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A Bayesian Approach to Estimate Reliability under Exponential Failure Time Distribution

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Abstract-*In this paper we consider an adaptive approach for estimating reliability growth, based on prior information which is motivated from practical considerations. We discuss two situations: in the first, both the prior distribution and the posterior distributions of the mean time to failure of an exponential distribution are stochastically ordered; in the second situation, the prior distribution is stochastically ordered with respect to the last posterior distribution. The former situation leads us to procedure which is not fully Bayesian, and is therefore termed "pseudo-Bayesian". Since we do not know the properties of this pseudo-Bayesian approach, we can best describe our work here as being a "pseudo-Bayesian scheme". The second situation leads us to an approach which is fully Bayesian under certain assumptions. Our work along the lines indicated in this paper is still in progress, and we invite the attention of other researchers to some of the problems we have posed, and the questions we have raised.*

Keywords: *-Bayesian estimation, Reliability growth, Prior and Posterior distributions.*

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I. Introduction

A complex, newly designed system generally undergoes several stages of testing before it is put into operation. After each stage of testing changes are made to the design (or the operating conditions are re-specified) with the hope that the new design would lead to a longer period of performance this procedure is referred to as reliability growth; this is because a longer period of performance implies an improvement in reliability. Often, in practice a change in the system design may result in a deterioration of the system performance, so that the term reliability growth may not be an appropriate description of what is actually happening to the system. However, since the intent of a design change is to improve the reliability, we shall continue to use the term reliability growth throughout this paper.Suppose that the system has been tested at stages 0, 1, 2,.... At the end of each stage an evaluation of the reliability of the system is made using any of the conventional procedures which are used to measure reliability. At each stage of testing, we may test either a single copy of the system, or several copies of the system, depending on the particular situation being considered. Having tested the system over, say, i, i = 0, 1, 2,..., stages, we would like to obtain the best (in a sense to be made precise) estimate of the reliability of the system at the ith stage; in doing so we would like to use all the testing information that we have acquired over the various stages.Information of the type indicated above will enable a decision-maker to determine if the design changes do indeed result in an improvement in reliability, and perhaps determine the rate (with respect to the stages) at which there is a growth in reliability. It will also enable a decision-maker to decide if the system is ready to be put into operation, and to arrive at a suitable cost-warranty agreement with the user.Since in the 1966 there have been several papers written under the heading of reliability growth, each emphasizing a different point of view. For a recent survey of these papers we refer the reader to Balaban [1]. The approach that we shall take in this paper is suggested by Bayesian considerations, with the analysis being undertaken in a manner which will be described in section 2. Others who have adopted a Bayesian point of view, but with a direction different from ours, are Smith [5], Weinrich and Gross [6], and Jewell [3].

As stated in the abstract, our paper is expository in nature; we do not claim to offer a definitive solution to the problem of estimating reliability growth. What we give here is an outline of a few potential approaches and discuss their ramifications, both from a philosophical and a practical point of view. We raise several statistical issues pertinent to each approach, and invite the attention of other researchers to the questions that we have raised. Toward the end of the paper we also describe some additional practical problems pertinent to reliability growth which are of interest, especially to management scientists and operations research analysts, since they involve cost considerations and optimization. The contribution of this paper, then, is the introduction of a comprehensive and unified Bayesian view of the problem of reliability growth, and an outline of some possible approaches that we can take with their ramifications. Our hope is that the text of this paper will stimulate more thought along the lines we described.

II. Bayesian Scheme for Estimating Reliability Growth

Suppose that the failure distribution of the system after the i[th] design change, i = 0, 1,..., is exponential with mean θ_i. That is, $f(t;\, \theta_i) = \theta_i^{-1}\exp(-t/\theta_i),\ \theta_i > 0,\ t > 0$. In particular, at stage 0, when the system is newly built, it's time to failure has an exponential distribution with mean θ_0. Based upon our knowledge of the performance of similarly designed systems, we assign a prior distribution to θ_0, say $G(\theta_0;\, \cdot)$. Without any loss in generality, we let $G(\theta_0;\, \cdot)$ be an inverted gamma distribution with parameters α_0 and β_0; this is the natural conjugate prior distribution for θ_0. Thus, we have $dG(\theta_0;\, \cdot) = g(\theta_0,\, \alpha_0,\, \beta_0)$ as

$$(\alpha_0^{\beta_0} / \Gamma(\beta_0))\theta_0^{-(\beta_0+1)}\exp(-\alpha_0/\theta_0);\ \text{for}\ \alpha_0,\, \beta_0,\, \theta_0 > 0 \qquad (1)$$

It is easy to verify (Mann, Schafer and Singpurwalla [4, p. 112]) that the mean and the variance of θ_0 with respect to the above prior distribution are $E(\theta_0) = \alpha_0/(\beta_0 - 1)$ and $Var(\theta_0) = \alpha_0^2/(\beta_0-1)(\beta_0-2)^2$, respectively. Thus, to be assured that both the prior mean and variance of θ_0 are finite, we must have $\beta_0 > 2$.

One way of choosing α_0 and β_0, the prior parameters, is to set $E(\theta_0)$ equal to our best guess of the mean time to failure θ_0, and to set Var(θ_0) equal to a number which reflects the quality of our guess about θ_0.

Having assigned our prior distribution, $G(\theta_0;\, \cdot)$, we test $n_0(\geq 1)$ copies of the system, and observe their times to failure, $t_{0,i}\ \ i = 1,...n_0$. Let $T_0 = \Sigma_i^{n_0} t_{0,i}$, be the total time on test. Then conditional on T_0, the posterior distribution of θ_0 is also an inverted gamma distribution; that is, $g(\theta_0 \mid T_0;\, \alpha_0,\, \beta_0,\, n_0)$ is

$$((\alpha_0 + T_0)^{\beta_0+n_0} / \Gamma(\beta_0 + n_0))\theta_0^{-(\beta_0+n_0+1)}\exp(-(\alpha_0 + T_0)/\theta_0) \quad (2)$$

It is well known that under the assumption of a squared error loss function, the Bayesian estimate of the mean time to failure at stage 0 is the mean of the posterior distribution of θ_0. Thus,

$$E(\theta_0 \mid T_0) = (\alpha_0 + T_0)/(\beta_0 + n_0 - 1) \qquad (3)$$

is our estimate of the mean time to failure at stage 0, given the failure data T_0. To express our uncertainty about this estimate, we compute the posterior (1–γ) credibility interval for θ_0. This interval is provided by specifying two numbers $(\theta_0 \mid T_0)_L$ and $(\theta_0 \mid T_0)_U$, $(\theta_0 \mid T_0)_L < (\theta_0 \mid T_0)_U$, such that

$$\mathrm{I}[(\theta_0 \mid T_0)_L,\, (\theta_0 \mid T_0)_U;\, g(\theta_0 \mid T_0;\ \alpha_0,\, \beta_0,\, n_0)] = 1 - \gamma \quad (4)$$

Where

$$I[a,\, b;\, g(s)] = \int_a^b g(s)ds$$

The parameter θ_0, with its posterior distribution, is a measure of the reliability of the system at stage 0. Having observed $E(\theta_0 \mid T_0)$, we may want to increase it either by making design changes or by re-specifying the operating conditions of the system. Suppose that we do decide to make these changes; let θ_1 denote the mean time to failure after the changes have been made. The system is now at stage 1, and we are ready to test the system and verify if thechanges have resulted in an improvement of the system.Since we have adopted a Bayesian point of view, our first task is to assign a prior distribution for θ_1 say $G(\theta_1;\, \cdot)$. The novelty of our approach pertains to the manner in which we go about choosing $G(\theta_1;\cdot)$, and this is motivated by the following consideration.Even though the changes that we have instituted have been undertaken with a view towards increasing the reliability of the system, there is a possibility that the changes could be deleterious to the system. In order to account for this possibility we shall choose $G(\theta_1;\cdot)$ in such a manner that θ_1 is stochastically larger than $G(\theta_0 | T_0)$; this is written as

$$\theta_1 \overset{st}{\geq} (\theta_0 | T_0)$$

Thus, we must have:

$$P(\theta_1 \geq x)P((\theta_0 | T_0 \geq x) \quad \text{for all } x \geq 0$$

The usual approach for situations involving reliability growth of the type described above is to take $\theta_1 \geq (\theta_0 | T_0)$ with probability 1, as has been done by Barlow, Bartholomew, Bremner and Brunk [2] and by Smith [5]. However, a prior chosen to satisfy the above condition does not place any mass in the region $\theta_1 < (\theta_0 | T_0)$, and therefore would exclude from our analysis the possibility of an adverse effect of the changes. Hence, the requirement that

$$\theta_1 \overset{st}{\geq} (\theta_0 | T_0)$$

appears to be more realistic.

Following our previous discussion, suppose that the prior distribution $G(\theta_1;\cdot)$ is also an inverted gamma with parameters α_1 and β_1. That is, $g(\theta_1; \alpha_1, \beta_1)$ is

$$(\alpha_1^{\beta_1} / \Gamma(\beta_1))\theta_1^{-(\beta_1+1)} \exp(=-\alpha_1 / \theta_1), \text{ for } \alpha_1, \beta_1, \theta_1 > 0 \qquad (5)$$

A sufficient condition for ensuring that

$$\theta_1 \overset{st}{\geq} (\theta_0 | T_0)$$

is to have $\alpha_1 \geq \alpha_0 + T_0$, and $\beta_1 = \beta_0 + n_0$. We can verify that under these conditions $E(\theta_1) \geq E(\theta_0 | T_0)$, an inequality we would hope to achieve under the hypothesis that a change is beneficial to the system. One possibility is to let $\alpha_1 = \alpha_0 + T_0 + a_1$, where a_1 is a measure of our prior belief about the magnitude of the improvement in the mean as a result of the changes. A possible strategy for choosing a_1 is to set $E(\theta_1) = (\alpha_0 + T_0 + a_1)/(\beta_1 - 1)$ equal to our best guess of the mean time to failure θ_1. For reasons to be given in section 2.1, we would also want $a_1 \geq \alpha_0$. With α_1 thus chosen, it is important that at stage 1 we treat T_0 as being a constant. If this is not done, then the posterior distribution of θ_1 will have to be obtained by treating α_1 as a hyper-parameter with a prior distribution which is related to the unconditional distribution of T_0. Under these circumstances the posterior distribution of θ_1 will not be an inverted gamma, and our procedure will become computationally quite involved. Furthermore, we shall also assume that $g(\theta_1; \alpha_1, \beta_1)$ is independent of $g(\theta_0 | T_0; \cdot)$.

Having chosen $g(\theta_1; \alpha_1, \beta_1)$, we test $n_1 \geq 1$ copies of the system (under stage 1) and observe T_1, the corresponding total time on test. Since we now treat T_0 as a constant, the posterior distribution of θ_1 conditioned on T_1, $g(\theta_1 | T_1; \cdot)$, is:

$$((\alpha_0 | T_0 + a_1 + T_1)^{\beta_0+n_0+n_1} / \Gamma(\beta_0 + n_0 + n_1))(1/\theta_1)^{\beta_0+n_0+n_1-1}$$
$$\exp(-(\alpha_0 + T_0 + a_1 + T_1)/\theta_0) \quad (6)$$

Analogous to (3), the Bayes estimator of θ_1 is:

$$E(\theta_1 | T_1) = (\alpha_0 + T_0 + a_1 + T_1)/(\beta_0 + n_0 + n_1 - 1) \qquad (7)$$

Under our postulate of reliability growth, we would also want to have

$(\theta_1 | T_1) \overset{st}{\geq} (\theta_0 | T_0)$;

a necessary (but not sufficient) condition for the above inequality is that:

$$(\alpha_0 + T_0 + a_1 + T_1) / (\beta_0 + n_0 + n_1 - 1) \geq (\alpha_0 + T_0) / (\beta_0 + n_0 - 1) \quad (8)$$

Which reduces to the requirement that:

$$(a_1 + T_0) / n_1 \geq (\alpha_0 + T_0) / (\beta_0 + n_0 - 1) \quad (9)$$

If (9) is not satisfied, we proceed to section 2.1 wherein we indicate a pooling procedure which gives us a desired inequality. If (9) is satisfied, then $E(\theta_1 | T_1)$, the Bayes estimator of θ_1, with a $(1-\gamma)$ posterior credibility interval given by

$I[(\theta_1 | T_1)_L, (\theta_1 | T_1)_U; g(\theta_1 | T_1; \cdot)] = 1 - \gamma$ represents our evaluation of the reliability of the system at stages 0 and 1.

If the value of $E(\theta_1 | T_1)$ and the credibility interval $[(\theta_1 | T_1)_L, (\theta_1 | T_1)_U]$ meet our desired reliability goals, then we choose not to make any further improvements to the system, and stop the testing procedure. If not, we proceed to stage 2 by making appropriate modifications to the system.

If we need to proceed to stage 2, then the prior distribution of θ_2 must be such that

$$\theta_2 \overset{st}{\geq} (\theta_1 | T_1). \quad (10)$$

One way of achieving the requirement specified by (10) is to choose α_2 and β_2, the parameters of the inverted gamma prior distribution of θ_2, in such a manner that $\alpha_2 = \alpha_0 + T_0 + a_1 + T_1 + a_2$ and $\beta_2 = \beta_0 + n_1 + n_2$. Once again, a_2 reflects our belief about the magnitude of the improvement in the mean as a consequence of the changes at stage 2, and now both T_0 and T_1 are treated as constants. Furthermore, for reasons given in section 2.2, we shall want to have $a_2 \geq \alpha_0 + T_0 + a_1$. We shall continue with our discussion of this procedure in section 2.2

2.1. Procedures when the posterior distributions at stages 0 and 1 are not stochastically ordered

If (9) is not satisfied, i.e. if

$$(\alpha_1 + T_1) / n_1 < (\alpha_0 + T_0) / (\beta_n + n_0 - 1), \quad (11)$$

and if we have no prior reason to believe that the reliability growth postulate is not true, then we consider that the randomness of T_0 and T_1 are causes for (11), and propose the following two strategies as a means of overcoming this.

Strategy 1. We essentially ignore (11), and for the time being do not concern ourselves with the fact that $E(\theta_1 | T_1) < E(\theta_0 | T_0)$ Thus, we proceed to stage 2 by choosing the prior distribution of θ_2 in such a manner that (10) is satisfied. After completing the testing over all the stages, we will perform an isotonic regression of the posterior means $E(\theta_i | T_i),\ i = 0, 1, \ldots.$. This will represent our final evaluation of the reliability of the system based upon tests performed at the various stages of testing. More about this strategy will be said in section 3.

Strategy 2. By ignoring (11) and directly proceeding to stage 2 we will, through the prior distribution of θ_2, allow the effects of (11) to perpetuate over the succeeding stages. We can avoid this by pooling (the violators) T_0 and T_1 and n_0 and n_1. That is, we replace

(a) both T_0 and T_1 by T_{01}, where $T_{01} = (T_0 + T_1)/2$, and
(b) both n_0 and n_1 by n_{01}, where $n_{01} = (n_0 + n_1)/2$.

When we pool as indicated, we must test to see if

$$(a_1 + T_{01}) / n_{01} \geq (\alpha_0 + T_0) / (\beta_0 + n_{01} - 1) \quad (12)$$

a condition analogous to (9) except that the n_i and the T_i, i = 0, 1, have been replaced by their pooled values. Since $(\beta_0 - 1) > 0$, a sufficient condition for (12) is that $a_1 \geq \alpha_0$.

To summarize, we must choose $a_1 \geq \alpha_0$, and when (9) is violated, we shall pool as indicated above, and be assured that:

$$(\alpha_0 + 2T_{01} + a_1) / (\beta_0 + 2n_{01} - 1) \geq (\alpha_0 + T_{01}) / (\beta_0 + n_{01} - 1), \quad (13)$$

a condition motivated by (8).

Note that since (13) can also be written as

$$(\alpha_0+T_0+T_1+a_1)/(\beta_0+n_0+n_1-1)\geq(2\alpha_0+T_0+T_1)/(2\beta_0+n_0+n_1-2)$$

and since the left-hand side of the above equation is identical to the left-hand side of (8), the effect of pooling is to lower the magnitude of the right-hand side of (8), the posterior mean of θ_0 given T_0. Thus, the effect of pooling is to lower the posterior mean at the previous stage in the light of information obtained at the current stage, the previous stage, and the prior conviction that reliability growth has indeed taken place.

From (3) replacing T_0 by T_{01}, and n_0 and n_{01}, we see that our revised Bayes estimator of θ_0 is:

$$E(\theta_0\,|\,T_0,\,T_1)=(2\alpha_0+T_0+T_1)/(2\beta_0+n_0+n_1-2) \quad (14)$$

and the revised (1 $-\gamma$) posterior credibility intervals for θ_0 are given by $(\theta_0\,|\,T_0,\,T_1)_L$ an $(\theta_0\,|\,T_0,\,T_1)_U$, where

$$I[(\theta_0\,|\,T_0,\,T_1)_L,\,(\theta_0\,|\,T_0,\,T_1)_U;\;g(\theta_0\,|\,T_0,\,T_1;\,\cdot)]=1-\gamma\,,$$

with $g(\theta_0\,|\,T_0,\,T_1;\,\cdot)$ being an inverted gamma distribution with scale parameter $(2\alpha_0+T_0+T_1)/2$ and shape parameter $(2\beta_0+n_0+n_1)/2$. To obtain a revised estimator of θ_1, the one which incorporates the effect of pooling, we replace, in (7), T_0, T_1, n_0, and n_1 by their pooled values. We can now immediately verify that the Bayes estimator of θ_1 remains unchanged and is therefore given by (7). Consequently, the credibility interval for θ_1 is also unchanged. We can now either stop at this point or proceed to stage 2, our choice being determined by the criteria described prior to and following (10). If we choose to stop, then our assessment of the reliability of the system at stages 0 and 1 is given by (14) and (7), respectively.

We remark that instead of pooling the T_i's and the n_i's, we could have pooled the quantities $T_i\,/\,n_i; i=0,1$. The reason we choose the former is that the resulting expressions are much simpler to work with, and are also easier to interpret.

2.2. Procedures for analysis at stage 2

In the previous section we saw that whether (9) is satisfied or not (i.e. irrespective of pooling), the prior distribution of θ_2 could be an inverted gamma with parameters α_2 and β_2, where

$\alpha_2=\alpha_0+T_0+a_1+T_1+a_2$ and $\beta_2=\beta_0+n_0+n_1$. Having chosen $g\left(\theta_2;\;\alpha_2,\;\beta_2\right)$, we test $n_2\geq 1$ copies of the system (under stage 2) and observe T_2, the corresponding total time on test. If $g\left(\theta_2;\;\alpha_2,\;\beta_2\right)$, is assumed to be independent of $g(\theta_1\,|\,T_1;\,.)$, then the posterior distribution of θ_2 given T_2 is also an inverted gamma with parameters $\left(\alpha_0+T_0+a_1+T_1+a_2+T_2\right)$ and $\left(\beta_0+n_0+n_1+n_2\right)$

Here again, in order to be assured that $(\theta_2\,|\,T_2)\overset{st}{\geq}(\theta_1\,|\,T_1)$

We need, as a necessary condition,

$$(\alpha_2+T_2)/n_2\geq(\alpha_0+T_0+a_1+T_1)/(\beta_0+n_0+n_1-1)\,. \quad (15)$$

If the above condition is satisfied, then we can either stop and (because of the independence assumption) use $(\alpha_0+T_0+a_1+T_1+a_2+T_2)/(\beta_0+n_0+n_1+n_2-1)$ as our Bayes estimator of θ_2, with a $100(1-\gamma)$ credibility interval for θ_2 given by the two numbers $(\theta_2\,|\,T_2)_L$ and $(\theta_2\,|\,T_2)_U$, where

$$I[(\theta_2\,|\,T_2)_L,\,(\theta_2\,|\,T_2)_U;\;g(\theta_2\,|\,T_2)]=1-\gamma$$

or we can proceed to stage 3 by making the appropriate modifications to the system. If we proceed to stage 3, we repeat our cycle so that the prior distribution of θ_3 satisfies the condition

$$\theta_3\overset{st}{\geq}(\theta_2\,|\,T_2)$$

If condition (15) is not satisfied, then we can, following strategy 1 of section 2.1, either ignore the inequality (15) and proceed directly to stage 3, or follow strategy 2 and pool T_1 and T_2 to obtain $T_{12}=(T_1+T_2)/2$ and $n_{12}=(n_1+n_2)/2$. If we choose to pool then, analogous to (12), we must have

$$(\alpha_2 + T_{12})/n_{12} \geq (\alpha_0 + T_0 + a_1 + T_{12})/(\beta_0 + n_0 + n_{12} - 1) \qquad (16)$$

A sufficient condition for the above is that $a_2 > \alpha_0 + T_0 + a_1$. Thus, at this stage, the prior parameter a_2 has to be related to T_0 the total time on test at stage 0.

As we emphasized in section 2.1, the effect of pooling is a lowering of the posterior mean at the previous stage. In the present case, we have changed the posterior mean at stage 1 from $(\alpha_0 + T_0 + a_1 + T_1)/(\beta_0 + n_0 + n_1 - 1)$; (eq. (7)) to

$$E(\theta_1 | T_0, T_1, T_2) = \frac{\alpha_0 + T_0 + a_1 + T_{12}}{\beta_0 + n_0 + n_1 - 1} = \frac{2(\alpha_0 + T_0 + a_1) + T_1 + T_2}{2(\beta_0 + n_0) + n_1 + n_2 - 2} \qquad (17)$$

Furthermore, our credibility intervals for θ_1, will now be given by the equation:

$$I[(\theta_1 | T_0, T_1, T_2)_L, (\theta_1 | T_0, T_1 + T_2)_U; g(\theta_1 | T_0, T_1, T_2; \cdot)] = 1 - \gamma$$

where $g(\theta_1 | T_0, T_1, T_2)$ is an inverted gamma distribution with a scale parameter $(2(\alpha_0 + T_0 + a_1) + T_1 + T_2)/2$ and a shape parameter $(2(\beta_0 + n_0) + n_1 + n_2 - 2)/2$. Of course, our Bayes estimator of θ_2 remains unchanged as $(\alpha_0 + T_0 + a_1 + T_1 + a_2 + T_2)/(\beta_0 + n_0 + n_1 + n_2 - 1)$. Since we have revised our estimator of θ_1 from that given by (7) to that given by (17), we will have to see if

$$(\theta_1 | T_0, T_1, T_2) \overset{st}{\geq} (\theta_0 | T_0) \qquad (18)$$

if we did not have to pool at stage 1, or $(\theta_1 | T_0, T_1, T_2) \overset{st}{\geq} (\theta_0 | T_0, T_1)$ (19)

if we had to pool at stage 1. A necessary condition for (18) is that

$$(\alpha_0 + T_0 + a_1 + T_{12})/(\beta_0 + n_0 + n_{12} - 1) \geq (\alpha_0 + T_0)/(\beta_0 + n_0 - 1) \qquad (20)$$

which reduces to the requirement that

$$(a_1 + T_{12})/n_{12} \geq (\alpha_0 + T_0)/(\beta_0 + n_0 - 1). \qquad (21)$$

If (21) is not violated, we proceed to stage 3. If (21) is violated, then we shall pool T_0 and T_{12} and n_0 and n_{12} to form

$T_{0,12} = (T_0 + T_{12})/2$ and $n_{0,12} = (n_0 + n_{12})/2$

and replace the appropriate quantities in (20) by their pooled values. Having done this, we shall need to have

$$\frac{\alpha_0 + T_{0,12} + a_1 + T_{0,12}}{\beta_0 + n_{0,12} + n_{0,12} - 1} \geq \frac{\alpha_0 + T_{0,12}}{\beta_0 + n_{0,12} - 1} \qquad (22)$$

which, because of $a_1 \geq \alpha_0$, is always true.

Since $T_{0,12} = (1/2)(T_0 + (1/2)(T_1 + T_2))$ and $n_{0,12} = (1/2)(n_0 + (1/2)(n_1 + n_2))$, condition (22) reduces to

$$\frac{\alpha_0 + T_0 + a_1 + T_{12}}{\beta_0 + n_0 + n_{12} - 1} \geq \frac{4\alpha_0 + 2T_0 + T_1 + T_2}{4\beta_0 + 2n_0 + n_1 + n_2 - 4}$$

Thus, if we did not have to pool at stage 1, and if condition (21) is violated, our Bayes estimator of θ_0 conditioned on T_0, T_1 and T_2 is

$$E(\theta_0 | T_0, T_1, T_2) = (4\alpha_0 + 2T_0 + T_1 + T_2)/(4\beta_0 + 2n_0 + n_1 + n_2 - 4). \qquad (23)$$

The credibility intervals for θ_0 are now given by an inverted gamma distribution with scale parameter $\left(4\alpha_0 + 2T_0 + T_1 + T_2\right)/4$ and shape parameter $\left(4\beta_0 + 2n_0 + n_1 + n_2\right)/4$. We can now either stop or proceed to stage 3.

Reverting to (19), we note that a necessary condition for satisfying this equation is that

$$(\alpha_0 + T_0 + a_1 + T_{12})/(\beta_0 + n_0 + n_{12} - 1) \geq (2\alpha_0 + T_0 + T_1)/(2\beta_0 + n_0 + n_1 - 1) \qquad (24)$$

If condition (24) is satisfied, we proceed to stage 3. Note that for $n_1 \geq n_2$ condition (24) reduces to

$$(a_1 + T_{12})/(\beta_0 + n_0 + n_{12} - 1) \geq (\alpha_0 + T_1)/(2\beta_0 + n_0 + n_1 - 1); \qquad (25)$$

note that when $n_1 \geq n_2, 2\beta_0 + n_0 + n_1 - 1 \geq \beta_0 + n_0 + n_{12} - 1$. Clearly, condition (25) is satisfied whenever $T_{12} \geq T_1$, i.e. when $T_2 > T_1$.Thus, in view of the above arguments, whenever condition (24) is violated, i.e. whenever $n_1 < n_2$ or $T_2 < T_1$ or both, we shall replace T_1 and n_1 by their pooled values T_{12} and n_{12}. Thus, after pooling, (24) becomes

$$(\alpha_0 + T_0 + a_1 + T_{12})/(\beta_0 + n_0 + n_1 - 1) \geq (4\alpha_0 + 2T_0 + T_1 + T_2)/(4\beta_0 + 2n_0 + n_1 + n_2 - 2)$$

which, because $a_1 \geq \alpha_0$, is always true.

To summarize, if we had to pool at stage 1, and if condition (24) is violated, our Bayes estimator of θ_0 conditioned on T_0, T_1 and T_2 is

$$E(\theta_0 \mid T_0, T_1, T_2) = (4\alpha_0 + 2T_0 + T_1 + T_2)/(4\beta_0 + 2n_0 + n_1 + n_2 - 2)$$

Note that its estimator is identical to the one given by eq. (23), which was based on the fact that there was no pooling at stage 1.

However, in the present case the credibility intervals for θ_0 are given by an inverted gamma distribution with scale parameter $(4\alpha_0 + 2T_0 + T_1 + T_2)/2$ and shape parameter $(4\beta_0 + 2n_0 + n_1 + n_2)/2$. We can now either stop or proceed to stage 3.

We contrast these to the credibility intervals for θ_0 given after (23); these pertained to the case of no pooling at stage 1. We note that pooling at both stages 1 and 0 has a tendency to make the credibility intervals wider than those obtained when there is pooling at stage 0 only. Thus, based on the above analysis, we claim that excessive pooling results in wider credibility intervals.

Our analysis of the failure data at the succeeding stages follows along the lines mentioned above. Our evaluation of the reliability of the system at stage *i*, *i* = 1, 2,..., is given by the appropriately conditioned Bayes estimator of θ, and its associated credibility interval.

2.3. Some remarks on the pooling procedure

Here it is clear that condition (7) is likely to be violated whenever (T_1 / n_1) is not much larger than (T_0 / n_0). Note that (T_i / n_i), *i* = 0, 1, is the classical maximum likelihood estimator of θ_i, *i* = 0, 1. Thus, (7) will be violated if the improvement in reliability in going from stage 0 to stage 1 is not significantly large. Hence, pooling will be necessary whenever the effect of the design changes is not substantial (or if the design changes have produced a significant deterioration).The pooling procedure advocated here is one among several others that can be used. For example, we could have pooled the estimated mean times to failure, (T_0 / n_0) and (T_1 / n_1), or we could have just pooled the observed total time on test, T_0 and T_1. However we pool, the important question is whether pooling the data is a legitimate Bayesian procedure.

An orthodox Bayesian might argue that by pooling we have violated the "likelihood principle" of statistical inference. He will object on the grounds that our decision rule is not based on the information provided to us by the true posterior distribution, but is instead based on a posterior distribution which is modified to suit our hypothesis. He would recommend that instead of pooling, it would be better to choose $a_1 > \alpha_0$, so that condition (7) will always be satisfied, or to choose a joint prior distribution on θ_0 and θ_1 in such a manner that there is no probability mass in the region $\theta_1 < \theta_0$ (as was done by Barlow et al. [2]).Our response to the above arguments is that not allowing any prior or posterior probability in the region $\theta_1 < \theta_0$ is too strong, and is perhaps an unreasonable requirement, and that pooling is necessitated by the randomness of the data. Thus, whenever the posterior distributions violate our requirement, i.e. $(\theta_1 \mid T_1) \overset{st}{\geq} (\theta_0 \mid T_0)$

it is preferable to pool the variables rather than to change the prior parameters in order to make $a_1 > \alpha_0$. As a compromise, we may want to delete the requirement that $\theta_1 \overset{st}{\geq} \theta_0$ with respect to the posterior distributions, and just work with the requirement that

$(\theta_1) \overset{st}{\geq} (\theta_0 \mid T_0)$; (see section 4).

Another comment about our procedure pertains to our rationale for requiring that subsequent to pooling, conditions of the type given by (13) be satisfied. Note that (12) is analogous to the necessary condition (8). However, the terms which comprise condition (13) are not the means of the true posterior distribution after pooling. For instance, after we replace T_0 by T_{01}, and n_0 by n_{01}, the mean of the posterior distribution of θ_0 conditioned on T_{01} is not $\left(\alpha_0 + T_{01}\right)/(\beta_0 + n_{01} - 1)$, as is implied by the right-hand side of (13). The actual mean of the true posterior distribution is quite complicated, and in view of the resulting computational difficulties we choose $\left(\alpha_0 + T_{01}\right)/\ (\beta_0 + n_{01} - 1)$ as being analogous to the mean of the posterior distribution of θ_0 given T_{01}.We approximate the mean of the posterior distribution of θ_1 given T_{01} in a similar manner, and thus write condition (13). Since the above approximations are motivated by the arguments which lead us to pool, we feel that they are inherently satisfactory.We close this section by stating that in the light of the above discussions, our approach should be called a "pseudo-Bayesian approach".

III. An isotonic regression of the raw posterior means

Our strategy 1 of section 2.1 specifies that the inequality (11) and similar inequalities be ignored whenever the posterior means do not have the correct order. As a result, we will have at the end of testing over, say (τ + 1) stages, the (τ + 1) posterior means

$E(\theta_0 \mid T_0),\ E(\theta_1 \mid T_0,\ T_1), \ldots, E(\theta_\tau \mid T_0, T_1, \ldots, T_\tau)$; Where $E(\theta_0 \mid T_0) = (\alpha_0 + T_0)/\beta_0 + n_0 - 1)$ and

$$E(\theta_i \mid T_0, T_1, \ldots, T_i) = \frac{\alpha_0 + T_0 + \sum_{j=1}^{i} (a_j + T_j)}{\beta_0 + n_0 + \sum_{j=0}^{i} n_j - 1} \qquad ;\ i = 1, 2\ldots, \tau$$

Under our postulate of reliability growth, we would need to have (as a necessary condition)

$$E(\theta_i \mid T_0, \ldots, T_i) \le E(\theta_{i+1} \mid T_0, \ldots, T_{i+1}) \qquad ; i = 0, 1, \ldots, \tau - 1 \qquad (26)$$

If condition (26) is satisfied, then our evaluation of the reliability growth curve is given by these posterior means, and our Bayes estimator of the reliability at stage τ is simply $E(\theta_\tau \mid T_0, T_1, \ldots, T_\tau)$.Note that because of the adaptive nature of our scheme $E(\theta_\tau \mid T_0, \ldots, T_\tau)$ is based on the failure data over all the previous and present stages of testing and our prior knowledge about the magnitude of the improvement over each stage. If condition (26) is violated by any one or more of the indices *i*, *i* = 0, 1,...,τ, then we shall, following Barlow et al. [2], pool the adjacent violators to obtain the isotonic regression of $E(\theta_i \mid T_0, \ldots, T_i)$, *i* = 0, ...,τ, say $E^*(\theta_i \mid T_0, \ldots, T_i)$ .We shall use the $E^*(\theta_\tau \mid T_0, \ldots, T_\tau)$, *i* = 0,...,τ as our evaluation of the reliability growth, and $E^*(\theta_i \mid T_0, \ldots, T_i)$ as our estimate of the reliability at stage τ. Note that like $E(\theta_\tau \mid T_0, \ldots, T_\tau),\ E^*(\theta_\tau \mid T_0, \ldots, T_\tau)$ is based on the failure data over all the previous stages, our prior knowledge about the magnitude of the improvements at each stage, and the postulate of reliability growth.

The remarks of section 2.3 are also appropriate for the isotonic regression estimators $E^*(\theta_i \mid T_0, \ldots, T_i)$, since

(a) by performing an isotonic regression of the true posterior means we have violated the likelihood principle; and

(b) the estimators $E^*(\theta_i \mid T_0, \ldots, T_i)$ are not the true posterior means of the θ_i, *i* = 0, ..., τ, and thus are not fully Bayesian.

IV. Estimation when the ordering is with respect to the priors only

In section 2 we considered the case when the mean lifetimes were stochastically ordered with respect to both the prior and the posterior distributions. In this section we delete the requirement that the means be ordered with respect to the posterior distribution. When this is done we will not have to pool the violators, nor will we have to perform an isotonic regression of the posterior means should we choose not to pool.

We start by choosing a prior distribution of $\theta_0,\ g(\theta_0;\ \alpha_0,\ \beta_0)$, as given by (1). The posterior distribution of θ_0 conditioned on T_0, $g(\theta_0 \mid T_0;\ \alpha_0,\ \beta_0 n_0)$ is given by (2). Assuming a squared error loss, the Bayes estimator of θ_0 is $E(\theta_0 \mid T_0)$, and this is given by (3); the credibility intervals for θ_0 are given by (4).

We now choose a prior distribution of $\theta_1,\ g(\theta_1;\ \alpha_1,\ \beta_1)$, in such a manner that

$$\theta_i \overset{st}{\geq} (\theta_0 \mid T_0)$$

Following our discussion in section 2, we take $g\left(\theta_2; a_2, \beta_2\right)$ as our prior distribution of θ_2 with $\alpha_2 = \alpha_0 + T_0 + a_1 + T_2 + a_2$ and $\beta_2 = \beta_0 + n_0 + n_1$; here again we treat T_0 and T_1 as constants. As before, if $g(\theta_2; \alpha_2, \beta_2)$ is taken to be independent of $g(\theta_1 \mid\ T_1;\ .)$, then the mean of the posterior distribution of θ_2 conditioned on T_2 is our Bayes estimator of θ_2 .We continue in this manner going from one stage to the next, obtaining at each stage the Bayes estimator of θ_i , $i = 3,...,\tau$.

V. Summary and conclusions

In this paper we have considered an adaptive approach for estimating reliability growth based on prior information.In section 2 we imposed a strong requirement on our approach by requiring that the mean times to failure at the various stages be stochastically ordered with respect to both the prior and the posterior distributions. The latter requirement can be satisfied if we pool the violators; however, pooling results in a violation of the likelihood principle and creates other computational difficulties. Even though the computational difficulties can be avoided by using some approximations (see section 2.3), the pooling makes our procedure not fully Bayesian. Thus, what we present in section 2 can best be described as a pseudo-Bayesian scheme for estimating reliability growth. A formal investigation of the properties of our scheme, despite the fact that it is not fully Bayesian, is a matter which needs further attention. Our scheme, however, does produce results which are reasonable and intuitively satisfying.

Our review of the literature in Bayesian statistics indicates that there is no discussion or even mention of the problem of estimating parameters which are stochastically ordered. As mentioned before, our strategy of pooling the violators to obtain the stochastic order may be unacceptable to a Bayesian. We therefore hope that this paper can stimulate some basic research into this general problem area.

In view of the difficulties mentioned above, in section 4 we weaken the specifications on our approach by deleting the requirement that the parameters by stochastically ordered with respect to the posterior distributions. This simplification obviates the need for pooling the violators, and thus would make our procedure fully Bayesian and therefore optimal in the usual sense of minimizing the square error loss function. However, the adaptive nature of our problem imposes certain computational difficulties. We circumvent these by treating the observed statistic T_i as constant at stage $(i + 1)$, $i = 0, 1, 2, ...,\tau - 1$, and by assuming the prior distribution at stage j to be independent of the posterior distribution at stage $(j - 1)$, $j = 1, 2,..., \tau$. Then, within the context of the above assumptions, the procedure of section 4 is fully Bayesian.

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