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Area under A Curve

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PREFACE:-

Firstly, I want to introduce myself, my Name is Shelbistar Marbaniang. I am a citizen of India (State Meghalaya). I have been graduated since 2018.

As a student I enjoy solving mathematical problems and my greatest wish for myself is one day I will be able to fix or resolve the Riemann Sum of Definite Integral as it is widely known as an **Approximation area under a curve**.

ABSTRACT:-
In this paper Δx is considered as an equal length (constant length) of each rectangles and triangles under a closed curve define by a function $f(x)$.
And $f(x_0)$, $f(x_1)$, $f(x_2)$, … … … … … … ., $f(x_{i-1})$, $f(x_i)$ are the points traced by a function $f(x)$ at $(x_0 , x_1, x_2 , \dots \dots \dots \dots, x_{i-1}, x_i)$.

Keywords: Definite Integral, Area under a curve by summation.

Definition:-
If $f(x)$ is a continuous function defined on the interval $[a , b]$, then the area under the curve $f(x)$ above the $x -$ axis, and between $x = a$ and $x = b$ is given by

$$\int_a^b f(x)dx = \frac{1}{2} \lim_{n\to\infty} \left\{ \sum_{i=1}^{n} \big(f(x_{i-1}) + f(x_i) \big)\Delta x \right\}$$

Where $f(x_{i-1})$ is a point traced by a function $f(x)$ at x_{i-1} and $f(x_i)$ is a point traced by a function $f(x)$ at x_i , and Δx is the length of each sub-interval of equal length.

Also $x_{i-1} = a + (i-1)\Delta x$ and $x_i = a + i\Delta x$

Proof:
Suppose the interval of a function $f(x)$ is divided into n number of sub-intervals of equal length Δx as shown in Fig.1.
Let $P = \{a = x_0 , x_1 , x_2, \dots \dots \dots \dots, x_{n-1} , x_n = b\}$ is called a **partition** of $\{a \ , b\}$.
Then,
The total sum of n number of subintervals of equal length of function $f(x)$ is given by

$$n\Delta x = b - a \qquad (1.1)$$

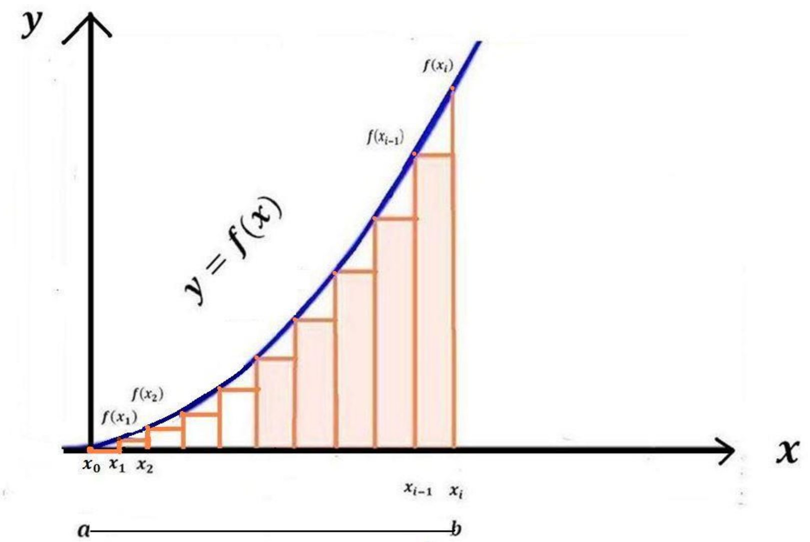

Fig.1

Step.1:- Finding the area of a closed curve at $(x_o\ , x_1)$:
Let $f(x_0) = 0$ be a point traced by the function $f(x)$ at x_0 as shown in Fig.2

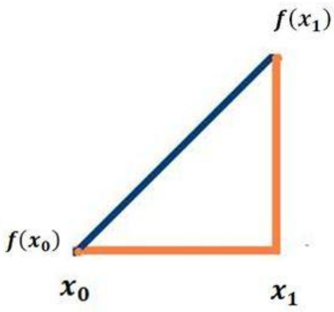

Fig.2

Again, consider a rectangle ABCD of width $f(x_0)$= 0 and length $= x_1 - x_0 = \Delta x$ exists in a closed curve of Fig.2 as shown in Fig.3

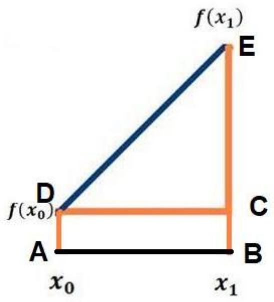

Fig. 3

From Fig.3, we have
AD=BC= $f(x_0)$, but $f(x_0) = 0$
AB = CD= $x_1 - x_0 = \Delta x$
EB= $f(x_1)$

EC=EB−BC $= f(x_1) - f(x_0)$

Now,

Area of a Rectangle, ABCD = $f(x_0)\Delta x$ (1.2)

Area of a Rectangle, ABCD =$0 \times \Delta x$, Since $f(x_0) = 0$

∴ Area of a Rectangle, ABCD = 0

And,

Area of Triangle EDC= $\frac{1}{2}$ CD×EC

Area of Triangle EDC= $\frac{1}{2}\left(\Delta x \times \left(f(x_1) - f(x_0)\right) \right)$

Area of Triangle EDC= $\frac{1}{2} f(x_1)\Delta x - \frac{1}{2} f(x_0)\Delta x$ (1.3)

Thus,

Total Area of a closed curve ABED = Area of a Rectangle, ABCD + Area of Triangle EDC

Total Area of a closed curve ABED = $f(x_0)\Delta x + \frac{1}{2} f(x_1)\Delta x - \frac{1}{2} f(x_0)\Delta x$ from (1.2) and (1.3)

Total Area of a closed curve ABED = $\frac{1}{2} f(x_0)\Delta x + \frac{1}{2} f(x_1)\Delta x$

∴ Total Area of a closed curve ABED = $\frac{1}{2}\{f(x_0)\Delta x + f(x_1)\}\Delta x$ (1.4)

Step.2:- Finding the area of a closed curve at (x_1, x_2)

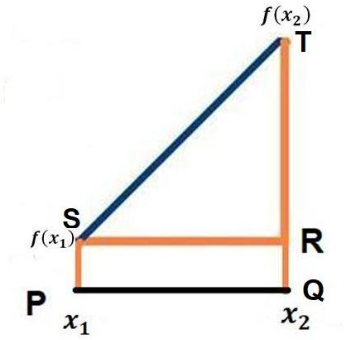

Fig.4

From Fig.4, we have

PS = QR = $f(x_1)$

PQ=RS= $x_2 - x_1 = \Delta x$

QT= $f(x_2)$

RT =QT−QR But QR=PS

RT =QT− PS

∴ RT =$f(x_2) - f(x_1)$

Now,

Area of Rectangle PQRS =PQ × PS

Area of Rectangle PQRS = $f(x_1) \times \Delta x$

∴ Area of Rectangle PQRS = $f(x_1)\Delta x$ (1.5)

And,

Area of a Triangle TSR =$\frac{1}{2}$ RS × RT

Area of a Triangle TSR =$\frac{1}{2}\{\Delta x \times (f(x_2) - f(x_1))\}$

∴ Area of a Triangle TSR =$\frac{1}{2} f(x_2)\Delta x - \frac{1}{2} f(x_1)\Delta x$ (1.6)

Thus,

Total Area of a closed curve PQTS = Area of Rectangle PQRS + Area of a Triangle TSR

Total Area of a closed curve PQTS = $f(x_1)\Delta x + \frac{1}{2} f(x_2)\Delta x - \frac{1}{2} f(x_1)\Delta x$

Total Area of a closed curve PQTS = $\frac{1}{2}f(x_1)\,\Delta x + \frac{1}{2}f(x_2)\Delta x$

∴ Total Area of a closed curve PQTS = $\frac{1}{2}\{f(x_1)\Delta x + f(x_2)\}\Delta x$ (1.7)

Step.3:- Finding the area of a closed curve at (x_{i-1}, x_i)

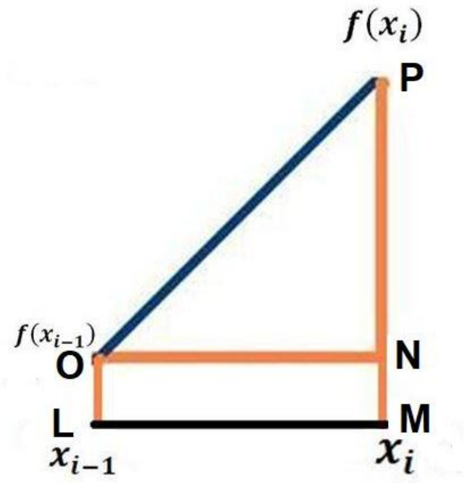

Fig.5

From Fig.5, we have
OL=MN= $f(x_{i-1})$
LM=ON= $x_i - x_{i-1} = \Delta x$
PM= $f(x_i)$
PN=PM−MN
∴ PN = $f(x_i) - f(x_{i-1})$
Now,
Area of Rectangle LMNO =OL × LM
Area of Rectangle LMNO = $f(x_{i-1})\times\Delta x$
∴ Area of Rectangle LMNO = $f(x_{i-1})\Delta x$ (1.8)

And
Area of Triangle PON = $\frac{1}{2}$ ON × PN
Area of Triangle PON = $\frac{1}{2}\Delta x \times\{f(x_i) - f(x_{i-1})\}$
∴ Area of Triangle PON = $\frac{1}{2}f(x_i)\Delta x - \frac{1}{2}f(x_{i-1})\Delta x$ (1.9)

Thus,
Total Area of a closed curve LMPO = Area of Rectangle LMNO + Area of Triangle PON
Total Area of a closed curve LMPO = $f(x_{i-1})\Delta x + \frac{1}{2}f(x_i)\Delta x - \frac{1}{2}f(x_{i-1})\Delta x$
Total Area of a closed curve LMPO = $\frac{1}{2}f(x_{i-1})\Delta x + \frac{1}{2}f(x_i)\Delta x$
∴ Total Area of a closed curve LMPO = $\frac{1}{2}\{f(x_{i-1}) + f(x_i)\}\,\Delta x$ (2.0)

Step.4:- Finding the total area under the closed curve of function $f(x)$ on interval $[a, b]$
Total area under a curve = Total Area of a closed curve ABED + Total Area of a closed curve PSTQ + …………………… + Total Area of a closed curve POLM

Total area under a curve = $\frac{1}{2}\{f(x_0)\Delta x + f(x_1)\}\Delta x + \frac{1}{2}\{f(x_1)\Delta x + f(x_2)\}\Delta x \ldots\ldots\ldots + \frac{1}{2}\{f(x_{i-1}) + f(x_i)\}$ Δx
(Using (1.4) , (1.7) , (2.0))

∴ Total area under a curve = $\frac{1}{2}\sum_{i=1}^{n}\{\{f(x_{i-1}) + f(x_i)\}\,\Delta x\}$

Hence,
$\int_a^b f(x)dx = \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n}\{\{f(x_{i-1}) + f(x_i)\}\,\Delta x\}$
Where n is the number of sub-intervals of equal length.

………
………....

Summary:

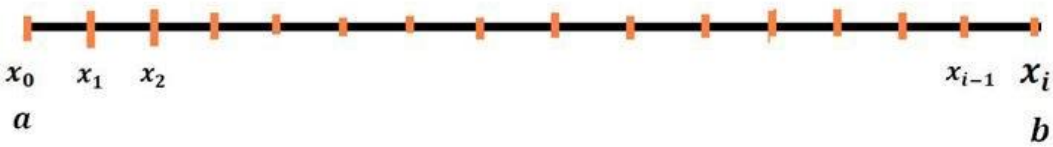

Fig.6

From Fig.6, we clearly see that
Total length of interval from x_0 to $x_{i-1} = x_{i-1} - x_0$
$=> \quad (i-1)\Delta x = x_{i-1} - x_0$
Where, $(i-1) = numbers\ of\ subintervals\ from\ x_0\ to\ x_{i-1}$
$=> \quad (i-1)\Delta x = x_{i-1} - a$ Since, $x_0 = a$
$\therefore \quad x_{i-1} = a + (i-1)\Delta x$

Total length of interval from x_0 to $x_i = x_i - x_0$
$=> \quad i\Delta x = x_i - x_0$ Where, $i = numbers\ of\ subintervals\ from\ x_0\ to\ x_i$
$=> \quad i\Delta x = x_{i-1} - a$ Since, $x_0 = a$
$\therefore \quad x_i = a + i\Delta x$

………

Worked Example:

Evaluate: $\int_a^b x\,dx$
Solution: Let $f(x) = x$

Now,
$$\int_a^b f(x)dx = \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n}\{\{f(x_{i-1}) + f(x_i)\}\,\Delta x\}$$

$$\int_a^b f(x)dx = \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x + \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$\int_a^b f(x)dx = \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n} f(a+(i-1)\Delta x)\,\Delta x + \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\,\Delta x$$

$$\int_a^b f(x)dx = \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n} (a+(i-1)\Delta x)\,\Delta x + \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n} (a+i\Delta x)\,\Delta x$$

$$\int_a^b f(x)dx = \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n} (a+(i-1)\Delta x)\,\Delta x + \frac{1}{2}\lim_{n\to\infty} \sum_{i=1}^{n} (a+i\Delta x)\,\Delta x$$

Here,
$$\lim_{n\to\infty} \sum_{i=1}^{n} (a+(i-1)\Delta x)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (a\Delta x + (i-1)\Delta x^2)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} (a+(i-1)\Delta x)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (a\Delta x + (i-1)\Delta x^2\ )$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} (a+(i-1)\Delta x)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} a\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} (i-1)\Delta x^2$$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=\lim_{n\to\infty}na\Delta x+\lim_{n\to\infty}\frac{n(n-1)}{2}\Delta x^2$ Since $\Delta x=constant$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=a\lim_{n\to\infty}n\Delta x+\lim_{n\to\infty}\frac{n(n-1)}{2}\Delta x^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=a\lim_{n\to\infty}(b-a)+\lim_{n\to\infty}\frac{n(n-1)}{2}\left(\frac{b-a}{n}\right)^2$ Since $\Delta x=\frac{b-a}{n}$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=a(b-a)+\lim_{n\to\infty}\frac{(n^2-n)}{2}\frac{(b-a)^2}{n^2}$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=a(b-a)+\lim_{n\to\infty}\frac{(n^2-n)}{2n^2}(b^2-2ab+a^2)$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=a(b-a)+\frac{1}{2}\lim_{n\to\infty}\frac{(n^2-n)}{n^2}(b^2-2ab+a^2)$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=a(b-a)+\frac{1}{2}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)(b^2-2ab+a^2)$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=a(b-a)+\frac{1}{2}\left(1-\frac{1}{\infty}\right)(b^2-2ab+a^2)$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=a(b-a)+\frac{1}{2}(1-0)(b^2-2ab+a^2)$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=a(b-a)+\frac{1}{2}(b^2-2ab+a^2)$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=ab-a^2+\frac{b^2}{2}-ab+\frac{a^2}{2}$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=\frac{b^2}{2}+\frac{a^2}{2}-a^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=\frac{b^2}{2}-\frac{a^2}{2}$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x=\frac{1}{2}(b^2-a^2)$

And

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}(a\Delta x+i\Delta x^2)$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}a\Delta x+\lim_{n\to\infty}\sum_{i=1}^{n}i\Delta x^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=\lim_{n\to\infty}na\Delta x+\lim_{n\to\infty}\frac{n(n+1)}{2}\Delta x^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=a\lim_{n\to\infty}n\Delta x+\lim_{n\to\infty}\frac{n^2+n}{2}\Delta x^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=a\lim_{n\to\infty}(b-a)+\lim_{n\to\infty}\frac{n^2+n}{2}\left(\frac{b-a}{n}\right)^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=a\,(b-a)+\lim_{n\to\infty}\frac{n^2+n}{2n^2}(b-a)^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=ab-a^2+\frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)(b-a)^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=ab-a^2+\frac{1}{2}\left(1+\frac{1}{\infty}\right)(b-a)^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=ab-a^2+\frac{1}{2}(1+0)(b-a)^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=ab-a^2+\frac{1}{2}(b-a)^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=ab-a^2+\frac{1}{2}(b^2-2ab+a^2)$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=ab-a^2+\frac{b^2}{2}-ab+\frac{a^2}{2}$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=\frac{b^2}{2}+\frac{a^2}{2}-a^2$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=\frac{b^2}{2}-\frac{a^2}{2}$

$\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta x=\frac{1}{2}(b^2-a^2)$

Thus

$\int_a^b f(x)dx=\frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n}(a+(i-1)\Delta x)\,\Delta x+\frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n}(a+i\Delta x)\,\Delta$

$\int_a^b f(x)dx=\frac{1}{2}\times\frac{1}{2}(b^2-a^2)+\frac{1}{2}\times\frac{1}{2}(b^2-a^2)$

$\int_a^b f(x)dx=\frac{1}{4}(b^2-a^2)+\frac{1}{4}(b^2-a^2)$

$\int_a^b f(x)dx=\frac{2}{4}(b^2-a^2)$

$\therefore\quad \int_a^b f(x)dx=\frac{1}{2}(b^2-a^2)$

Conclusion:-

In this manuscript, we have obtained interesting result on solving Definite Integral by summation .We have shown that the end result is very useful to solve the system of Definite Integral.

Data Availability:-

No data were used to support this study.

Conflicts of Interest:-

The author declares that there is no conflict of interest.

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