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Even Sum Labeling Of Subdivision, Super Subdivision And Arbitrary Super Subdivision Of Graphs

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Abstract:
This Study Focuses On Even Sum Property Of Graphs In The Context Of Subdivision, Super Subdivision And Arbitrary Super Subdivision Of Graphs. We Have Established That A Subdivision Of A Star Graph $K_{1,n}$ And A Complete Bipartite Graph $K_{2,n}$ Are Even Sum Graphs. We Have Also Proved That A Super Subdivision Of A Cycle C_{4n} When Each Edge Is Replaced By $K_{2,t}$ And An Arbitrary Super Subdivision Of Path P_n When Each Edge Of The Path Is Replaced By K_{2,m_i} With Arbitrary m_i Are Even Sum Graphs.

***Key Word**:Even Sum Labeling, Even Sum Graph, Subdivision, Super Subdivision, Arbitrary Super Subdivision.*

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I. Introduction

A word 'graph'is used to mean a finite, simple and undirected graph, the number of vertices in a graph is denoted by p or $|V(G)|$and the number of edges in a graph is denoted by q or $|E(G)|$throughout this paper. Graph labeling was initiated by Rosa[1]and a detailed survey on graph labeling is updated every year by Gallian[2].

In 2013, the idea of odd sum labeling was set by Arockiaraj and Mahalakshmi[3]. Recently, Trivedi and Chaudhary[4] presented odd sum labeling of a complete bipartite graph and its splitting and subdivision. Monika and Murugan[5] have introduced odd-even sum labeling in 2017. Monika and Murugan[6] further concluded that the subdivision of a star graph, subdivision of a bistar graph and an H graph of a path P_n are odd-even sum graphs. The notion of even sum labeling was first introduced by Andharia and Kaneria[7], which was a motivation from odd sum labeling and odd-even sum labeling of graph. They concluded that a slanting ladder graph SL_n is an even sum graph for $1 < n < 9$. Later, Kaneria and Andharia[8] discussed even sum labeling of path, cycle, complete bipartite graph, grid graph and mirror graph. Kaneria and Andharia[9] have also proved that Jelly fish graph, splitting graph of $K_{1,n}$, $K_{2,n}$ and$K_{1,n,n}$ and degree splitting graph of $K_{1,n}$ are even sum graphs. Recently, Andharia and Kaneria[10] proved that the graphsJ_n, $B(3,n)$, TB_n, $P_m(+)\overline{K_n}$ and $(\overline{K_n} \cup P_3) + 2K_1$ are even sum graphs.

This paper deals with even sum labeling of subdivision of a star $K_{1,n}$ and a complete bipartite graph $K_{2,n}$, super subdivision of a cycle C_{4n} when each edge is replaced by $K_{2,t}$ and an arbitrary super subdivision of pathP_n when each edge of the path is replaced by K_{2,m_i} with arbitrary m_i.

Definition 1: A (p,q) graph $G = (V,E)$ is said to admit even sum labeling[8] if there exists an injective function $f: V(G) \to \{0, \pm 2, \pm 4, \dots, \pm 2q\}$ such that the induced mapping $f^*: E(G) \to \{2, 4, \dots, 2q\}$ defined by $f^*(uv) = f(u) + f(v), \forall uv \in E(G)$ is bijective. The function f is called an even sum labeling of$G$. The graph which admits even sum labeling is called even sum graph.

Definition 2: If each edge of a graph G is broken into two edges by exactly one vertex, then the resultant graph is said to be a subdivision of G and it is denoted by $S(G)$.

Definition 3: A super subdivision of a graph G[11], denoted by $SS(G)$is a graph obtained from G by replacing every edge xy of G with a complete bipartite graph$K_{2,t}$, for some t in such a way that the end vertices x, y of each edge are merged with the two vertices of 2-vertices part of $K_{2,t}$ after removing the edge xy from G.

Definition 4: A super subdivision of a graph G[11] is said to be an arbitrary super subdivision of a graph G if every edge of a graph G is replaced by an arbitrary $K_{2,m}$ where m varies arbitrarily. We shall denote it by $ASS(G)$.

II. Main Results

Theorem 1: Subdivision of a star graph $K_{1,n}$ is an even sum graph.

Proof: Consider a star graph $K_{1,n}$ with vertices $u, u_1, u_2, \ldots\ldots, u_n$ and edges uu_i; $i = 1, 2, \ldots\ldots, n$. Suppose for each $i = 1, 2, \ldots\ldots, n$, the edge uu_i is broken into two edges by a vertex v_i. It will divide each edge uu_i of $K_{1,n}$ into two edges uv_i and v_iu_i; $\forall\, i = 1, 2, \ldots\ldots, n$. The resultant graph G is a subdivision of a star graph i.e. $S(K_{1,n})$. The ordinary labeling of $S(K_{1,5})$ is shown in Figure 1.
Clearly, $V(G) = \{u, u_i,\ v_i \mid 1 \le i \le n\}$, $E(G) = \{uv_i,\ v_iu_i \mid 1 \le i \le n\}$ and hence $|V(G)| = p = 2n + 1$ and $|E(G)| = q = 2n$.

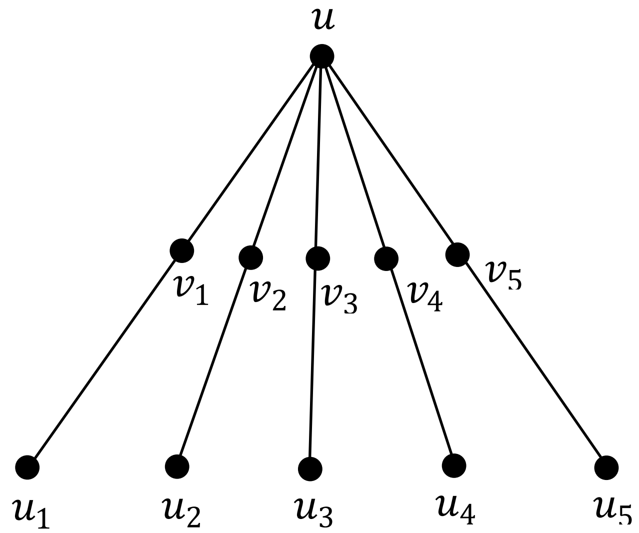

Figure – 1: Ordinary labeling of $S(K_{1,5})$

Now, define a function $f: V(G) \to \{0, \pm 2, \pm 4, \ldots\ldots, \pm 2q\}$ as

$$f(u) = -2,$$
$$f(u_1) = 0,$$
$$f(u_i) = 4(i - n - 1);\ \forall\, i = 2, 3, \ldots\ldots, n,$$
$$f(v_i) = 2(2n + 1 - i);\ \forall\, i = 1, 2, \ldots\ldots, n.$$

Then the induced edge labeling f^* for the graph $G = S(K_{1,n})$ is given by

$$f^*(xy) = f(x) + f(y);\ \ \forall\, xy \in E(G).$$
$$\therefore\ f^*(uv_i) = f(u) + f(v_i)$$
$$= 4n - 2i;\ \ \forall\, i = 1, 2, \ldots\ldots, n$$
$$= \{4n - 2, 4n - 4, \ldots\ldots, 2n + 2, 2n\},$$
$$f^*(v_1u_1) = f(v_1) + f(u_1)$$
$$= 4n,$$
$$f^*(v_iu_i) = f(v_i) + f(u_i)$$
$$= 2i - 2;\ \forall\, i = 2, 3, \ldots\ldots, n$$
$$= \{2, 4, 6, \ldots\ldots, 2n - 2\}.$$

Thus, $f^*(xy) \in \{2, 4, 6, \ldots\ldots 2n, \ldots\ldots, 4n\} = \{2, 4, 6, \ldots\ldots, 2q\}$; $\forall\, xy \in E(G)$.
So, $f^*\big(E(G)\big) = \{2, 4, 6, \ldots\ldots, 2q\}$.
Therefore, f is the even sum labeling of G, and hence, $S\big(K_{1,n}\big)$ is an even sum graph.

Illustration 1: Subdivision of $K_{1,5}$ with its even sum labeling is shown in Figure 2.

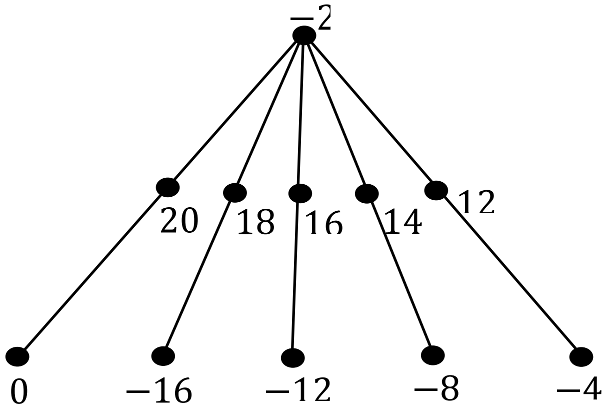

Figure – 2: Even sum labeling of $S(K_{1,5})$

Theorem 2: Subdivision of a complete bipartite graph $K_{2,n}$ is an even sum graph.
Proof: Let $V\left(K_{2,n}\right) = \{u, v, w_i \mid 1 \le i \le n\}$ and $E\left(K_{2,n}\right) = \{uw_i, vw_i \mid 1 \le i \le n\}$.

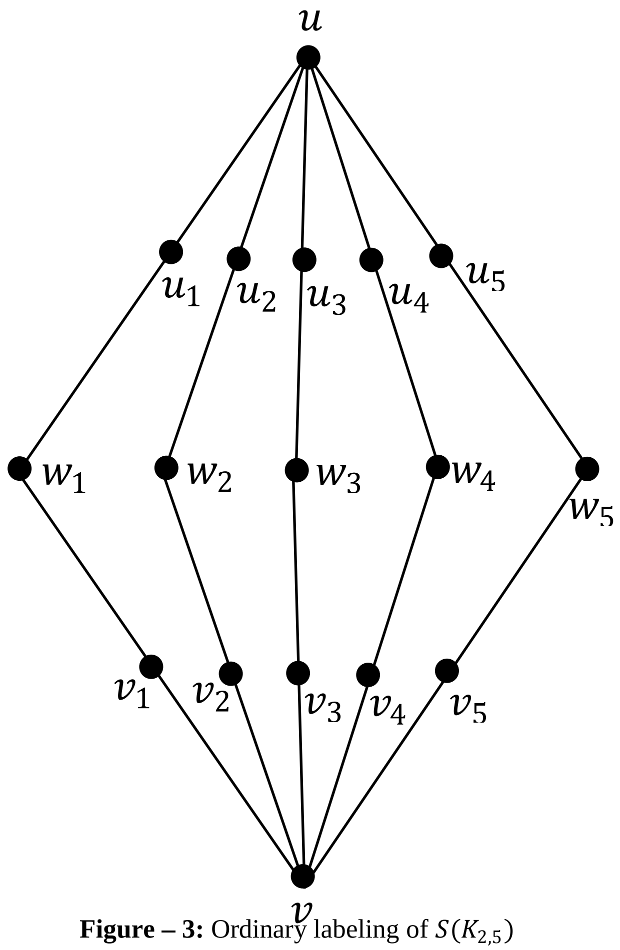

Figure – 3: Ordinary labeling of $S(K_{2,5})$

Let G be a graph obtained by breaking edges uw_i into two edges by a vertex u_i and breaking edges vw_i into two edges by a vertex v_i. Then G is a subdivision of $K_{2,n}$ i.e. $G = S(K_{2,n})$. Figure 3 show the subdivision of $K_{2,5}$ with its ordinary labeling.
Clearly, $V(G) = \{u, v, u_i, v_i w_i \mid 1 \le i \le n\}$ and $E(G) = \{uu_i, u_i w_i, vv_i, v_i w_i \mid 1 \le i \le n\}$.
Hence, $|V(G)| = p = 3n + 2$ and $|E(G)| = q = 4n$.
Define vertex labeling function $f: V(G) \to \{0, \pm 2, \pm 4, \dots \dots, \pm 2q\}$ as

$$f(u) = 0,$$

$$f(v) = -4n,$$
$$f(u_i) = 2(3n+i);\ \forall\, i = 1, 2, \ldots\ldots, n,$$
$$f(w_i) = 2(1-2i);\ \forall\, i = 1, 2, \ldots\ldots, n,$$
$$f(v_i) = 2(2n+i);\ \forall\, i = 1, 2, \ldots\ldots, n.$$

The induced edge labeling f^* for the graph G is given by

$f^*(xy) = f(x) + f(y);\ \forall\, xy \in E(G).$

$\therefore\ f^*(uu_i) = f(u) + f(u_i) = 6n + 2i;\ \forall\, i = 1, 2, \ldots\ldots, n$

$= \{6n+2, 6n+4, \ldots\ldots, 8n\},$

$f^*(u_i w_i) = f(u_i) + f(w_i) = 6n + 2 - 2i;\ \forall\, i = 1, 2, \ldots\ldots, n$

$= \{6n, 6n-2, \ldots\ldots, 4n+2\},$

$f^*(vv_i) = f(v) + f(v_i) = 2i;\ \forall\, i = 1, 2, \ldots\ldots, n$

$= \{2, 4, 6, \ldots\ldots, 2n\},$

$f^*(v_i w_i) = f(v_i) + f(w_i) = 4n + 2 - 2i;\ \forall\, i = 1, 2, \ldots\ldots, n$

$= \{4n, 4n-2, \ldots\ldots, 2n+2\}.$

Thus, $f^*(xy) \in \{2, 4, 6, \ldots\ldots, 8n\} = \{2, 4, 6, \ldots\ldots, 2q\};\ \forall\, xy \in E(G).$

So, $f^*\big(E(G)\big) = \{2, 4, 6, \ldots\ldots, 2q\}.$

Therefore, f is the even sum labeling of G, and hence, $S(K_{2,n})$ is an even sum graph.

Illustration 2: Subdivision of $K_{2,5}$ and its even sum labeling is shown in Figure 4.

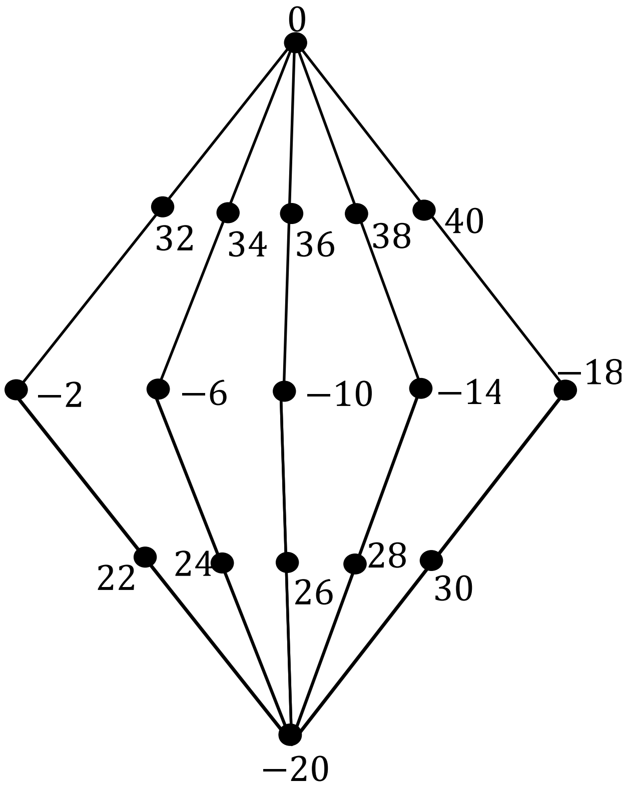

Figure – 4: Even sum labeling of $S(K_{2,5})$

Theorem 3: A super subdivision of cycle C_{4n} when each edge is replaced by $K_{2,t}$ is an even sum graph.

Proof: Suppose $u_1, u_2, \ldots\ldots\ldots, u_{4n-1}, u_{4n}$ are the vertices of given cycle C_{4n}.

Let $e_i = u_i u_{i+1};\ \forall\, i = 1, 2, \ldots\ldots, 4n-1$ and $e_{4n} = u_{4n} u_1$ be the edges of the cycle C_{4n}. Then it's super subdivision $SS(C_{4n})$is obtained by replacing each edge $e_i;\ i = 1,2, \ldots\ldots, 4n$ by a complete bipartite graph $K_{2,t}$ for some positive integer t as shown in Figure 5. If we take $G = SS(C_{4n})$ then,

$V(G) = \{u_i \mid i = 1,2, \ldots\ldots, 4n\} \cup \{v_{i,j} \,\big|\, i = 1,2, \ldots\ldots, 4n;\ j = 1,2, \ldots\ldots, t\}$ and

$E(G) = \{u_i v_{i,j}, v_{i,j} u_{i+1} \mid i = 1,2, \ldots\ldots, 4n-1;\ j = 1,2, \ldots\ldots, t\}$

$\cup \{u_{4n} v_{4n,j}, v_{4n,j} u_1 \,\big|\, j = 1,2, \ldots\ldots, t\}.$

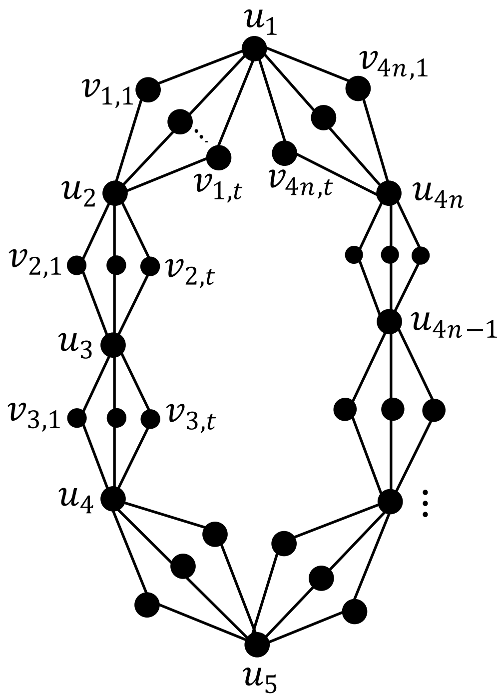

Figure – 5: Ordinary labeling of $SS(C_8)$ when each edge is replaced by $K_{2,3}$

Clearly, $q = |E(G)| = 8nt$.
Now, we define a vertex labeling function $f: V(G) \to \{0, 1, 2, \ldots\ldots, q\}$ as follow:

$$f(u_i) = 2t(1 - i);\ \forall\ i = 1, 2, \ldots\ldots, 2n,$$
$$f(u_i) = -2ti;\ \forall\ i = 2n + 1, 2n + 2, \ldots\ldots, 4n,$$
$$f(v_{i,j}) = 2q - 2(j - 1) - 2t(i - 1);\ \forall\ i = 1, 2, \ldots\ldots, 4n;\ \forall\ j = 1, 2, \ldots\ldots, t.$$

The above vertex labeling pattern with the induced edge labeling function $f^*: E(G) \to \{2, 4, 6, \ldots\ldots, 2q\}$given by $f^*(xy) = f(x) + f(y);\ \forall\ xy \in E(G)$ implies the even sum labeling of$G$. Hence, a super subdivision of $C_{4n}$when each edge of the cycle is replaced by $K_{2,t}$ is an even sum graph.

Illustration 3: Figure 6 shows the even sum labeling of the super subdivision of cycle C_8 when each edge of C_8 is replaced by $K_{2,3}$.

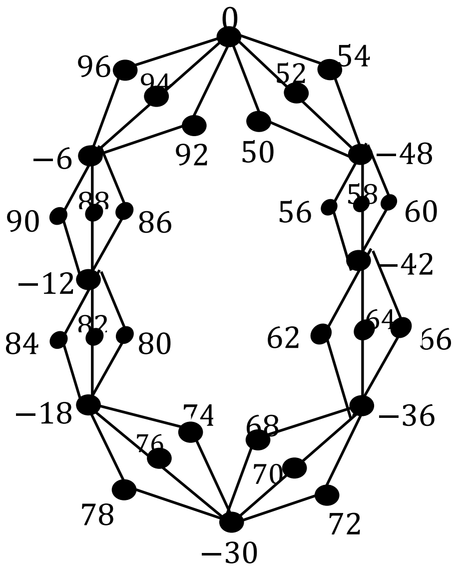

Figure – 6: Even sum labeling of $SS(C_8)$ when each edge is replaced by $K_{2,3}$

Theorem 4: An arbitrary super subdivision of path P_n when each edge e_i of the path is replaced by K_{2,m_i} with arbitrary m_i admits even sum labeling.
Proof: Suppose $u_1, u_2, \ldots\ldots, u_n$ are the vertices of given pathP_n. Let $e_i = u_i u_{i+1}$; $\forall\ i = 1, 2, \ldots\ldots, n-1$ be the edges of P_n. Let G be an arbitrary super subdivision of P_n which is obtained from P_n by replacing each edge e_i of P_n by a complete bipartite graph $K_{2,m_i}$with arbitrary positive integer m_i; $\forall 1 \leq i \leq n-1$,in a way that the end vertices of e_i are merged with the two vertices of the two vertices part of K_{2,m_i}, after excluding the edge e_i from P_n.

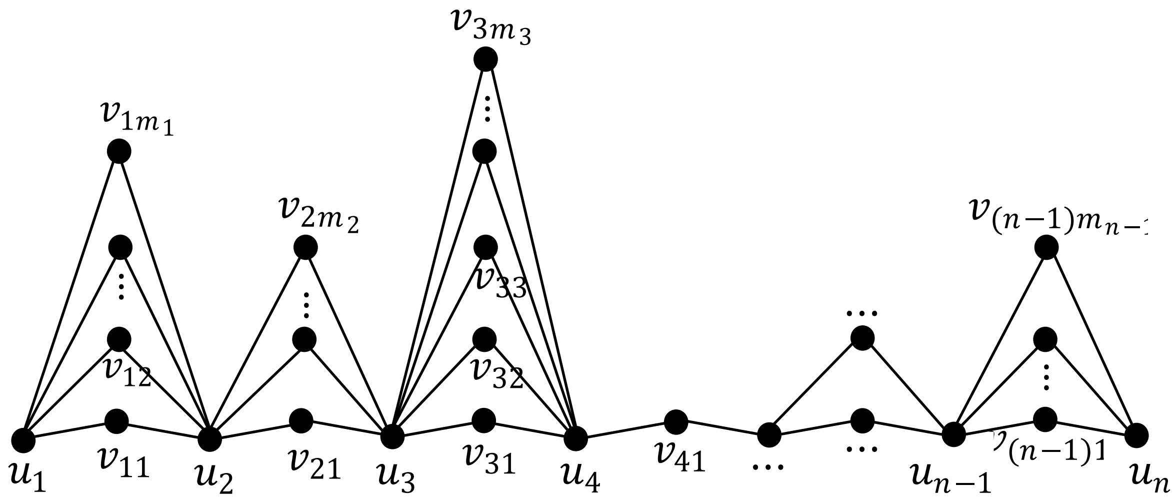

Figure – 7: Ordinary labeling of $ASS(P_7)$

The ordinary vertex labeling of thus obtained graph G is shown in Figure 7 which depicts that,
$V(G) = \{u_i \mid i = 1, 2, \ldots\ldots, n\} \cup \{v_{ij} \mid i = 1, 2, \ldots\ldots, n-1;\ j = 1, 2, \ldots\ldots, m_i\}$and
$E(G) = \{u_i v_{ij},\ u_{i+1} v_{ij} \mid i = 1, 2, \ldots\ldots, n-1;\ \ j = 1, 2, \ldots\ldots, m_i\}$.
Also, $|V(G)| = q = 2\sum_{i=1}^{n-1} m_i$.
Now, if we define a vertex labeling function $f: V(G) \to \{0,1,2, \ldots\ldots, q\}$ as

$$f(u_1) = 2q,$$

$$f(u_i) = 2q - 2\sum_{j=1}^{i-1} m_j\ ;\ \forall\ i = 2, 3, \ldots\ldots, n,$$

$$f(v_{1j}) = 2 - 2j;\ \forall\ j = 1,2, \ldots\ldots, m_1,$$

$$f(v_{ij}) = 2 - 2j - 2\sum_{k=1}^{i-1} m_k\ ;\ \forall\ i = 2,3, \ldots\ldots, n-1;\ \forall\ j = 1,2, \ldots\ldots, m_i\ ,$$

then this labeling pattern with the induced edge labeling function $f^*: E(G) \to \{2, 4, 6, \ldots\ldots, 2q\}$given by $f^*(xy) = f(x) + f(y)$; $\forall\ xy \in E(G)$ implies the even sum labeling of$G$. Hence, an arbitrary super subdivision of a path $P_n$when each edge e_i of the path is replaced by K_{2,m_i} with arbitrary $m_i$admits even sum labeling.

Illustration 4: Even sum labeling of the arbitrary super subdivision of a path P_7 is shown in Figure 8.

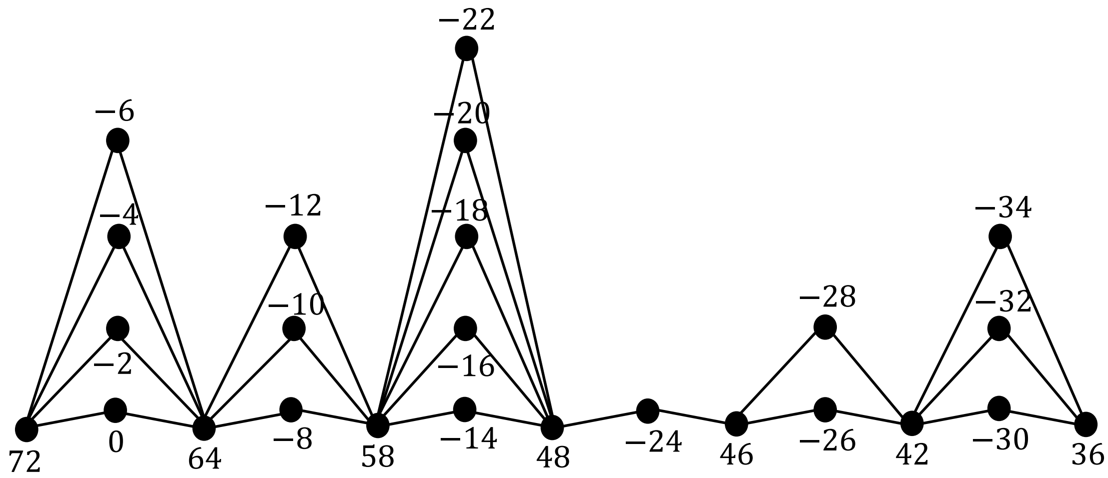

Figure – 8: Even sum labeling of $ASS(P_7)$

III. Conclusion

This chapter initiates even sum labeling of subdivision, super subdivision and arbitrary super subdivision of some graphs.

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