l(y,y)) \\ &= \varphi(l^s(x,z) + l^s(z,w) + l^s(y,z)). \end{aligned}$$

Hence, conditions (Bdr1), (Bdr2), and (Bdr3) in definition 2.4 is fulfilled and l is a extended rectangular b metric space on X. ■

Definition 3.5: Let$(X, l)$is a extended rectangular partial b metric space then

(i) A sequences$\{x_n\}$in$(X, l)$is convergent to$x \in X$if

$$\lim_{n\to\infty} l(x_n, x) = l(x, x) = l(x_n, x_n).$$

(ii) A Sequences$\{x_n\}$in$(X, l)$is a Cauchy sequences if there is

$$lim_{n,m\to\infty} l(x_n, x_m).$$

(iii) Extended rectangular partial b metric spaces is complete if each sequences cauchy$\{x_n\}$in$(X, l)$is convergent.

Theorem 3.6: Let (X, l) is an extended rectangular b metric space. If $\{x_n\}$ is convergent sequence in (X, l) then limit $\{x_n\}$ is unique.

Proof. Let $\{x_n\}$ is a convergent sequence in $\{X, l\}$ has a limit that is not unique, then

$\lim_{n\to\infty} l(x_n, x) = l(x, x)and\lim_{n\to\infty} l(x_n, x) = l(y, y)$

Let arbitarynumber of$\varepsilon > 0$ so that for each $n \geq n_o$ and for each $n \geq n_1$:

Take $n^* = \max\{n_0, n_1\}$, then for each $n \geq n^*$

$$\begin{aligned} |l(x,x) - l(y,y)| &= |l(x,x) - l(x_n x) + l(x_n,x) - l(y,y)| \\ &\leq |l(x,x) - l(x_n,x)| + |l(x_n,x) - l(y,y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Because $\varepsilon > 0$ for all arbitrary number, then $l(x, x) = l(y, y)$ and limit $\{x_n\}$ is unique.■

Theorem 3.7: Let (X, l) is an extended rectangular partial b metric space. If $\{x_n\}$ is convergent sequence in (X, l) then $\{x_n\}$ is finite.

Proof. Let $\{x_n\}$ is sequence in (X, l) and convergent to x then

$$\lim_{n\to\infty} l(x_n, x) = l(x, x).$$

Let arbitarynumber of$\varepsilon > 0$ so that for each $n \geq n_o$ and $n_0 \in \mathbb{N}$ then $|l(x_n, x) - l(x, x)| < \varepsilon$. Let choose $\varepsilon = 1$ so $|l(x_n, x) - l(x, x)| < 1$ for each $n \geq n_0$.

Next, take a number $M = maks\,\{1, l(x_1, x), l(x_2, x), l(x_3, x), \dots, l(x_{n_0-1}, x)\}$, then we get $|l(x_n, x) - l(x, x)| \leq M$.

Because $l(x_n, x) < 1 \leq M$ and for each $n \geq n_0$. So $l(x_n, x\,) \leq M$, for each $n \in \mathbb{N}$ and it is proved that sequence $\{x_n\}$ is finite.■

Lemma 3.8: Let (X, l^s) is an extended rectangular partial b metric space. Then

(i) A sequence $\{x_n\}$ is Cauchy sequence in (X, l^s) iff $\{x_n\}$ is Cauchy sequence in $\{X, l\}$.
(ii) A sequence $\{x_n\}$ is complete sequence in (X, l^s) iff $\{x_n\}$ is complete sequence in (X, l) and $\lim_{n\to\infty} l^s(x, x_n) = 0$ iff $\lim_{n,m\to\infty} l(x_n, x_m) = l(x, x)$.

Proof. It will be shown that every Cauchy sequences in$(X, l)$is Cauchy sequences in(X, l^s).Let$\{x_n\}$ is Cauchy sequences in(X, l)then there is$\propto \in \mathbb{R}$ such that for$\varepsilon > 0$there is$n_0 \in \mathbb{N}$with $|l(x_n, x_m) - \propto| < \varepsilon/4$.

For each n, with$n \geq n_0$then

$$\begin{aligned} l^s(x,y) &= 2l(x_n, x_m) - l(x_n, x_n) - l(x_m, x_m) \\ &= l(X_n, x_m) - \propto + \propto - l(x_n, x_n) + l(x_m, x_m) - \propto + \propto - l(x_m, x_m) \\ &\leq |l(x_n, x_m) - \propto| + |\propto - l(x_n, x_n)| + |l(x_m, x_n) - \propto| + |\propto - l(x_m, x_m)| \\ &< \varepsilon \end{aligned}$$

For each$n, m \geq n_0$.

So, we get$\{x_n\}$ is a Cauchy sequence in(X, l^s).

Next, we will shownthat if$(X, l)$is complete then$(X, l^s)$is complete. If $\{x_n\}$is Cauchy sequence in(X, l)then there is$y \in X$such that

$$\lim_{n\to\infty} l^s(y, x_n) = 0.$$

Hence,

$$\begin{aligned} \lim_{n\to\infty}\big(l(x_n, y) - l(y,y) + l(y, x_n) - l(x_n, x_n)\big) &= 0 \\ \lim_{n\to\infty}\big(l(x_n, y) - l(y,y)\big) &= 0 \\ \lim_{n\to\infty}\big(l(y, y_n) - l(x_n, x_n)\big) &= 0 \end{aligned}$$

So,

$$\lim_{n\to\infty} l(x_n, x) = l(y,y) = \lim_{n\to\infty}(x_n, x_n)$$

With (L3) we get

$$l(y,y) \leq \lim_{n,m\to\infty} l(x_n, y) = \lim_{n,m\to\infty} l(x_n, x_n) \leq \lim_{n,m\to\infty} l(x_n, x_m).$$

So, we get $\{x_n\}$ is Convergen sequence in (X, l).

Next, we will shown that every Cauchy sequence $\{x_n\}$ in (X, l^s) is Cauchy sequence in (X, l).

Let $\varepsilon = \frac{1}{2}$, Then there is $n_0 \in \mathbb{N}$ such that $l^s(x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_0$. Because

$$l(x_n, x_n) \leq l(x_n, x_{n_0}) \leq l^s(x_{n_0}, x_{n_0}) + l(x_n, x_{n_0}) < \frac{1}{2} + l(x_n, x_{n_0}).$$

$l(x_n, x_m)$is finite in real number then there is $a \in \mathbb{R}$ such that subsequence $l(x_{n_k}, x_{n_k})$ from $l(x_n, x_n)$ convergent to a or $\lim_{k\to\infty}(x_{n_k}, x_{n_k}) = a$.

Now, we prove that $l(x_n, x_n)$ is cauchy sequence in $\mathbb{R}$. Because $\{x_n\}$ is cauchy sequence in (X, l^s) for $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $l^s(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$

$$\begin{aligned} l(x_n, x_m) - l(x_m, x_m) &\leq l(x_n, x_n) - l(x_m, x_m) \\ &\leq l^S(x_m, x_n) \\ &< \varepsilon \end{aligned}$$

Next, $\lim_{n\to\infty} l(x_n, x_n) = a$ then on the other hand,

$$\begin{aligned} |l(x_n, x_m) - a| &= |l(x_n, x_m) - l(x_n, x_n) + l(x_n, x_n) - a| \\ &\leq l^s(x_m, x_n) + |l(x_n, x_n) - a| \end{aligned}$$

for each $n, m \geq n_0$. *Then* $lim_{n,m\to\infty} l(x_n, x_m) = a$ *and result and resulted* $\{x_n\}$ *is Cauchy sequence in* (X, l) *and the otherwise, let* $\{x_n\}$ *is Cauchy sequence in* (X, l^s). *Then* $\{x_n\}$ *is Cauchy sequence in* (X, l)*and convergent to*$x \in X$ *with*

$$\lim_{n\to\infty} l(x, x_n) = \lim_{n\to\infty} l(x_m, x_n) = l(x, x)$$

Then for $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$l(x, x_n) - l(x, x) < \frac{\varepsilon}{4}$$

and

$$|l(x_n, x_n) - l(x, x) \leq l(x_m, x_n) - l(x, x) < \frac{\varepsilon}{4}.$$

Next,

$$\begin{aligned} |l^s(x_n, x)| &= |l(x_n,, x) - l(x_n, x_n) + l(x_n, x) - l(x, x)| \\ &\leq |l(x_n, x) - l(x, x)| + |l(x, x) - l(x_n, x_n)| + |l(x_n, x) - l(x, x)| \\ &< \varepsilon \end{aligned}$$

With $n \geq n_0$. Then (X, l^S) is complete.

Let $\lim_{n\to\infty} l^s(x_n, x) = 0$then

$$\lim_{n\to\infty}\big(l(x_n, x) - l(x_n, x_n)\big) + \lim_{n\to\infty}\big(l(x_n, x) - l(x, x)\big) = 0.$$

Such that

$$\begin{aligned} \lim_{n\to\infty}(l(x_n, x_m) - l(x, x)) &\leq \lim_{n\to\infty} \varphi(l(x, z) + l(z, w) + l(w, w) - l(z, z) - l(w, w) \\ &\quad -l(x, x)) \\ &= \varphi(0) \\ &= 0. \blacksquare \end{aligned}$$

IV. CONCLUSION

Extended rectangular partial$b$ metric space is generaliation of rectangular partial b metric space by changing the constans s in the axiom to a strictly increasing continuous function andevery rectangular pastial b metric space is extended rectangular pastial b metric space with $\varphi(t) = st$ but the otherwise not necessarily. A sequence $\{x_n\}$ is Cauchy Sequence in (X, l^S) iff $\{X_n\}$ is Cauchy sequence in $(X, l)$and a sequence $\{x_n\}$ is complete in (X, l^S) iff $\{X_n\}$ is complete.

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