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A Study on Pell and Pell-Lucas Numbers

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Abstract: *In this paper, we have presented few properties of Pell and Pell-Lucas numbers. Then the matrices related to these numbers are given in this paper. Next some identities satisfied by these numbers with proofs are discussed in this paper.*

Keywords: *Recurrence relation, Pell numbers, Pell-Lucas numbers, Binet formula.*

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I. Introduction

The Pell numbers are named after English mathematician John Pell (1611-1685) and the Pell-Lucas numbers are named after the mathematician Edouard Lucas (1842-1891). The details about Pell and Pell-Lucas numbers can be found in [1, 2, 3]. In [3, 4] authors showed that the Pell numbers can be represented in matrices. The identities satisfied by Pell and Pell-Lucas numbers are stated in [5]. Both the Pell numbers and Pell-Lucas numbers can be calculated by recurrence relations.

The sequence of Pell numbers $\{P_n\}$ is defined by recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2 \text{ with } P_0 = 0 \text{ and } P_1 = 1 \quad (1)$$

Where P_n denotes nth Pell number. The sequence of Pell numbers starts with 0 and 1 and then each number is the sum of twice its previous number and the number before its previous number.

The sequence of Pell-Lucas numbers $\{Q_n\}$ is defined by recurrence relation

$$Q_n = 2Q_{n-1} + Q_{n-2} \text{ for } n \geq 2 \text{ with } Q_0 = 2 \text{ and } Q_1 = 2 \quad (2)$$

Where Q_n denotes nth Pell-Lucas number. In the sequence of Pell-Lucas numbers each of the first two numbers is 2 and then each number is the sum of twice its previous number and the number before its previous number.

The first few Pell and Pell-Lucas numbers calculated from (1) and (2) are given in the following Table no.1.

Table no.1: First few Pell and Pell-Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
P_n	0	1	2	5	12	29	70	169	408	985	2378
Q_n	2	2	6	14	34	82	198	478	1154	2786	6726

The rest of the paper is organized as follows. The properties of Pell and Pell-Lucas numbers are mentioned in Section-II. Matrix representations of Pell and Pell-Lucas numbers are given in Section-III. The identities satisfied by Pell and Pell-Lucas numbers are stated in Section-IV. Finally conclusion is given in Section-V.

II. Properties of Pell and Pell-Lucas numbers

1. The Pell numbers $\{P_n\}$ are either even or odd but Pell-Lucas numbers $\{Q_n\}$ are all even.
2. *Binet formula*:
   The Binet formulas satisfied by Pell and Pell-Lucas numbers are given by

$$P_n = \frac{a^n - b^n}{a - b} \quad (3)$$

$$Q_n = a^n + b^n \quad (4)$$

Where a and b are the roots of quadratic equation $x^2 - 2x - 1 = 0$. Solving this equation, we get

$$a = 1 + \sqrt{2} \quad (5)$$

and

$$b = 1 - \sqrt{2} \quad (6)$$

Then

$$a + b = 2 \quad (7)$$

$$a - b = 2\sqrt{2} \quad (8)$$

$$ab = -1 \quad (9)$$

3. Let α and β are the solutions of equation

$$x^2 - 2y^2 = \pm 1 \quad (10)$$

The sets of values of α and β satisfying (10) are given by
$(\alpha, \beta) \equiv (1,1), (3,2), (7,5), (17,12), (41,29), (99,70), (239,169), (577,408) \dots \dots etc.$
The ratio α/β is approximately equal to $\sqrt{2} \approx 1.414$. Larger are the values of α and β, the more closer is the value of α/β to $\sqrt{2}$. For example, $\frac{41}{29} \approx 1.413793$ and $\frac{239}{169} \approx 1.414201$. The sequence of the ratio α/β closer to $\sqrt{2}$ is

$$\frac{\alpha}{\beta} \approx \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \dots \dots \quad (11)$$

In the above sequence (11), the denominator of each fraction is a Pell number and the numerator is the sum of a Pell number and its predecessor in Pell sequence. Hence in general, we can write

$$\sqrt{2} \approx \frac{P_{n-1}+P_n}{P_n} \quad (12)$$

4. *Pythagorean triples:*
If $A, B \& C$ are the integer sides of a right angled triangle satisfying the Pythagoras theorem $A^2 + B^2 = C^2$, then the integers (A, B, C) are known as Pythagorean triples. These triples can be formed by Pell numbers .The Pythagorean triples has the form

$$(A, B, C) \equiv (2P_nP_{n+1},\ P_{n+1}^2 - P_n^2,\ P_{2n+1}) \quad (13)$$

For example, if $n = 2$, $A = 2P_2P_3 = 2 \times 2 \times 5 = 20$, $B = P_3^2 - P_2^2 = 5^2 - 2^2 = 21$ and $C = P_5 = 29$. That is, the Pythagorean triple for $n = 2$ is $(20,21,29)$.
The sequence of Pythagorean triples obtained by putting $n = 1,2,3, \dots. etc.$ in (13) is given by
(4,3,5), (20,21,29), (120,119,169), (696,697,985),.........etc.

5. *Pell Primes*:
A Pell number which is a prime number is called *Pell Prime*. The first few Pell Primes are
2, 5, 29, 5741, 33461, ...
The indices of the Pell Primes in the sequence of Pell numbers respectively are
2, 3, 5, 11, 13, ...
That is, $P_2 = 2, P_3 = 5, P_5 = 29, P_{11} = 5741, \dots \dots. etc.$ The indices of Pell Primes are also prime numbers.

6. *Pell-Lucas Primes:*
The number $\frac{Q_n}{2}$ is called Pell-Lucas Prime. The Pell-Lucas Primes are
3, 7, 17, 41, 239, 577,.....etc.
The indices of the above Pell-Lucas numbers in Pell-Lucas sequence are
2, 3, 4, 5, 7, 8,.........etc.
That is, $\frac{Q_2}{2} = 3, \frac{Q_3}{2} = 7, \frac{Q_4}{2} = 17, \dots..$ etc.

7. The relation between Pell and Pell-Lucas numbers is given by

$$Q_n = \frac{P_{2n}}{P_n} \quad (14)$$

Proof: Using the Binet formulas (3) and (4) we get

$$P_nQ_n = \left(\frac{a^n - b^n}{a-b}\right)(a^n + b^n)$$
$$= \frac{a^{2n} - b^{2n}}{a-b}$$
$$= P_{2n} \text{ [By (3)]}$$
$$\Rightarrow Q_n = \frac{P_{2n}}{P_n}$$

Thus (14) is proved.
For example, $Q_3 = \frac{P_6}{P_3} = \frac{70}{5} = 14$.

8. *Simpson formula*:
The Pell numbers satisfy *Simpson's formula* given by

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n \quad (15)$$

Proof: Using Binet formula (3), we get

$$P_{n+1}P_{n-1} - P_n^2 = \frac{(a^{n+1} - b^{n+1})(a^{n-1} - b^{n-1})}{(a-b)^2} - \frac{(a^n - b^n)^2}{(a-b)^2}$$
$$= \frac{(a^{2n} - a^{n+1}b^{n-1} - b^{n+1}a^{n-1} + b^{2n}) - (a^{2n} + b^{2n} - 2a^nb^n)}{(a-b)^2}$$
$$= \frac{-a^{n-1}b^{n-1}(a^2 + b^2 - 2ab)}{(a-b)^2}$$

$$= -(ab)^{n-1} = -(-1)^{n-1} \quad \text{[By (9)]}$$

$$\Rightarrow P_{n+1}P_{n-1} - P_n^2 = (-1)^n$$

Thus Simpson formula (15) is proved.

9. As proved in [6] the sum of Pell numbers up to $(4n+1)$ is a perfect square as given below.

$$\sum_{i=0}^{4n+1} P_i = (P_{2n} + P_{2n+1})^2 \qquad (16)$$

For example if $n = 1$, LHS = $\sum_{i=0}^{5} P_i = P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 0 + 1 + 2 + 5 + 12 + 29 = 49$ and RHS $= (P_2 + P_3)^2 = (2+5)^2 = 49$. Hence the above relation (16) is verified.

III. Matrix representation of Pell and Pell-Lucas numbers

In this section some matrices are represented in terms of Pell and Pell-Lucas numbers.

1. Consider a 2×2 matrix R given by

$$R = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad (17)$$

Writing the matrix R in terms of Pell numbers, we have

$$R = \begin{pmatrix} P_2 & P_1 \\ P_1 & P_0 \end{pmatrix} \qquad (18)$$

Now let us find out the matrices R^2 and R^3.

$$R^2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{[By (17)]} \qquad (19)$$

$$R^3 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 5 \\ 5 & 2 \end{pmatrix} \quad \text{[By (17) \&(19)]} \qquad (20)$$

In terms of Pell numbers the above matrices (19) and (20) can be written as

$$R^2 = \begin{pmatrix} P_3 & P_2 \\ P_2 & P_1 \end{pmatrix} \qquad (21)$$

$$R^3 = \begin{pmatrix} P_4 & P_3 \\ P_3 & P_2 \end{pmatrix} \qquad (22)$$

Considering (17), (21) and (22) one can write the nth power of the matrix R in general as

$$R^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} \qquad (23)$$

$$for\ n = 1,2,3, \ldots\ldots etc.$$

2. Consider a 2×2 matrix E given by

$$E = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \qquad (24)$$

The Pell numbers for even index n are expressed in terms of the matrix E by the following relation.

$$[P_n] = \frac{1}{2^{n/2}}[1 \quad 0]E^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{if } n \text{ is even} \qquad (25)$$

Example: The above relation can be verified taking an example with $n = 4\ (even)$. Now let us find out the value of E^4.

$$E^2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{[By (24)]}$$

$$= \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} = 2\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \qquad (26)$$

$$E^4 = 4\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{[By (26)]}$$

$$= 4\begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix} \qquad (27)$$

For $n = 4$, (25) can be written as

$$[P_4] = \frac{1}{2^2}[1 \quad 0]E^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{4}[1 \quad 0]4\begin{bmatrix} 29 & 12 \\ 12 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{[Using (27)]}$$

$$= [1 \quad 0]\begin{bmatrix} 12 \\ 5 \end{bmatrix} = [12]$$

That is, $P_4 = 12$, which is true. Hence the relation (25) is verified.

3. The 4th power of matrix E in (27) can be written in terms of Pell numbers as

$$E^4 = 2^2\begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix} = 2^2\begin{pmatrix} P_5 & P_4 \\ P_4 & P_3 \end{pmatrix}$$

In general the above relation can be written as

$$E^n = 2^{n/2}\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} \quad \text{if } n \text{ is even} \qquad (28)$$

Where the matrix E is given by (24).

4. The matrix E given by (24) is an invertible matrix since $\det E \neq 0$.
We have

$$E^2 = \begin{pmatrix} 10 & 4 \\ 4 & 2 \end{pmatrix} \quad \text{[By (26)]}$$

Then,

$$\det E^2 = \begin{vmatrix} 10 & 4 \\ 4 & 2 \end{vmatrix} = 4 \tag{29}$$

Now consider a matrix B, whose elements are cofactors of E^2. Hence

$$B = \begin{pmatrix} 2 & -4 \\ -4 & 10 \end{pmatrix}$$

The transpose of above matrix B is given by

$$B^T = \begin{pmatrix} 2 & -4 \\ -4 & 10 \end{pmatrix} \tag{30}$$

The matrix B is a symmetric matrix as $B = B^T$. Now E^{-2} can be calculated using the relation

$$E^{-2} = \frac{1}{\det E^2} B^T$$

$$= \frac{1}{4}\begin{pmatrix} 2 & -4 \\ -4 & 10 \end{pmatrix} \quad \text{[By (29) \& (30)]}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} P_1 & -P_2 \\ -P_2 & P_3 \end{pmatrix} \tag{31}$$

In general the above expression (31) can be written as

$$E^{-n} = \frac{1}{2^{n/2}}\begin{pmatrix} P_{n-1} & -P_n \\ -P_n & P_{n+1} \end{pmatrix} \quad \text{if } n \text{ is even} \tag{32}$$

5. Consider a 2×2 matrix F given by

$$F = 2E = \begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix} \quad \text{[By (24)]} \tag{33}$$

Now let us find out the matrix F^2.

$$F^2 = \begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 40 & 16 \\ 16 & 8 \end{pmatrix} = 8\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \tag{34}$$

In terms of Pell numbers the above expression can be written as

$$F^2 = 2^3\begin{pmatrix} P_3 & P_2 \\ P_2 & P_1 \end{pmatrix} \tag{35}$$

Now let us calculate the matrix F^4.

$$F^4 = F^2 \times F^2 = 64\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = 2^6\begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix} \text{[By (34)]}$$

In terms of Pell numbers the above expression can be written as

$$F^4 = 2^6\begin{pmatrix} P_5 & P_4 \\ P_4 & P_3 \end{pmatrix} \tag{36}$$

In general noting the above relations (35) and (36) the nth power of the matrix F can be written as

$$F^n = 2^{3n/2}\begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix} \quad \text{if } n \text{ is even} \tag{37}$$

6. Let us now calculate the matrix F^3 in terms of Pell-Lucas numbers.

$$F^3 = F \times F^2 = \begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix} 8\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = 8\begin{pmatrix} 34 & 14 \\ 14 & 6 \end{pmatrix} \text{[By (33) \& 34)]} \tag{38}$$

The above relation in terms of Pell-Lucas numbers can be written as

$$F^3 = 2^3\begin{pmatrix} Q_4 & Q_3 \\ Q_3 & Q_2 \end{pmatrix} \tag{39}$$

Now let us find out the matrix F^5.

$$F^5 = F^2 \times F^3 = 8\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} 8\begin{pmatrix} 34 & 14 \\ 14 & 6 \end{pmatrix} \quad \text{[By (34) \& (38)]}$$

$$= 2^6\begin{pmatrix} 198 & 82 \\ 82 & 34 \end{pmatrix} \tag{40}$$

In terms of Pell-Lucas numbers the above expression can be written as

$$F^5 = 2^6\begin{pmatrix} Q_6 & Q_5 \\ Q_5 & Q_4 \end{pmatrix} \tag{41}$$

In general using the above relations (39) and (41), the nth power of matrix F in terms of Pell-Lucas numbers can be written as

$$F^n = 2^{3(n-1)/2}\begin{pmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{pmatrix} \quad \text{if } n \text{ is odd} \tag{42}$$

IV. Identities satisfied by Pell and Pell-Lucas numbers

The following are some identities satisfied by Pell and Pell-Lucas numbers.

1. $$P_{n+1} + P_{n-1} = Q_n \tag{43}$$

Proof: Applying Binet formula (3) to LHS we get

$$P_{n+1} + P_{n-1} = \frac{a^{n+1}-b^{n+1}}{a-b} + \frac{a^{n-1}-b^{n-1}}{a-b}$$
$$= \frac{a^{n+1}-b^{n+1}+\frac{a^n}{a}-\frac{b^n}{b}}{a-b}$$
$$= \frac{a^{n+1}-b^{n+1}-a^n b+b^n a}{a-b} \quad \text{[By (9)]}$$
$$= \frac{a^n(a-b)+b^n(a-b)}{(a-b)} = a^n + b^n$$
$$= Q_n \quad \text{[By (4)]}$$

Thus (43) is proved.

2. $$Q_n = 2(P_n + P_{n-1}) \quad (44)$$

Proof: Using (43), we obtain

$$Q_n = P_{n+1} + P_{n-1}$$
$$= (2P_n + P_{n-1}) + P_{n-1} \quad \text{[By (1)]}$$
$$= 2(P_n + P_{n-1})$$

Hence (44) is proved.

3. $$P_{n+2} - P_{n-2} = 2Q_n \quad (45)$$

Proof:

$$\text{LHS} = P_{n+2} - P_{n-2} = (2P_{n+1} + P_n) - P_{n-2} \quad \text{[By (1)]}$$
$$= 2P_{n+1} + (P_n - P_{n-2})$$
$$= 2P_{n+1} + 2P_{n-1} \quad \text{[By (1)]}$$
$$= 2(P_{n+1} + P_{n-1})$$
$$= 2Q_n \quad \text{[By (43)]}$$

Thus (45) is proved.

4. $$Q_{n-1} + Q_{n+1} = 8P_n \quad (46)$$

Proof:

$$Q_{n-1} + Q_{n+1} = 2(P_{n-1} + P_{n-2}) + 2(P_{n+1} + P_n) \quad \text{[By (44)]}$$
$$= (2P_{n-1} + P_{n-2}) + P_{n-2} + 2P_{n+1} + 2P_n$$
$$= P_n + P_{n-2} + 2(2P_n + P_{n-1}) + 2P_n \quad \text{[By (1)]}$$
$$= (2P_{n-1} + P_{n-2}) + 7P_n$$
$$= P_n + 7P_n \quad \text{[By (1)]}$$
$$= 8P_n$$

Hence (46) is proved

5. $$P_{2^m} = Q_{2^{m-1}} Q_{2^{m-2}} \dots\dots Q_4 Q_2 Q_1 \quad (47)$$

Proof:

$$P_{2^m} = P_{2\times 2^{m-1}} = P_{2^{m-1}} Q_{2^{m-1}} \quad [\because\ P_{2n} = P_n Q_n \ by\ (14)]$$
$$= Q_{2^{m-1}} P_{2\times 2^{m-2}}$$

By repeated application of (14) in RHS of above expression, we get

$$P_{2^m} = Q_{2^{m-1}} Q_{2^{m-2}} P_{2^{m-2}}$$
$$= Q_{2^{m-1}} Q_{2^{m-2}} P_{2\times 2^{m-3}}$$
$$= Q_{2^{m-1}} Q_{2^{m-2}} Q_{2^{m-3}} P_{2^{m-3}}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$= Q_{2^{m-1}} Q_{2^{m-2}} \dots\dots Q_{2^{m-(m-2)}} Q_{2^{m-(m-1)}} Q_{2^{m-m}} P_{2^{m-m}}$$
$$= Q_{2^{m-1}} Q_{2^{m-2}} \dots\dots Q_4 Q_2 Q_1 P_1$$
$$= Q_{2^{m-1}} Q_{2^{m-2}} \dots\dots Q_4 Q_2 Q_1 \quad [\because\ P_1 = 1]$$

Thus (47) is proved.

6. $$P_{-n} = (-1)^{n+1} P_n \quad (48)$$

Proof:

$$P_{-n} = \frac{a^{-n} - b^{-n}}{a-b} \quad \text{[By (3)]}$$
$$= \frac{\left(-\frac{1}{b}\right)^{-n} - \left(-\frac{1}{a}\right)^{-n}}{a-b} \quad \text{[By (9)]}$$
$$= \frac{[(-b)^{-1}]^{-n} - [(-a)^{-1}]^{-n}}{a-b}$$
$$= \frac{(-1)^n (b^n - a^n)}{a-b}$$
$$= \frac{(-1)^{n+1} (a^n - b^n)}{(a-b)}$$
$$= (-1)^{n+1} P_n \quad \text{[By (3)]}$$

Hence (48) is proved

7. $$Q_{-n} = (-1)^n Q_n \qquad (49)$$

Proof:

$$\begin{aligned} Q_{-n} &= a^{-n} + b^{-n} && \text{[By (4)]} \\ &= \left(-\tfrac{1}{b}\right)^{-n} + \left(-\tfrac{1}{a}\right)^{-n} && \text{[By (9)]} \\ &= [(-b)^{-1}]^{-n} + [(-a)^{-1}]^{-n} \\ &= (-1)^n (a^n + b^n) \\ &= (-1)^n Q_n && \text{[By (4)]} \end{aligned}$$

Thus (49) is proved.

8. $$P_m Q_n + P_n Q_m = 2P_{m+n} \qquad (50)$$

Proof:

$$\begin{aligned} P_m Q_n + P_n Q_m &= \left(\frac{a^m - b^m}{a-b}\right)(a^n + b^n) + \left(\frac{a^n - b^n}{a-b}\right)(a^m + b^m) \quad \text{[By (3) \& (4)]} \\ &= \frac{a^{m+n} + a^m b^n - b^m a^n - b^{m+n} + a^{m+n} + a^n b^m - b^n a^m - b^{m+n}}{a-b} \\ &= \frac{2(a^{m+n} - b^{m+n})}{a-b} \\ &= 2P_{m+n} \text{ [By (3)]} \end{aligned}$$

Hence (50) is proved

9. $$Q_m P_{n-m-1} + Q_{m-1} P_{n-m} = Q_n \qquad (51)$$

Proof:

$$\begin{aligned} Q_m P_{n-m+1} + Q_{m-1} P_{n-m} &= (a^m + b^m)\left(\frac{a^{n-m+1} - b^{n-m+1}}{a-b}\right) + (a^{m-1} + b^{m-1})\left(\frac{a^{n-m} - b^{n-m}}{a-b}\right) \\ & \qquad \text{[By (3) \& (4)]} \\ &= \frac{a^{n+1} - a^m b^{n-m+1} + b^m a^{n-m+1} - b^{n+1} + a^{n-1} - a^{m-1} b^{n-m} + b^{m-1} a^{n-m} - b^{n-1}}{a-b} \\ &= \frac{a^{n+1} - b^{n+1} + a^{n-1} - b^{n-1} - a^{m-1} b^{n-m}(ab+1) + a^{n-m} b^{m-1}(ab+1)}{a-b} \\ &= \frac{a^{n+1} - b^{n+1}}{a-b} + \frac{a^{n-1} - b^{n-1}}{a-b} \quad [\because ab = -1 \; by \; (9)] \\ &= P_{n+1} + P_{n-1} \quad \text{[By (3)]} \\ &= Q_n \quad \text{[By (43)]} \end{aligned}$$

Thus (51) is proved.

10. $$P_{3n} = Q_n P_{2n} - (-1)^n P_n \qquad (52)$$

Proof:

$$\begin{aligned} RHS = Q_n P_{2n} - (-1)^n P_n &= (a^n + b^n)\left(\frac{a^{2n} - b^{2n}}{a-b}\right) - (-1)^n P_n \\ &= \frac{a^{3n} - a^n b^{2n} + b^n a^{2n} - b^{3n}}{a-b} - (-1)^n P_n \\ &= \frac{a^{3n} - b^{3n}}{a-b} + \frac{a^n b^n (a^n - b^n)}{a-b} - (-1)^n P_n \\ &= P_{3n} + (-1)^n P_n - (-1)^n P_n \quad \text{[By (3)]} \; [\because ab = -1 \; by \; (9)] \\ &= P_{3n} = LHS \end{aligned}$$

Hence (52) is proved

11. $$P_{3n} = P_n\{Q_{2n} + (-1)^n\} \qquad (53)$$

Proof:

$$\begin{aligned} RHS = P_n\{Q_{2n} + (-1)^n\} &= P_n Q_{2n} + (-1)^n P_n \\ &= \left(\frac{a^n - b^n}{a-b}\right)(a^{2n} + b^{2n}) + (-1)^n P_n \\ &= \frac{a^{3n} + a^n b^{2n} - b^n a^{2n} - b^{3n}}{a-b} + (-1)^n P_n \\ &= \frac{a^{3n} - b^{3n}}{a-b} - \frac{a^n b^n (a^n - b^n)}{a-b} + (-1)^n P_n \\ &= P_{3n} - (-1)^n P_n + (-1)^n P_n \quad \text{[By (3)]} \; [\because ab = -1 \; by \; (9)] \\ &= P_{3n} = LHS \end{aligned}$$

Thus (53) is proved.

12. $$Q_{3n} = Q_n\{Q_{2n} - (-1)^n\} \qquad (54)$$

Proof:

$$\begin{aligned} RHS = Q_n\{Q_{2n} - (-1)^n\} &= Q_n Q_{2n} - (-1)^n Q_n \\ &= (a^n + b^n)(a^{2n} + b^{2n}) - (-1)^n Q_n \\ &= a^{3n} + a^n b^{2n} + b^n a^{2n} + b^{3n} - (-1)^n Q_n \\ &= (a^{3n} + b^{3n}) + a^n b^n (a^n + b^n) - (-1)^n Q_n \\ &= Q_{3n} + (-1)^n Q_n - (-1)^n Q_n \quad \text{[By (4)]} \; [\because ab = -1 \; by \; (9)] \\ &= Q_{3n} = LHS \end{aligned}$$

Hence (54) is proved.

13. $$8{P_n}^2 = {Q_n}^2 - 4(-1)^n \qquad (55)$$

Proof:

$$RHS = {Q_n}^2 - 4(-1)^n = (a^n + b^n)^2 - 4(ab)^n \quad [By\ (4)\ \&\ ab = -1\ by\ (9)]$$
$$= (a^n - b^n)^2$$
$$= \left(\frac{a^n - b^n}{a-b}\right)^2 (a-b)^2$$
$$= P_n^2\left(2\sqrt{2}\right)^2 \text{ [By (3) and (8)]}$$
$$= 8{P_n}^2 = LHS$$

Thus (55) is proved.

14. $$Q_{4n} = 8{P_{2n}}^2 + 2 \qquad (56)$$

Proof:

$$LHS = Q_{4n} = (a^{4n} + b^{4n}) = (a^{2n} - b^{2n})^2 + 2(ab)^{2n} \text{[By (4)]}$$
$$= \left(\frac{a^{2n} - b^{2n}}{a-b}\right)^2 (a-b)^2 + 2(-1)^{2n} [\because ab = -1\ by\ (9)]$$
$$= P_{2n}^2\left(2\sqrt{2}\right)^2 + 2 \text{ [By (3) and (8)] } [\because (-1)^{2n} = 1]$$
$$= 8{P_{2n}}^2 + 2 = RHS$$

Hence (56) is proved.

15. $$Q_{4n+2} = 8{P_{n+1}}^2 - 2 \qquad (57)$$

Proof:

$$LHS = Q_{4n+2} = (a^{4n+2} + b^{4n+2}) = (a^{2n+1} - b^{2n+1})^2 + 2(ab)^{2n+1} \text{[By (4)]}$$
$$= \left(\frac{a^{2n+1} - b^{2n+1}}{a-b}\right)^2 (a-b)^2 + 2(-1)^{2n+1} [\because ab = -1\ by\ (9)]$$
$$= P_{2n+1}^2\left(2\sqrt{2}\right)^2 - 2 \text{ [By (3) and (8)] } [(-1)^{2n+1} = -1]$$
$$= 8{P_{2n+1}}^2 - 2 = RHS$$

Thus (57) is proved.

16. $$Q_n^2 + Q_{n+1}^2 = Q_{2n} + Q_{2n+2} \qquad (58)$$

Proof:

$$LHS = Q_n^2 + Q_{n+1}^2 = (a^n + b^n)^2 + (a^{n+1} + b^{n+1})^2$$
$$= (a^{2n} + b^{2n} + 2a^n b^n) + (a^{2n+2} + b^{2n+2} + 2a^{n+1}b^{n+1})$$
$$= (a^{2n} + b^{2n}) + 2(ab)^n + (a^{2n+2} + b^{2n+2}) + 2(ab)^{n+1}$$
$$= Q_{2n} + Q_{2n+2} + 2\{(-1)^n + (-1)^{n+1}\} \text{ [By (4) \& (9)]}$$
$$= Q_{2n} + Q_{2n+2}\ [\because (-1)^n + (-1)^{n+1} = 0\ for\ n = 0,1,2,\ldots]$$
$$= RHS$$

Thus (58) is proved.

17. $$a^n = P_n a + P_{n-1} \qquad (59)$$

Proof: Using the values of Pell numbers we have

$$a = a + 0 = P_1 a + P_0 \qquad (59a)$$
$$a^2 = \left(1+\sqrt{2}\right)^2 = 1 + 2 + 2\sqrt{2} = 2\left(1+\sqrt{2}\right) + 1 = P_2 a + P_1 \qquad (59b)$$
$$a^3 = \left(1+\sqrt{2}\right)^3 = 1 + 3\sqrt{2} + 6 + 2\sqrt{2} = 5\left(1+\sqrt{2}\right) + 2$$
$$= P_3 a + P_2 \qquad (59c)$$

Looking at the forms of the above three expressions, one can write in general that

$$a^n = P_n a + P_{n-1}$$

Thus (59) is proved.

18. $$b^n = P_n b + P_{n-1} \qquad (60)$$

Proof: Using the values of Pell numbers we get

$$b = b + 0 = P_1 b + P_0 \qquad (60a)$$
$$b^2 = \left(1-\sqrt{2}\right)^2 = 1 + 2 - 2\sqrt{2} = 2\left(1-\sqrt{2}\right) + 1 = P_2 b + P_1 \qquad (60b)$$
$$b^3 = \left(1-\sqrt{2}\right)^3 = 1 - 3\sqrt{2} + 6 - 2\sqrt{2} = 5\left(1-\sqrt{2}\right) + 2$$
$$= P_3 b + P_2 \qquad (60c)$$

Looking at the forms of the above three expressions, one can write in general that

$$b^n = P_n b + P_{n-1}$$

Hence (60) is proved.

19. $$a^{-n} = \begin{cases} aP_n - P_{n+1}, & if\ n\ odd \\ P_{n+1} - aP_n, & if\ n\ even \end{cases} \qquad (61)$$

Proof: Replacing n by $(-n)$in (59) we have

$$a^{-n} = P_{-n}a + P_{-(n+1)}$$
$$= a(-1)^{n+1}P_n + (-1)^{n+2}P_{n+1} \text{ [By (48)]}$$
$$= (-1)^{n+1}[aP_n - P_{n+1}]$$
$$\Rightarrow a^{-n} = \begin{cases} aP_n - P_{n+1}, & if\ n\ odd \\ P_{n+1} - aP_n, & if\ n\ even \end{cases}$$

Thus (61) is proved.

20. $$b^{-n} = \begin{cases} bP_n - P_{n+1}, & if\ n\ odd \\ P_{n+1} - bP_n, & if\ n\ even \end{cases} \quad (62)$$

Proof: Replacing n by $(-n)$in (60) we get

$$b^{-n} = P_{-n}b + P_{-(n+1)}$$
$$= b(-1)^{n+1}P_n + (-1)^{n+2}P_{n+1} \text{ [By (48)]}$$
$$= (-1)^{n+1}[bP_n - P_{n+1}]$$
$$\Rightarrow b^{-n} = \begin{cases} bP_n - P_{n+1}, & if\ n\ odd \\ P_{n+1} - bP_n, & if\ n\ even \end{cases}$$

Hence (62) is proved.

21. $$a^m P_{n-m+1} + a^{m-1}P_{n-m} = a^n \quad (63)$$

Proof:

$$LHS = a^m P_{n-m+1} + a^{m-1}P_{n-m} = a^{m-1}(aP_{n-m+1} + P_{n-m}) \quad (64)$$

Replacing n by $(n - m + 1)$ in (59) we get

$$a^{n-m+1} = P_{n-m+1}a + P_{n-m} \quad (65)$$

Substituting (65) in (64) we obtain

$$LHS = a^{m-1} \times a^{n-m+1} = a^n = RHS$$

Hence (63) is proved.

22. $$b^m P_{n-m+1} + b^{m-1}P_{n-m} = b^n \quad (66)$$

Proof:

$$LHS = b^m P_{n-m+1} + b^{m-1}P_{n-m} = b^{m-1}(bP_{n-m+1} + P_{n-m}) \quad (67)$$

Replacing n by $(n - m + 1)$ in (60) we have

$$b^{n-m+1} = P_{n-m+1}b + P_{n-m} \quad (68)$$

Substituting (68) in (67) we get

$$LHS = b^{m-1} \times b^{n-m+1} = b^n = RHS$$

Hence (66) is proved.

23.

$$\sum_{i=0}^{n} P_{ki+j} = \begin{cases} \frac{P_{nk+k+j} - (-1)^k P_{nk+j} - P_j - (-1)^j P_{k-j}}{Q_k - (-1)^k - 1} & if\ \ j < k \\ \frac{P_{nk+k+j} - (-1)^k P_{nk+j} - P_j + (-1)^k P_{j-k}}{Q_k - (-1)^k - 1} & if\ j > k \end{cases} \quad (69)$$

Proof:

$$\sum_{i=0}^{n} P_{ki+j} = \sum_{i=0}^{n} \frac{a^{ki+j} - b^{ki+j}}{a-b} \quad \text{[By (3)]}$$
$$= \frac{1}{a-b}\left[a^j \sum_{i=0}^{n} a^{ki} - b^j \sum_{i=0}^{n} b^{ki}\right] \quad (70)$$

The summation of terms in geometric series is given by

$$\sum_{p=l}^{n} z^p = \frac{z^l - z^{n+1}}{1-z} \quad (71)$$

Putting $p = i, l = 0$ and $z = a^k$ in the above expression (71) we get

$$\sum_{i=0}^{n} a^{ki} = \frac{1 - a^{k(n+1)}}{1-a^k} = \frac{a^{nk+k} - 1}{a^k - 1} \quad (72)$$

Similarly putting $p = i, l = 0$ and $z = b^k$ in (71) we have

$$\sum_{i=0}^{n} b^{ki} = \frac{1 - b^{k(n+1)}}{1-b^k} = \frac{b^{nk+k} - 1}{b^k - 1} \quad (73)$$

Substituting (72) and (73) in RHS of(70) we get

$$\sum_{i=0}^{n} P_{ki+j} = \frac{1}{a-b}\left[a^j \left(\frac{a^{nk+k} - 1}{a^k - 1}\right) - b^j \left(\frac{b^{nk+k} - 1}{b^k - 1}\right)\right]$$
$$= \frac{1}{a-b}\left[\frac{a^j(a^{nk+k} - 1)(b^k - 1) - b^j(b^{nk+k} - 1)(a^k - 1)}{(a^k - 1)(b^k - 1)}\right]$$
$$= \frac{1}{a-b}\left[\frac{a^j(a^{nk+k}b^k - a^{nk+k} - b^k + 1) - b^j(b^{nk+k}a^k - b^{nk+k} - a^k + 1)}{a^k b^k - a^k - b^k + 1}\right]$$
$$= \frac{1}{a-b}\left[\frac{a^{j+nk+k}b^k - a^{j+nk+k} - a^j b^k + a^j - b^{j+nk+k}a^k + b^{j+nk+k} + b^j a^k - b^j}{(ab)^k - (a^k + b^k) + 1}\right]$$
$$= \frac{1}{(-1)^k - Q_k + 1}\left[-\left(\frac{a^{nk+k+j} - b^{nk+k+j}}{a-b}\right) + \left(\frac{a^k b^j - a^j b^k}{a-b}\right) + \left(\frac{a^j - b^j}{a-b}\right) + (ab)^k\left(\frac{a^{nk+j} - b^{nk+j}}{a-b}\right)\right]$$

[By (4) and (9)]

$$= \frac{1}{(-1)^k - Q_k + 1}\left[-P_{nk+k+j} + \left(\frac{a^k b^j - a^j b^k}{a-b}\right) + P_j + (-1)^k P_{nk+j}\right] \quad (74)$$

[By(3) and (9)]

Slightly modifying the 2nd part within the brackets of RHS of the above expression (74) we get

$$\frac{a^k b^j - a^j b^k}{a-b} = \begin{cases} \frac{(ab)^j}{a-b}(a^{k-j} - b^{k-j}) & if\ j < k \\ \frac{(ab)^k}{a-b}(b^{j-k} - a^{j-k}) & if\ j > k \end{cases}$$

$$= \begin{cases} (-1)^j P_{k-j} & if\ j < k \\ -(-1)^k P_{j-k} & if\ j > k \end{cases} \quad \text{[By (3) \& (9)]} \quad (75)$$

Using (75) in (74) we obtain

$$\sum_{i=0}^{n} P_{ki+j} = \begin{cases} \frac{1}{(-1)^k - Q_k + 1}\left[-P_{nk+k+j} + (-1)^j P_{k-j} + P_j + (-1)^k P_{nk+j}\right] & if\ j < k \\ \frac{1}{(-1)^k - Q_k + 1}\left[-P_{nk+k+j} - (-1)^k P_{j-k} + P_j + (-1)^k P_{nk+j}\right] & if\ j > k \end{cases}$$

$$\Rightarrow \sum_{i=0}^{n} P_{ki+j} = \begin{cases} \frac{P_{nk+k+j} - (-1)^k P_{nk+j} - P_j - (-1)^j P_{k-j}}{Q_k - (-1)^k - 1} & if\ j < k \\ \frac{P_{nk+k+j} - (-1)^k P_{nk+j} - P_j + (-1)^k P_{j-k}}{Q_k - (-1)^k - 1} & if\ j > k \end{cases}$$

Hence (69) is proved.

24. $$\sum_{i=0}^{n} P_{ki} = \frac{P_{nk+k} - (-1)^k P_{nk} - P_k}{Q_k - (-1)^k - 1} \quad (76)$$

Proof: Putting $j = 0$ in (69) for the case $j < k$ we get

$$\sum_{i=0}^{n} P_{ki} = \frac{P_{nk+k} - (-1)^k P_{nk} - P_0 - P_k}{Q_k - (-1)^k - 1}$$

$$\Rightarrow \sum_{i=0}^{n} P_{ki} = \frac{P_{nk+k} - (-1)^k P_{nk} - P_k}{Q_k - (-1)^k - 1} \quad [\because P_0 = 0]$$

Thus (76) is proved.

25.

$$\sum_{i=0}^{n} P_{i+j} = \begin{cases} \frac{P_{n+1+j} + P_{n+j} - P_j - (-1)^j P_{1-j}}{2} & if\ j < 1 \\ \frac{P_{n+1+j} + P_{n+j} - P_j - P_{j-1}}{2} & if\ j > 1 \end{cases} \quad (77)$$

Proof: For $k = 1$, we can write (69) as

$$\sum_{i=0}^{n} P_{i+j} = \begin{cases} \frac{P_{n+1+j} + P_{n+j} - P_j - (-1)^j P_{1-j}}{Q_1 + 1 - 1} & if\ j < 1 \\ \frac{P_{n+1+j} + P_{n+j} - P_j - P_{j-1}}{Q_1 + 1 - 1} & if\ j > 1 \end{cases}$$

$$\Rightarrow \sum_{i=0}^{n} P_{i+j} = \begin{cases} \frac{P_{n+1+j} + P_{n+j} - P_j - (-1)^j P_{1-j}}{2} & if\ j < 1 \\ \frac{P_{n+1+j} + P_{n+j} - P_j - P_{j-1}}{2} & if\ j > 1 \end{cases} \quad [\because Q_1 = 2]$$

Hence (77) is proved.

V. Conclusion

Pell and Pell-Lucas numbers can be represented by matrices. The identities satisfied by these numbers can be derived using Binet formula. This study on Pell and Pell-Lucas numbers will inspire curious mathematicians to extend it further.

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