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# Characterizations Of $S_\xi$ −Super Strictly Singular Operator

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***Abstract**:*
The notion introduced in [1] for $S_\xi -$ Strictly singular operator in Banach spaces. We extend this notion for super strict singular operator in Locally convex spaces (Lcs). Some properties and characterization for these operators are derived in hereditarily indecomposable(HI) complex Banach spaces.
**Keywords:**Strictly singular operator, Super strictly singular operator, Hereditarily indecomposable(HI).
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## I. Introduction

The concept of a Strictly singular operator was introduced by T.Kato[2] in conection with perturbation theory of Fredholm operators. The concept of a strictly singular operator was extended by many mathematician and researchers. The spaces $\mathcal{X}$ and $\mathcal{Y}$ be denote Locally convex spaces and$A : \mathcal{X} \to \mathcal{Y}$ be denote a bounded linear operator from $D(A)$ to $R(A)$. A linear operator $A$ is called Strictly singular if it does not have bounded inverse on any infinite dimensional closed subspace of $\mathcal{X}$. An operator $A$ is called pre-compact if $A(b_x)$is totally bounded in , where $b_x$ denote open unit ball in $\mathcal{X}$. If closure of $A(b_x)$ is compact ( $\overline{A(b_x)}$ is compact ) then $A$ is compact operators. $SS(\mathcal{X},\mathcal{Y})$ denote the collection of Strictly singular operators (SS) which form norm closed ideal in $B(\mathcal{X},\mathcal{Y})$. Strictly singular operator has close connection with compact operator. T.Kato proved that every strictly singular operator mapping a Hilbert space $\mathcal{X}$ in to Hilbert space$\mathcal{Y}$ is compact.

## II. Definitions

**Definition 2.1** A continuous linear operator $A : \mathcal{X} \to \mathcal{Y}$ is said to be Super strictly singular operator (SSS) if there exist a zero neighbourhood ( absolutely convex and closed ) $V$ in $\mathcal{X}$ with has property that for every infinite dimensional subspace $X_1 \subset \mathcal{X}$ such that $X_1 \cap N(A) = \{0\}$ and for every zero neighbourhood $W$ in $\mathcal{Y}$ there exist an infinite dimensional subspace $X_2 \subset X_1$ such that $X_2 \not\subset V$ and $A(V \cap X_2) \subset W$.
Let $C_u$ be the gauge of zero neighbourhood in $\mathcal{X}$. Then $\mathcal{X}_u$ is the quotient space $\mathcal{X} / C_u^{-1}(0)$ equipped with norm

$$\|\{x\}\| = C_u(x) \qquad ( x \in \{x\},\ \{x\} \in \mathcal{X}_u ) \qquad (1)$$

$\overline{\mathcal{X}_u}$ denote the complexion of $\mathcal{X}_u$.
A Locally convex space $\mathcal{X}$ is said to be nuclear space (generalised Hilbert space) if there exist a base $B$ of zero neighbourhood $V$ such that for all $V \in B$ the spaces $\overline{\mathcal{X}_u}$ are Hilbert.
Lemma 2.2 Let $\mathcal{X}$ and $\mathcal{Y}$ be nuclear(Hilbert) spaces then every Strictly singular operator $A : \mathcal{X} \to \mathcal{Y}$ is compact operator.
Proof: Due to T.Kato [2].

Schreier families 2.3 We memorise the definition of Schreier families $(S_\xi)(1 \le \xi < w_1)$ introduced by Alspach and Argyros [9]. Let $\mathcal{A}$ and $\mathcal{B}$ be two finite subset of natural number $\mathbb{N}$. By $\mathcal{A} \le \mathcal{B}$ means if $max\mathcal{A} \le min\mathcal{B}$.For$n \in \mathbb{N}$ and $\mathcal{A} \subset \mathbb{N}$, $n \le \mathcal{A}$ if $(n) \le min\mathcal{A}$. We suppose that , $\emptyset < A$ and $\mathcal{A} < \emptyset$ for all non empty finite subset $\mathcal{A}$ of $\mathbb{N}$ . For any ordinal number $0 \le \xi < w_1$, by transitive recursive process, the schreier family $S_\xi$ which is collection of finite subset of $\mathbb{N}$ , defined by following way ,let$S_o = \{(n)/n \in \mathbb{N}\} \cup (\emptyset)$ .
Assume $\xi$ be a successor ordinal and $S_\varsigma$ is defined for $\varsigma + 1 = \xi$, let,

$$S_\xi = \left(\bigcup_{i=1}^{n} E_i | n \ge 1, n \le E_1 < E_2 < \cdots < E_n \ and \ E_i \in S_\varsigma \ for 1 \le i < n\right) \cup (\emptyset) \qquad (2)$$

For every $n \in \mathbb{N}$ and $\xi < w_1$,and let $S_\xi\{[n,\infty)\} = \left(E \in S_\xi | n \leq E\right)$. If $\xi < w_1$ is limit ordinal and $S_\varsigma$ be defined for all $\varsigma < \xi$ by fix an increasing sequence$(\xi_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \xi_n = \xi$ and$S_\xi = \bigcup_{n=1}^{\infty} S_{\xi_n}\{[n,\infty)\}$,i.e

$$S_\xi = \left\{E | n \leq E \text{ and } E \in S_{\xi_n} \text{ for some } n \in \mathbb{N}\right\}$$

b

Let $1 \leq \xi < w_1$, and $A : \mathcal{X} \to \mathcal{Y}$ be continuous linear operator,then $A$ is $S_\xi$ –strictly singular operator if every basic sequence $(x_n)$ there exist a set $E \in S_\xi$ and a vector $z \in [x_i]_{i\in E} / (0)$, there is no positive number $c$ such that$\|z\| \leq c\|Az\|$( where$[x_i]_{i\in E}$ be closed linear span of $(x_i)_{i\in E}$ ).

An operator $A \in L(\mathcal{X},\mathcal{Y})$ is $S_\xi$–Super strictly singular operator if every normalize basic sequence $(x_n)$ there exist a set $E \in S_\xi$ and a given vector $z \in [x_i]_{i\in E}/(0)$, there exist other vector $y \in [x_i]_{i\in E}/(0)$and no positive scalar $c$ such that $Az = Ay$ and$\|y\| \leq c\|Ay\|$.

Let $SS_\xi(\mathcal{X},\mathcal{Y})$, $SSS_\xi(\mathcal{X},\mathcal{Y})$, $SS(\mathcal{X},\mathcal{Y})$ are denote the$S_\xi$- Strictly singular operators, $S_\xi$- Super strictly singular operators and Strictly singular operators respectively. These sets are norm closed two sided operator ideals on the ring of the set of all bounded linear operators. Any unexplained notions and definitions can be found in [5,6].

## III. Characterizations, Properties and Results

A knowncharacterization of $S_\xi$ –Strictly singular operator is given in theorem 3.1. A similar new characterization of$S_\xi$ −Super strictly singular operator is proved and some new result is derived.

Theorem 3.1 Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces (HI) and $1 \leq \xi,\ \varsigma < w_1$ then

(i) $SS_\xi(\mathcal{X},\mathcal{Y}) \subseteq SS(\mathcal{X},\mathcal{Y})$.

(ii) If $1 \leq \xi < \varsigma < w_1$ then $SS_\xi(\mathcal{X},\mathcal{Y}) \subseteq SS_\varsigma(\mathcal{X},\mathcal{Y})$.

(iii)$SS_\xi(\mathcal{X})$ is norm closed.

(iv) If $A \in SS_\xi(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{X})$ then $AoB$ and $AoB$ belong to $SS_\xi(\mathcal{X})$.

(v) If $A \in SS_\xi(\mathcal{X})$ and $B \in SS_\varsigma(\mathcal{X})$ then $(A + B) \in SS_{\xi+\tau}(\mathcal{X})$. In particular if $A, B \in SS_\xi(\mathcal{X})$ then $(A + B) \in SS_{\xi 2}(\mathcal{X})$.

Proof: The following above result is automatic continuity due to Androulakis, Dodos, Sirotkin, and Trositsky [1].

Theorem 3.2 Let $\mathcal{X}$ and $\mathcal{Y}$ be two infinite dimensional complex Banach spaces (HI) and $1 \leq \xi,\ \varsigma < w_1$ then

(i) $SS_\xi(\mathcal{X},\mathcal{Y}) \subseteq SSS_\xi(\mathcal{X},\mathcal{Y}) \subseteq SS(\mathcal{X},\mathcal{Y})$.

(ii) If $1 \leq \xi < \varsigma < w_1$ then $SSS_\xi(\mathcal{X},\mathcal{Y}) \subseteq SSS_\varsigma(\mathcal{X},\mathcal{Y})$.

(iii) $SSS_\xi(\mathcal{X})$ is norm closed.

(iv) If $A \in SSS_\xi(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{X})$ then $AoB$ and $AoB$ belong to $SSS_\xi(\mathcal{X})$.

(v) If $A \in SSS_\xi(\mathcal{X})$ and $B \in SSS_\varsigma(\mathcal{X})$ then $(A + B) \in SSS_{\xi+\tau}(\mathcal{X})$. In particular if $A, B \in SSS_\xi(\mathcal{X})$ then $(A + B) \in SSS_{\xi 2}(\mathcal{X})$.

Proof: (i) Let $A \in SS_\xi(\mathcal{X},\mathcal{Y})$ and $X_0$ be any subspace of $\mathcal{X}$ such that every $x + N(A) \in X_o$, then $\|x + \mathcal{NA} \leq \varepsilon \mathcal{A}(x) + \mathcal{NA}$ for some positive number $\varepsilon$.

Then, for any $\varepsilon' > \varepsilon$ , a given$x \in \tau_A^{-1}(X_o)$ ( where $\tau_A$ is quotient map ), there is an element $y \in \tau_A^{-1}(X_o)$ such that $Ax = Ay$ and $x + N(A) = y + N(A)$ ,now we have

$$\|y\| \leq \left(\frac{\varepsilon'}{\varepsilon}\right)\|x + N\| \leq \varepsilon'\left\|A\left(x + N(A)\right)\right\| = \varepsilon'\left\|A\left(y + N(A)\right)\right\| = \varepsilon\|A(y)\| \qquad (3)$$

This implies $A \in SSS_\xi(\mathcal{X},\mathcal{Y})$, and $SS_\xi(\mathcal{X},\mathcal{Y}) \subseteq SS(\mathcal{X},\mathcal{Y})$ and $SSS_\xi(\mathcal{X},\mathcal{Y}) \subseteq SS(\mathcal{X},\mathcal{Y})$ is obvious.

(ii) since , then there exist $n \in \mathbb{N}$ such that if $n \in n_1 \in S_\xi$ then $n_1 \in S_\varsigma$ i.e means $S_\xi \cap [(n, n+1, \ldots)]^{<\infty} \subseteq S_\varsigma$ , now if $A \in SSS_\xi(\mathcal{X},\mathcal{Y})$ and $\varepsilon$ be a positive number and let $(x_n)$ be a basic sequence(normalize) in $\mathcal{X}$ then consider a sequence$(y_n)$ where$y_i = x_{n+i}$. There exist $n_1 \in S_\xi$ and $z \in [y_i]_{i\in n_1} / (0)$ such that $Ax = Az$ and $\|Az\| \leq \varepsilon\|z\|$. Since $n_1 \in S_\xi$ and $n_1 \subseteq (n, n+1, \ldots)$ implies $n_1 \in S_\varsigma$ .

(iii) Let $(x_n)$ be a basic sequence(semi-normalize) in $\mathcal{X}$ and $(A_n) \in SSS_\xi\ (\mathcal{X})A \in \mathcal{L}(\mathcal{X})$ with $\lim_{n\to\infty} A_n = A$ , given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\left\|A_{n_0} - A\right\| \leq \frac{\epsilon}{2}$ . since $A_{n_0} \in SSS_\xi(\mathcal{X})$ , there exist$n_1 \in S_\xi$ and $x \in [x_i]_{i\in n_1} / (0)$ such that $\left\|A_{n_0}x\right\| \leq \frac{\epsilon}{2}\|x\|$ . thus, $\|Ax\| \leq \left\|\left(A_{n_0} - A\right)x\right\| + \left\|A_{n_0}x\right\| \leq \frac{\epsilon}{2}\|x\| + \frac{\epsilon}{2}\|x\| = \epsilon\|x\|$ . (4)

(iv) Let $N$ be infinite dimensional subspace of $\mathcal{X}$ and $A_N$and $B_N$ be restriction to $N$. Let $A \in SSS_\xi(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{X})$, we prove $AoB \in SSS_\xi(\mathcal{X})$ and $BoA \in SSS_\xi(\mathcal{X})$. let $(x_n)$ be basic sequence and $\varepsilon$ be positive number ,then there exist $n_1 \in S_\xi$ and a given vector $x \in [x_i]_{i\in n_1} / (0)$ there is a vector $y \in [x_i]_{i\in n_1} / (0)$ such that $(AoB)_N x = (AoB)_N y$ and $\|A_N x\| \leq \frac{\varepsilon}{\|B_N\|}\|x\|$, thus,$\|(AoB)_N x\| \leq \|A_N x\|\|B_N\| \leq \varepsilon\|x\|$, this implies, $AoB \in SSS_\xi(\mathcal{X})$.Now , let $N$ be infinite dimensional subspace of $\mathcal{X}$ and $A$ is $S_\xi$- super strictly singular operator. let $\varepsilon$ be positive number, $J_0 \in n_1 \in S_\xi$, we find $i_0 \in I$, such that $\gamma_{ij_0}(A_N) \le \varepsilon$ for each $i \ge i_0$ and $(BoA)_N x = (BoA)_N x$, for some $x, y \in [x_n]_{n \in J_0}$. Thus we have, $\gamma_{ij}((BoA)_N) \le \varepsilon$, for each $i \ge i_0$ . ( where $\gamma_{ij}(A_N) = \sup\{q_j Ax/p_i x \le 1, x \in N\}\, q_j,\ p_i$ increasing semi norm). this implies $BoA$ is $S_\xi$- super strictly singular.

(v) let $A \in SSS_\xi(\mathcal{X})$ and $B \in SSS_\varsigma(\mathcal{X})$. let $(x_n)_{n\in\mathbb{N}}$ be basic sequence(normalize) in $\mathcal{X}$. then we find $n_1 < n_2 < \cdots \in S_\xi$ and vector $w_j \in [x_{ni}]_{i \in n_j}$ with $\|w_j\| = 1$ such that $\|Aw_j\| < \frac{\varepsilon}{2^c}$ where $c$ is a basic constant of $(x_n)$. Since $w_j$ is also a basic sequence, we find $m \in S_\varsigma$ and $y \in [w_j]_{j\in m}/(0)$ such that $\|By\| \le \frac{\varepsilon}{2}\|y\|$, suppose $m = (J_1 J_2 \dots J_m)$ and $y = \sum_{i=1}^{m} a_i w_{J_i}$ then $y = \sum_{i\in N} b_i x_{n_i}$ for some $N \in S_\xi[S_\xi]$ such that $(A+B)x = (A+B)y$ for some $y \in [w_j]_{j\in m}/(0)$. We choose a subsequence $(n_i)$ of natural number $\mathbb{N}$, such that $y \in [(x_i)_{i\in K}]$ for some $K \in S_{\xi+\varsigma}$. We have $|a_i| < 2c\|y\|$, it implies that ,

$$\|(A+B)y\| \le \sum_{i=1}^{m}|a_i|\|(A+B)w_j\| \le 2c.\frac{\varepsilon}{4c}\|y\| = \frac{\varepsilon}{2}\|y\| \qquad (5)$$

And hence, $\|(A+B)y\| < \varepsilon\|y\|$, $\Rightarrow (A+B) \in SSS_{\xi+\varsigma}$.

Lemma 3.3 Let $\mathcal{X}$ and $\mathcal{Y}$ be separableBanach spaces and $A \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ then

(i) $\mathcal{X}$ is( H.I) iff $\mathcal{X}$ is $HI_\xi$ for some $\xi < w_1$.

(ii) $A$ is strictly singular operator iff $A$ is $S_\xi$- strictly singular for some $\xi < w_1$ .

Proof: it proved due to [1].

Every bounded operator $A$ on a infinite dimensional Banach space over complex number $\mathbb{C}$ has a form $(A - \lambda I), \lambda \in \mathbb{C}$, which is infinitely singular. Using this fact, Gower and Maurey [3] proved every operator $A$ on a complex Banach space (HI) is of the form $(\lambda I + B)$ where $\lambda \in \mathbb{C}$ and $B$ is strictly singular operator.

Lemma 3.4 [1] If $1 \le \xi < w_1$and $\mathcal{X}$ is a complex $HI_\xi$space then every $A \in \mathcal{L}(\mathcal{X})$ is expressed in the form $\lambda I + B$, where $\lambda \in \mathbb{C}$ and $\in SS_\xi(\mathcal{X})$ . we extend this result for $S_\xi$- super strictly singular operator.

Theorem 3.6 Let $A \in \mathcal{L}(\mathcal{X})$, where $\mathcal{X}$ is a complex $HI_\xi$Banach space and if $1 \le \xi < w_1$, then $A$ is written in the form $A = \lambda I + B$, where $\lambda \in \mathbb{C}$ and $B \in SSS_\xi(\mathcal{X})$. The proof of this result based on lemma3.7, first we define lemma.

Lemma 3.7 Let $\mathcal{X}$be infinite dimensional complex $(HI)$space and $\mathcal{Y}$ be any Banach space, then $A \in SS(\mathcal{X},\mathcal{Y})$ if and only if for every $\epsilon > 0$,there is an infinite dimensional subspace $X_1$ of $\mathcal{X}$ such that $\|A|_{X_1}\| \le \epsilon$. This result follow from a well known property of Strictly singular operator [4, Page76].

Proof 3.6 : Since for every operator $B$, no restriction of $B$ to a subspace of infinite codimension is an isomorphism, i.e means there exist $\lambda \in \mathbb{C}$ such that $B = A - \lambda I$ is infinitely singular. Let $(l_n)$ be basic sequence (normalize) in $\mathcal{X}$ and $\epsilon$ be a number greater than zero, then there exist a subspace$X_1$ of $\mathcal{X}$ ,with $dim(X_1) = \infty$ such that $\|B|_{X_1}\| < \frac{\epsilon}{2}$ and $\mathcal{X}$ is $HI_\xi$ space ,then there exist $N \in S_\xi$, a unit vector $v \in X_1$, and a vector $u \in (l_n)_{n\in N}$ with $B(u) = B(v)$ such that $\|u - v\| \le \frac{\epsilon}{2\|B\|+\epsilon}$ it is easy to verify that $\left\|\frac{u}{\|u\|} - u\right\| < \frac{\epsilon}{2\|B\|}$ where$\frac{u}{\|u\|} \in (l_n)_{n\in N}$, now then,

$$\left\|B\frac{u}{\|u\|}\right\| \le \|B(v-u)\| + \|Bv\| + \left\|B\left(u - \frac{u}{\|u\|}\right)\right\|$$

$$\le \frac{\epsilon}{2} + \|B\|\frac{\epsilon}{2\|u\|} + \|B\|\left\|u - \frac{u}{\|u\|}\right\| < \epsilon \qquad (6)$$

This complete the proof.

Theorem 3.7 Let $1 \le \xi < w_1$ and $\mathcal{X}$ is a complex $HI_\xi$ Banach space ,then $SSS_\xi(\mathcal{X},\mathcal{Y}) = SS(\mathcal{X},\mathcal{Y})$,for every Banach space $\mathcal{Y}$.

Proof: Let $(l_n)$ be normalized basic sequence in $\mathcal{X}$,let $A \in SS(\mathcal{X},\mathcal{Y})$ ,thus $(Al_n)$ is Cauchy sequence in $\mathcal{Y}$, since $\mathcal{Y}$ is Banach space , $(Al_n)$ converge to some $m \in \mathcal{Y}$. We choose $\epsilon, \delta > 0$ such that $\frac{\delta(1+\|A\|)}{1-\delta} < \epsilon$. Since $\mathcal{X}$ has a infinite dimensional subspace $X_1$ of $\mathcal{X}$ such that $\|A|_{X_1}\| < \delta$.and there exist $N \in S_\xi$ (because $\mathcal{X}$ is $HI$ space) and a vector $l \in (l_n)_{n\in N}$ and $x_1 \in X_1$ with $\|x_1\| = 1$ and $\|l - x_1\| < \delta$ such that $Al = Ax_1$ for some $x_1 \in X_1$. This implies $\|l\| > 1 - \delta$ and

$$\left\|\lim_{n\to\infty} Al_n\right\| = \|Al\| \le \|Ax_1\| + \|A\|\|l - x_1\|$$

$$< \delta(1 + \|A\|) < \epsilon\|l\|. \qquad (7)$$

And since $SSS_\xi(\mathcal{X},\mathcal{Y}) \subseteq SS(\mathcal{X},\mathcal{Y})$ and hence $SSS_\xi(\mathcal{X},\mathcal{Y}) = SS(\mathcal{X},\mathcal{Y})$.

Theorem 3.8 Let $\mathcal{X}$ be infinite dimensional complex $HI$ Banach space and $\mathcal{Y}$ be a Banach space and $X_1 \subseteq N(A)$be a closed subspace of $\mathcal{X}$ .if then $B: \mathcal{X}/_{X_1} \to \mathcal{Y}$ is also $S_\xi$-Super strictly singular operator.

Proof: Since $A: \mathcal{X} \to \mathcal{Y}$ be $S_\xi$-Super strictly singular operator, let $(l_n)$ be normalized basic sequence in $\mathcal{X}$,then $(l_n + X_1)$ be bounded sequence in $\mathcal{X}/_{X_1}$,there is positive number $\epsilon > 0$ such that$\|l_n + X_1\| =$

$inf\{\|l_n + u\| \,/\, u \in X_1\} \leq \epsilon$ for each $n \in N \in S_\xi$, then it follows that there exist a sequence $(u_n)$ in $X_1$ with unit norm such that $\|l_n + u_n\| \leq \epsilon$ for each $n \in N \in S_\xi$. This implies, the sequence $A(l_n + u_n)$ has a convergent subsequence. Since $A(l_n + u_n) = B(l_n + X_1)$ for each $n \in N \in S_\xi$. And hence $B$ is $S_\xi$-Super strictly singular operator.

Theorem 3.9Let $A_n, n \in N \in S_\xi$, be sequence of$S_\xi$-Super strictly singular operator from $\mathcal{X}$ to $\mathcal{Y}$ with $D(A_i) = D(A_j)$ for all $i, j \in N$ such that $\|A_n - A\| \to 0$, for some $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with same domain. Then $A$ is $S_\xi$-Super strictly singular operator.

Proof : let $\epsilon$ be a positive number and $X_1$ be closed subspace of domain of $A_n$ such that $\|Ax\| \geq \epsilon\|x\|$ for every $x \in X_1$. For this, we prove $dim(X_1)$ is finite. Let $n \in N \in S_\xi$ be so large that $\|A_n - A\| < \epsilon$ , then ,$\|A_n x\| \geq \|Ax\| - \|(A_n - A)x\| \geq \in \|x\|$ for every $x \in X_1$, since $A_n$ is $S_\xi$-Super strictly singular operator, this implies $dim(X_1) < \infty$.

Conclusions: This paper has been to try to establish some result on $S_\xi$-Super strictly singular operator which are done with help of theorem on strictly singular operator and $S_\xi$- strictly singular operator. we utilizes the results from previous studies to satisfy the present results.

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