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A Study Of Submanifolds In A Contact Riemannian Manifold

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ABSTRACT
In this chapter, we investigate non-invariant submanifolds of almost para-contact Riemannian manifolds, we establish a necessary and sufficient condition for a sub manifold immersed in an almost Para contact Riemannian manifold to be invariant and show further properties of invariant sub manifolds in almost para-contact Riemannian manifolds.
Now we shall recollect an almost r-para-contact Riemannian manifold and treat the relations between this manifold and an almost product Riemannian manifold. Next, we study an invariant sub manifold immersed in an almost r-para contact Riemannian manifold and show that there exist the invariant sub manifolds of the three types in the almost r-para contact Riemannian manifold.
The purpose of the present note is to give a necessary and sufficient condition for a sub manifold M^3 of a conformally flat space to be conformally flat.
In this note, we generalize this result to K-contact Riemannian manifold and also study an invariant submani fold V immersed in almost paracontact Riemannian manifold to show that the V admits either an almost paracontact Riemannian structure or an almost product Riemannian structure (φ,g) excepting the case where φ is trivial.

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I. An Almost Para-contact Riemannian Manifold:

Let $\overline{M}$ be an m-dimensional manifold. If there exist on $\overline{M}$ a (1,1) tensor field φ a vector field ξ and a 1-form η satisfying

(1.1) $\eta(\xi)=1, \qquad \varphi^2 = I-\eta\otimes\xi,$

where *I* is the identity, then $\overline{M}$ is said to be an almost para contact manifold [3]. In the almost para contact manifold, the following relations hold good.

(1.2) $\varphi\xi = 0, \qquad \eta \circ \varphi = 0, \qquad \text{rank}(\varphi) = m-1$

Every almost para contact manifold has a positive definite Riemannian metric *G* such that

(1.3) $$\eta(\overline{X}) = G(\xi, \overline{X}),$$

(1.4) $$G(\varphi\overline{X}, \varphi\overline{Y}) = G(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}), \overline{X}, \overline{Y} \in \mathfrak{X}(\overline{M})$$

where $\mathfrak{X}(\overline{M})$ denotes the set of differentiable vector fields on $\overline{M}$. In this case, we say that $\overline{M}$ has an almost para contact metric structure (φ,ξ ,η, *G*) and $\overline{M}$ is said to be an almost para contact Riemannian manifold. Form (1.3) and (1.4), we can easily get the relation

(1.5) $$G(\varphi\overline{X}, \overline{y}) = G(\overline{X}, \varphi\overline{Y})$$

Here after, we assume that $\overline{M}$ is an almost para contact Riemannian manifold with a structure (φ,ξ ,η, *G*) . It is clear that the eigen values of the matrix (φ) are 0 and ±1, where the multiplicity of 0 is equal to 1.

Let *M* be an n-dimensional differentiable manifold (*S*= *m*-*n*) and suppose that *M* is immersed in the almost para contact Riemannian manifold $\overline{M}$ by the immersion *i*: $M \to \overline{M}$. We denote by i_* the differential of the immersion *i*. The induced Riemannian metric *g* of *M* is given by

$g(X,Y) = G(i_* X, i_* Y), \qquad X, Y \in \mathfrak{X}(M)$

where $\mathfrak{X}(M)$ is the set differentiable vector fields on *M*. We denote by $T_P(M)$ the tangent space of *M* at $P \in M$, by $T_P(M)^\perp$ the normal space of *M* at *P* and by { N_1, .N_2,…..,N_S} an orthonormal basis of the normal space $T_P(M)^{\perp}$. If $\varphi T_P(M) \subset T_P(M)$ for any point $P \in M$, then M is called an invariant submanifold. If $\varphi T_P(M) \subset T_P(M)^{\perp}$ for any point $P \in M$, then M is called an anti-invariant submanifold.

The transform $\varphi i_* X$ of $X \in T_P(M)$ by φ and φN_i of N_i by φ can be respectively written in the next forms:

(1.6) $$\varphi i_* X = i_* \psi X + \sum_{i=1}^{S} u_i(X) N_i, \qquad X \in \mathfrak{X}(M),$$

(1.7) $$\varphi N_i = i_* U_i + \sum_{j=1}^{S} \lambda_{ji} N_j,$$

where ψ, u_i, U_i and λ_{ji} are respectively a (1,1)-tensor, 1-forms, vector field and functions on M and Latin indices take values 1,2,……,S. And the vector field ξ can be expressed as follows:

(1.8) $$\xi = i_* V + \sum_{i=1}^{S} \alpha_i N_i,$$

where V and α_i are respectively a vector field and functions on M, from these equations we have [6].

(1.9) $$g(\psi X, Y) = g(X, \psi Y)$$
$$u_i(X) = g(U_i, X), \qquad \lambda_{ij} = \lambda_{ji}$$

If M is an invariant sub manifold, then we have $U_i = 0$. However, in the paper, we treat mainly a non-invariant sub manifold.

II. Sub Manifolds of an Almost Para Contact Riemannian Manifold Satisfying $\overline{\nabla}_{i_*X}\varphi = 0$:

Let M be a sub manifold of an almost para contact Riemannian manifold $\overline{M}$ with a structure (φ,ξ, ,η, G). Now we suppose that $\overline{\nabla}_{i_*X}\varphi = 0$ holds good along M. then from (a) and (b)

(a) $$(\overline{\nabla}_{i_*X}\varphi)i_*Y = i_*\left\{(\nabla_X\psi)Y - \sum_i u_i(y)H_iX - \sum_i h_i(X,y)U_i\right\} + \sum_i\left\{h_i(X,\psi y) + (\nabla_X u_i)(y) - \sum_i \mu_{ij}(X)u_j(y) - \sum_j \lambda_{ij}h_j(X,Y)\right\}N_i$$

(b) $$(\overline{\nabla}_{i_*X}\varphi)N_i = i_*\left\{\nabla_X U_i + \psi H_i X - \sum_j \mu_{ij}(X)U_j - \sum_j \lambda_{ij}H_jX\right\} + \sum_j\left\{h_j(X,U_i) + h_i(X,U_j) + \nabla_X\lambda_{ij} + \sum_k \lambda_{ik}\mu_{kj}(X) + \sum_k \lambda_{jk}\mu_{ki}(X)\right\}N_j,$$

we have

(2.1) $$(\nabla_X\psi)Y - \sum_i u_i(y)H_iX - \sum_i h_i(X,Y)U_i = 0,$$

(2.2) $$h_j(X,U_i) + h_i(X,U_j) + \nabla_X\lambda_{ij} + \sum_k \lambda_{ik}\mu_{kj}(X) + \sum_k \lambda_{jk}\mu_{ki}(X) = 0$$

from (2.1), we know that if M is totally geodesic, then an equation $\nabla_X \psi = 0$ holds good. Conversely, we have the following theorem

Theorem 2.1:

Let $\overline{M}$ be an almost para contact Riemannian manifold with a structure (φ,ξ,η, G) and M a sub manifold of $\overline{M}$ satisfying $\overline{\nabla}_{i_x X}\varphi = 0$, if U_i (i=1,2,.....,S) is linearly independent and $\nabla_X \psi = 0$, then M is totally geodesic.

Proof:

If $\nabla_X \psi = 0$, then we have from (2.1)

$$\sum_i u_i(Y) H_i X + \sum_i h_i(X,Y)U_i = 0.$$

from which,

$$\sum_i u_i(Y) h_i(X,Z) + \sum_i u_i(Z) h_i(X,Y) = 0, \qquad X,Y,Z \in \mathfrak{X}(M)$$

that is

$$\sum_i u_i(Y)h_i(X,Z) = -\sum_i u_i(Z)h_i(X,Y)$$

Thus, we know that $\sum_i u_i\,(Y)\,h_i\,(X,Z)$ is symmetric and at the same time skew symmetric in X,Y. Therefore we have $\sum_i u_i\,(Y)\,h_i\,(X,Z)=0$ and consequently we get $h_i\,(X,Z)=0$ because U_i (i=1,2,....,S) are linearly independent. Let $\{e_1,e_2,....,e_n\}$ be an orthonormal basis of $T_P(M)$ at any point $P\in M$. Then a trace of the matrix (ψ) is given by an equation

$$T_r(\psi) = \sum_{\lambda=1}^{n} g(\psi e_\lambda, e_\lambda),$$

where Greek indices takes values 1,2,.....,n.

Theorem 2.2:

$\nabla_X T_r(\psi) = T_r(\nabla_X\psi)$,

Proof:

$$\begin{aligned}\nabla_X T_r(\psi) &= \nabla_X \sum_\lambda g(\psi e_\lambda, e_\lambda)\\ &= \sum_\lambda \{g((\nabla_X\psi)e_\lambda, e_\lambda) + 2g(\psi e_\lambda, \nabla_X e_\lambda)\}\\ &= T_r(\nabla_X\psi) + 2\sum_\lambda g(\psi e_\lambda, \nabla_X e_\lambda).\end{aligned}$$

Now we get

$$\psi e_\lambda = \sum_\mu f_{\lambda\mu} e_\mu, \qquad \nabla_X e_\lambda = \sum_\mu l_{\lambda\mu} e_\mu,$$

Then we can see easily that $f_{\lambda\mu} = f_{\mu\lambda}$, $l_{\lambda\mu} + l_{\mu\lambda} = 0$ hold good. Therefore

$$\sum_\lambda g(\psi e_\lambda, \nabla_X e_\lambda) = \sum_\lambda g\left(\sum_\mu f_{\lambda\mu} e_\mu, \sum_V l_{\lambda V} e_V\right) = \sum_\lambda\sum_\mu\sum_V f_{\lambda\mu} l_{\lambda V}\delta_{\mu V} = \sum_\lambda\sum_\mu f_{\lambda\mu} l_{\lambda\mu} = 0$$

thus we get $\nabla_X T_r(\psi) = T_r(\nabla_X\psi)$.

III. Submanifolds of a P-Sasakian Manifold:

Let $\overline{M}$ be an m-dimensional Riemannian manifold, G be a positive definite metric and $\overline{\nabla}$ be the operator of Covariant differentiation. We suppose that there exists on $\overline{M}$ a vector field ξ and a 1-form η satisfying.

(3.1) $\eta(\xi)=1$, $\quad \eta(\overline{X}) = G(\xi, \overline{X}), \quad \overline{X} \in \mathcal{X}(\overline{M})$

when equations,

(3.2) $$G(\overline{\nabla}_{\overline{X}}\xi, \overline{Y}) = G(\overline{\nabla}_{\overline{y}}\xi, \overline{X}), \quad \overline{X},\overline{Y} \in \mathcal{X}(\overline{M}),$$

(3.3) $$\overline{\nabla}_X\overline{\nabla}_y\xi - \overline{\nabla}_z\xi = -G(\overline{X},\overline{Y})\xi - G(\xi,\overline{Y})\overline{X} + 2\eta(\overline{X})\eta(\overline{Y})\xi,$$

Where $\overline{Z} = \overline{\nabla}_{\overline{X}}\overline{Y}$, holds good , $\overline{M}$ is said to be a P-Sasakian mani fold. If we suppose that φ is a (1,1) tensor field, which represents a linear mapping : $\mathcal{X}(\overline{M}) \in \overline{X} \to \overline{\nabla}_{\overline{X}}\xi$, that is,

(3.4) $$\varphi\overline{X} = \overline{\nabla}_{\overline{X}}\xi,$$

then, equation (3.2) and (3.3) become

(3.5) $$G(\varphi\overline{X}, \overline{Y}) = G(\overline{X}, \varphi\overline{Y}),$$

(3.6) $$\begin{aligned}(\overline{\nabla}_{\overline{X}}\varphi)(\overline{Y}) &= -G(\overline{X},\overline{Y})\xi - G(\xi,\overline{Y})\overline{X} + 2\eta(\overline{X})\eta(\overline{Y})\xi\\ &= -\{-\overline{X} + \eta(\overline{X})\xi\}\eta(\overline{Y}) + \{-G(\overline{X},\overline{Y}) + \eta(\overline{X})\eta(\overline{Y})\}\xi\end{aligned}$$

respectively, [4]. Differentiating $\eta(\xi)=1$ covariantly, we have $\varphi\xi=0$. Further more, differentiating this equation covariantly , we get $\varphi^2\overline{X} = \overline{X} - \eta(\overline{X})\xi$, from which we have (1.4)

Theorem 3.1:

Let $\overline{M}$ be a P-Sasakian manifold admitting a vector filed ξ and a 1-form η which satisfy (3.1). If we denote by φ $a(1,1)$ tensor field which represents a linear mapping : $\mathfrak{X}\,\overline{M} \ni \overline{X} \mapsto \overline{\nabla}_X \xi$, then (φ,ξ,η,G) is an almost para contact metric structure.

Hereafter, in the P-Sasakian manifold $\overline{M}$. Let M be a sub manifold of dimension n (m - n=S) immersed in the P-Sasakian manifold $\overline{M}$ and g be the induced metric. from (3.4) and (1.6), we have

(3.7) $$\overline{\nabla}_{i_*X}\xi = i_*\psi X + \sum_i u_i(X)N_i,$$

Therefore from,

$$\overline{\nabla}_{i_*X}\xi = i_*(\nabla_X V - \sum_i \alpha_i H_i X) + \sum_j \left\{ h_j(X,V) + \nabla_X \alpha_j + \sum_i \alpha_i \mu_{ij}(X) \right\} N_j$$

we get

(3.8) $$\psi X = \nabla_X V - \sum_i \alpha_i H_i X, \qquad X \in \mathfrak{X}(M),$$

(3.9) $$u_j(X) = h_j(X,V) + \nabla_X \alpha_j + \sum_i \alpha_i \mu_{ij}(X).$$

Making use of (3.9), we have

Theorem 3.2:

Let M be sub manifold of an almost para contact Riemannian manifold $\overline{M}$ with a structure (φ,ξ,η,G) satisfying (3.4). If M is totally geodesic and ξ is tangent to M, then M is invariant.

from (3.6) we have.

$$(\overline{\nabla}_{i_*X}\varphi)(i_*Y) = -G(i_*X, i_*Y)\xi - \eta(i_*Y)i_*X + 2\eta(i_*X)\eta(i_*Y)\xi$$

$$= i_*\{g(X,Y)V - \nu(Y)X + 2v(X)v(Y)V\} - \sum_i \alpha_i \{g(X,Y) + 2v(X)v(Y)\} N_i$$

therefore from

$$(\overline{\nabla}_{i_*x}\varphi)(i_*Y) = i_*\left\{(\nabla_X\psi)Y - \sum_i u_i(Y)H_iX - \sum_i h_i(X,Y)U_i\right\}$$

$$+ \sum_i \left\{ h_i(X,\psi Y) + (\nabla_X u_i)(Y) - \sum_j \mu_{ij}(X)u_j(Y) - \sum_j \lambda_{ij} h_j(X,Y) \right\} N_i$$

we get

(3.10) $$(\nabla_X\psi)Y - \sum_i u_i(Y)H_iX - \sum_i h_i(X,Y)U_i = -g(X,Y)V - g(V,Y)X + 2v(X)v(Y)V.$$

Similarly, because we have from (3.6)

$$(\overline{\nabla}_{i_*X}\varphi)(N_i) = i_*\alpha_i\{X + 2v(X)V\} - 2\sum_i \alpha_i \alpha_j v(X)N_j,$$

We find

(3.11) $$h_j(X,U_i) + h_i(X,U_j) + \nabla_X \lambda_{ij} + \sum_k \lambda_{ik}\mu_{kj}(X) + \sum_k \lambda_{jk}\mu_{ki}(X) = 2\alpha_i\alpha_j v(X)$$

by

$$(\overline{\nabla}_{i_*X}\varphi)(N_i) = i_*\left\{ \nabla_X U_i + \psi H_i X - \sum_j \mu_{ij}(X)U_j - \sum_j \lambda_{ij} H_j X \right\}$$

$$+ \sum_j \left\{ h_j(X,U_i) + h_i(X,U_j) + \nabla_X \lambda_{ij} + \sum_k \lambda_{ik}\mu_{kj}(X) + \sum_k \lambda_{jk}\mu_{ki}(X) \right\} N_j$$

Now, we put

(3.12) $$\tilde{\psi}(X,Y) = (\nabla_X\psi)Y - \{g(X,Y)V - g(V,Y)X + 2v(X)v(Y)V\}$$

then from (3.10) we have

(3.13) $$\tilde{\psi}(X,Y) = \sum_i u_i(Y)H_iX + \sum_i h_i(X,Y)U_i$$

When $\tilde{\psi}(X,Y) = 0$ we have the following theorem:

Theorem 3.3:

Let $\overline{M}$ be a p–Sasakian manifold with a structure (φ,ξ,η,G), M be a sub manifold immersed in $\overline{M}$ and ξ be not tangent to M. If U_i $(i=1,2,.....,S)$ are linearly independent and

(3.14) $(\nabla_X\psi)\,Y = -g(X,Y)\,V - g(V,Y)\,X + 2v(X)\,v(Y)\,V.$

Then M is totally geodesic

Proof :

From (3.13) and (3.14) we have

$$\sum_i u_i(Y) H_i X + \sum_i h_i(X,Y) U_i = 0,$$

From which, we find $h_i(X,Y)=0$ (See proof of theorem 2.1)

Note :

When $\nabla_X \alpha_j + \sum_i \alpha_i \mu_{ij}(X) = 0,$ If M is totally geodesic, then we have $u_j(X)=0$ by virtue of (3.9) therefore in this case, theorem (3.3) is not true.

IV. Submanifolds of SP-Sasakian Manifolds:

Let $\overline{M}$ be an m-dimensional Riemannian manifold. We suppose that there exist on $\overline{M}$ a vector field ξ and a 1-form η satisfying (3.1) When an equation

(4.1) $$\overline{\nabla}_X \xi = \varepsilon(\overline{X} - \eta(\overline{X})\xi)\ (\varepsilon = \pm 1), \quad \overline{X} \in \mathfrak{X}(\overline{M})$$

holds good, $\overline{M}$ is said to be an SP- Sasakian manifold. Since from (4.1) we can get (3.2) and (3.3), an SP-Sasakian manifold is a P-Sasakian manifold. if we suppose that a (1,1) tensor field φ satisfies (3.4), then (φ,ξ,η,G) is an almost Para contact metric structure. In this section, we suppose that $\overline{M}$ is an SP-Sasakian manifold admitting *a*(1,1) tensor field φ which satisfies (3.4).

from (4.1) we have

$$\overline{\nabla}_{i_* X} \xi = \varepsilon(i_* X - \eta(i_* X)\xi) = \varepsilon\left\{ i_*(X - v(X)V) - \sum_j \alpha_j v(X) N_j \right\}$$

By mean of (3.7), we get

(4.2) $\psi X = \varepsilon\,(X - v(X)\,V),$

(4.3) $u_j(X) = -\varepsilon\,\alpha_j\,v(X),$

V. Linear Independence of Vector Fields U_i:

Let M be a sub manifold immersed in an almost paracontact Riemannian manifold $\overline{M}$ with a structure (φ,ξ,η,G). We transform the orthonormal basis $\{N_1, N_2,......,N_S\}$ of $T_P(M)^{\perp}$ to another orthonormal basis $\{\overline{N}_1, \overline{N}_2, ...,\overline{N}_S\}$ of $T_P(M)^{\perp}$ [7]. We put

(5.1) $$\overline{N}_l = \sum_{j=1}^{S} K_{jl} N_j$$

Then, (K_{jl}) is an orthogonal matrix and we have

$$N_j = \sum_{l=1}^{S} K_{jl} \overline{N}_l$$

making use of $\{\overline{N}_1, \overline{N}_2, ...,\overline{N}_S\}$, we get

(5.2)
$$\varphi i_* X = i_* \psi X + \sum_l \overline{u}_l(X) \overline{N}_l,$$
$$\varphi \overline{N}_l = i_* \overline{U}_l + \sum_h \overline{\lambda}_{lh} \overline{N}_h,$$
$$\xi = i_* V + \sum_l \overline{\alpha}_l \overline{N}_l,$$

Where

$$\overline{U}_l(X) = \sum_i K_{il} u_i(X), \quad \overline{U}_l = \sum_i k_{il} U_i, \quad \overline{\lambda}_{lh} = \sum_{i,j} k_{il} \lambda_{ij} k_{jh}, \overline{\alpha}_l = \sum_i k_{il} \alpha_i,$$

By a suitable transformation of the orthonormal basis $\{N_1, N_2, \ldots, N_S\}$, we can get

$$\overline{\lambda}_{ij} = \lambda_i \delta_{ij},$$

Where λ_i are eigen values of the matrix (λ_{ij}). In this case, we have

$$\varphi \overline{N}_l = i_* \overline{U}_l + \lambda_l \overline{N}_l, \tag{5.3}$$

$$\bar{u}_j(\overline{U}_j) = 1 - \overline{\alpha}_j^2 - \lambda_j^2, \tag{5.4}$$

$$\bar{u}_k(\overline{U}_j) = -\overline{\alpha}_k \overline{\alpha}_j \quad (k \neq j) \tag{5.5}$$

VI. Anti Invariant Submanifolds of an Almost Paracontact Riemannian Manifold:

Let M be an anti invariant sub manifold immersed in an almost paracontact Riemannian manifold $\overline{M}$. Then since, we have $\psi = 0$, from

$$\psi^2 X = X - v(X)V - \sum_{i=1}^{S} u_i(X) U_i, \text{ we get}$$

$$X - v(X)V - \sum_i u_i(X) U_i = 0.$$

From which

$$g(X, X) - v(X)^2 - \sum_i u_i(X)^2 = 0,$$

Substituting $X = e_\lambda$ and summing up in λ, we get

$$(S+1) - n = 2\sum_j \alpha_j^2 + \sum_{i,j} \lambda_{ij}^2 \tag{6.1}$$

by virtue of

$$u_k(U_j) = \delta_{kj} - \alpha_k \alpha_j - \sum_{i=1}^{S} \lambda_{ki}\lambda_{ji}, \text{ and } v(V) = 1 - \sum_{i=1}^{S} \alpha_i^2,$$

Thus we have η≦S+1,

When $n = S+1$, from (6.1), we have

$\lambda_{ij} = 0$, $\alpha_j = 0$

Consequently, we have $\varphi T_P(M)^\perp \subset T_P(M)$ and ξ is tangent to M. Thus, by means of $u_k(U_j) = \delta_{kj} - \alpha_k \alpha_j - \sum_{i=1}^{S} \lambda_{ki}\lambda_{ji}$, $u_i(V) + \sum_{j=1}^{S} \alpha_j \lambda_{ji} = 0$, and $v(V) = 1 - \sum_{i=1}^{S} \alpha_i^2$, we know that U_i (i=,1,2.....,S), V are mutually orthogonal unit vector fields.

In an almost para contact Riemannian manifold $\overline{M}$, when the equation

$$\varphi \overline{X} = \overline{\nabla}_{\overline{X}} \xi \tag{6.2}$$

holds good, $\overline{M}$ is a said to be a special para contact Riemannian manifold [4], If M is an anti-invariant submanifold of dimension $n = S+1$, then we have

$$\nabla_X V = 0, \quad u_j(X) = h_j(X, V)$$

VII. Transformation of the Orthonormal Basis $\{N_i\}$ of $T(M)^\perp$:

Let M be a sub manifold immersed in an almost para contact Riemannian manifold $\overline{M}$ and $\{N_1, N_2, \ldots, N_S\}$ be an orthonormal basis of the normal space $T_P(M)^\perp$ at $P \in M$ [7]. We assume that $\{\overline{N}_1, \overline{N}_2, \ldots, \overline{N}_S\}$ is the another orthonormal basis of $T_P(M)^\perp$ and put

$$\overline{N}_i = \sum_{l=1}^{S} k_{li} N_l \tag{7.1}$$

By means of $G(\overline{N}_i, \overline{N}_j) = \sum_{l=1}^{S} k_{li}k_{lj}$, we have $\sum_{l=1}^{S} k_{li}k_{lj} = \delta_{ij}$, from which $\sum_{h=1}^{S} k_{ih}k_{jh} = \delta_{ij}$. Consequently a matrix (k_{ij}) is an orthonogonal matrix. Thus from (7.1), we have $N_j = \sum_{l=1}^{S} k_{jl}\ \overline{N}_l$.

Making use of (7.1), equations (1.6), (1.7) and (1.8) are respectively written in the following forms:

(7.2)
$$\varphi i_* X = i_*\psi X + \sum_{l=1}^{S} \overline{u}_l(X)\,\overline{N}_l,$$
$$\varphi \overline{N}_l = i_*\overline{U}_l + \sum_{h=1}^{S} \overline{\lambda}_{lh}\overline{N}_h,$$
$$\xi = i_*V + \sum_{l=1}^{S} \overline{\alpha}_l \overline{N}_l,$$

where

(7.3)
$$u_l(X) = \sum_{i=1}^{S} k_{il}u_i(X), \quad \overline{U}_l = \sum_{i=1}^{S} k_{il}U_i,$$

(7.4)
$$\overline{\lambda}_{lh} = \sum_{i,j=1}^{S} k_{il}\lambda_{ij}k_{jh}, \quad \overline{\lambda}_{lh} = \overline{\lambda}_{hl},$$
$$\overline{\alpha}_l = \sum_{i=1}^{S} k_{il}\alpha_i$$

By virtue of (7.3), the linear independence of vectors U_i (i =1,2,.....,S) is invariant under the transformation (7.1) of the orthonormal basis {N_1, N_2,....,N_S}.

Further more, because λ_{ij} is symmetric in i and j, from (7.4) we can find that under a suitable transformation (7.1) λ_{ij} reduces to $\overline{\lambda}_{ij} = \lambda_i\delta_{ij}$, where λ_i (i = 1,2,...,s) are eigen values of matrix (λ_{ij}). In this case (7.2) and

$u_k(U_j) = \delta_{kj} - \alpha_k\alpha_j - \sum_{i=1}^{S} \lambda_{ki}\lambda_{ji}$, are respectively written in the next forms:

(7.5)
$$\varphi \overline{N}_l = i_*\overline{U}_l + \lambda_l \overline{N}_l,$$
$$\overline{u}_k(\overline{U}_j) = \delta_{kj} - \overline{\alpha}_k\overline{\alpha}_j - \lambda_k\lambda_j\delta_{kj},$$

from which we have

$$\overline{u}_j(\overline{U}_j) = 1 - \overline{\alpha}_j^2 - \lambda_j^2 \ \ and\ \overline{u}_k(\overline{U}_j) = -\overline{\alpha}_k\overline{\alpha}_j (k \neq j)$$

VIII. Invariant Submanifolds of an Almost Paracontact Riemannian Manifold :

Let M be a sub manifold immersed in an almost paracontact Riemannian manifold $\overline{M}$. If $\varphi T_P(M) \subset T_P(M)$ for any point $P \in M$, then M is called an invariant submanifold. In an invariant submanifold M, equations (1.6), (1.7) and (1.8) are written in the following forms:

(8.1)
$$\varphi i_*\ X = i_*\psi X, \quad X \in \mathfrak{X}(M),$$

(8.2)
$$\varphi N_i = \sum_{j=1}^{S} \lambda_{ij}N_j,$$

(8.3)
$$\xi = i_*V + \sum_{i=1}^{S} \alpha_i N_i,$$

Lemma 8.1:

In an invariant submanifold M which is immersed in an almost paracontact Riemannian manifold $\overline{M}$, the following equations hold good.

(8.4) $\psi^2 = 1 - v \otimes V$,

(8.5) $\alpha_i V = 0$.

(8.6) $$\delta_{kj} - \alpha_k \alpha_j - \sum_{i=1}^{S} \lambda_{ki}\lambda_{ji} = 0.$$

(8.7) $\psi V = 0$

(8.8) $$\sum_{i=1}^{S} \alpha_i \lambda_{ij} = 0,$$

(8.9) $$v(V) = 1 - \sum_{i=1}^{S} \alpha_i^2,$$

(8.10) $$g(\psi X, \psi Y) = g(X,Y) - v(X)v(Y), \quad X, Y \in \mathfrak{X}(M).$$

From (8.5) and (8.9), we get the following two cases: When V =0 (or $\sum_i \alpha_i^2 = 1$), that is, ξ normal to *M*, since from (8.4) and (8.10) we have $\psi^2 = I$, $g(\psi X, \psi Y) = g(X,Y)$, (ψ, g) is an almost product metric structure when ever ψ is non-trivial.

when $V \neq 0$ (or $\alpha_i = 0$), that is, ξ is tangent to *M*, by means of (8.4), (8.9), (8.10) and $v(X) = g(V,X)$, (ψ, V, v, g) is an almost para contact metric structure. Thus we have

Theorem 8.1:

Let M be an invariant sub manifold immersed in an almost para contact Riemannian manifold $\overline{M}$ with a structure (φ, ξ, η, G). Then one of the following cases occurs T. Miya Zawa[6].

Case (I) : ξ is normal to *M*. In this case, the induced structure (ψ, g) on *M* is an almost product metric structure when ever ψ is non-trivial.

Case (II): ξ is tangent to M. In this case, the induced structure (ψ, V, v, g) is an almost para contact metric structure.

Furthermore, we have the following theorems:

Theorem 8.2:

In order that, in an almost para contact. Riemannian manifold *M* with a structure (φ, ξ, η, G) the submanifold *M* of $\overline{M}$ is invariant, it is necessary and sufficient that the induced structure (ψ, g) on *M* is an almost product metric structure when ever ψ is non-trivial or the induced structure (ψ, V, v, g) on M is an almost paracontact metric structure.

Proof:

From theorem 8.1, the necessity is evident conversely, we first assume that the induced structure (ψ, g) is an almost product metric structure. Then from equation (c)

(c) $$\psi^2 X = X - v(X)V - \sum_{i=1}^{S} u_i(X)U_i \ \ or \ \ \psi^2 = I - v \otimes V - \sum_{i=1}^{S} u_i \otimes U_i, X \in \mathfrak{X}(M)$$

We have $v(X)V + \Sigma_i u_i(X) U_i = 0$ from which $g(v(X) V + \Sigma_i u_i(X) U_i, X) = 0$ that is $v(X)^2 + \Sigma_i u_i(X)^2 = 0$. Consequently, since we get $v(X) = u_i(X) = 0$ $(i = 1,2,....,s)$ the submanifold *M* is invariant and ξ is normal to *M*.

Next, we assume that the induced structure (ψ, V, v, g) is an almost para contact metric structure. Then, from Equation (c) we have $\Sigma_i u_i(X) U_i = 0$, from which $u_i(X) = 0$ $(i = 1,2,...,s)$ and from Equation (d).

(d) $$u_j(\psi X) + \sum_{i=1}^{S} \lambda_{ji} u_i(X) + \alpha_j v(X) = 0$$

We get $\alpha_i = 0$, thus *M* is invariant and ξ is tangent to *M*.

IX. Paracontact Riemannian Manifolds and P-Sasakian Manifolds:

Let $\overline{M}$ be an almost paracontact Riemannian manifold with a structure (φ, ξ, η, G) . If we put $\Phi(\overline{X}, \overline{Y}) = G(\varphi \overline{X}, \overline{Y})$ for $\overline{X}, \overline{Y} \in \mathfrak{X}(\overline{M})$, then from (1.5) we have $\Phi(\overline{X}, \overline{Y}) = \Phi(\overline{Y}, \overline{X})$.

We denote by $\overline{\nabla}_X$ the operator of covariant differentiation with respect to G along the vector field $\overline{X}$. For a vector field $\overline{Y}$, the covariant derivative $\overline{\nabla}_X \overline{Y}$ of $\overline{Y}$, has local components $\overline{X}^{\mu} \overline{\nabla}_{\mu} \overline{Y}^{\lambda}$, where $\overline{X}^{\mu}$ and $\overline{Y}^{\mu}$ are the local components of $\overline{X}$ and $\overline{Y}$ respectively and Greek indices λ, μ, v take values 1,2,.....,*m*.

When the equation

(9.1) $$2\Phi(\bar{X},\bar{Y}) = (\bar{\nabla}_X \eta)(\bar{Y}) + (\bar{\nabla}_y \eta)(\bar{X})$$

holds good, $\bar{M}$ is called a Para contact Riemannian manifold and (φ,ξ,η,G) a Para contact metric structure.

Especially, If the equation $(\bar{\nabla}_{\bar{X}}\eta)(\bar{Y}) = (\bar{\nabla}_{\bar{Y}}\eta)(\bar{X})$ holds good, then we have $\Phi(\bar{X},\bar{Y}) = (\bar{\nabla}_{\bar{X}}\eta)(\bar{Y})$

Consequently,

$$G(\varphi\bar{X},\bar{Y}) = \bar{\nabla}_{\bar{X}}\eta(\bar{Y}) - \eta(\bar{\nabla}_{\bar{X}}\bar{Y}) = \bar{\nabla}_{\bar{X}}G(\xi,\bar{Y}) - \eta(\bar{\nabla}_{\bar{X}}\bar{Y}) = G(\bar{\nabla}_{\bar{X}}\xi,\bar{Y})$$

Thus we find

(9.2) $$\varphi\bar{X} = \bar{\nabla}_{\bar{X}}\xi$$

when the above equation holds good, $\bar{M}$ is called a special paracontact Riemannian manifold and (φ,ξ,η,G) is referred as a special contact metric structure [1].

Now, we assume that $\bar{M}$ is a special paracontact Riemannian manifold. If the equation

(9.3) $$(\bar{\nabla}_{\bar{X}}\varphi)\bar{Y} = -G(\bar{X},\bar{Y})\xi - G(\xi,\bar{Y})\bar{X} + 2\eta(\bar{X})\eta(\bar{Y})\xi,$$

holds good where $G(\xi\, y) = \eta(y)$, then $\bar{M}$ is called a P-sasakian (or para Sasakian) manifold . By using local Components (9.2) and (9.3) are written as follows:

$$\varphi_\mu^\lambda = \bar{\nabla}_\mu\xi^\lambda, \quad \bar{\nabla}_\nu\bar{\nabla}_\mu\xi^\lambda = (-G_{\nu\mu} + \eta_\nu\eta_\mu)\xi^\lambda + G\delta_\nu^\lambda + \eta_\nu\xi^\lambda\eta_\mu$$

where $\varphi_\mu^\lambda, \xi^\mu, \eta_\mu$ and $G_{\mu\lambda}$ are local components of φ, ξ,η and G respectively, moreover, in a special para contact Riemannian manifold $\bar{M}$, if the equation

(9.4) $$\varphi\bar{X} = \bar{\nabla}_X\xi = \varepsilon(\bar{X} - \eta(\bar{X})\xi) \quad (\varepsilon = \pm 1), \text{i.e.}, \quad \varphi = \varepsilon(I - \eta\otimes\xi)$$

holds good, then $\bar{M}$ is called an SP-Sasakian (or special para Sasakian) manifold. It is clean that (9.4) satisfies (9.3).

X. An Almost r-paracontact Riemannian Manifold:

Let $\bar{M}$ be an m-dimensional Riemannian manifold with a positive definite metric G. If there exist a (1,1)-tensor field ψ on $\bar{M}$, r vector fields $\xi_1,\ldots,\xi_r$ ($r < m$), r 1-forms $\eta_1,\ldots,\eta_r$ such that

(10.1) $\eta_X(\xi_Y) = \delta_{xy}$ $(X,Y = 1,\ldots,r)$

(10.2) $$\psi^2 = I - \sum_{x=1}^{r}\eta_x\otimes\xi_x,$$

(10.3) $$\eta_x(\bar{X}) = G(\xi_x,\bar{X}),$$

(10.4) $$G(\psi\bar{X},\psi\bar{Y}) = G(\bar{X},\bar{Y}) - \sum_{X=1}^{r}\eta_X(\bar{X})\eta_X(\bar{Y}),$$

where $\bar{X},\bar{Y}$ are any vector fields on $\bar{M}$, then $(\psi, \xi_1\ldots,\xi_r,\eta_1,\ldots\eta_r,G)$ is said to be an almost r-paracontact Riemannian structure on $\bar{M}$ and $\bar{M}$ an almost r-paracontact Riemannian manifold, [5]. This structure is written (ψ, ξ_x, η_x, G) for short.

Theorem:

In an almost r-para contact Riemannian manifold with the structure (ψ, ξ_x, η_x, G), the following equations hold good:

(10.5) (a) $\psi\xi_X = 0$ (b) $\eta\, o\psi = 0$,

(10.6) $$\Phi(\bar{X},\bar{Y}) \overset{\text{def}}{=} G(\psi\bar{X},\bar{Y}) = G(\bar{X},\psi\bar{Y})$$

Proof :

(10.5) (a) using (10.4), we get

$$G(\psi\xi_X,\psi\xi_X) = G(\xi_X,\xi_X) - \sum_y\eta_y(\xi_X)\eta_y(\xi_X) = 0,$$

From which, we have $\psi\xi_x = 0$

(10.5) (b) using (10.2) for $\psi^2(\psi\bar{X}) = \psi(\psi^2\bar{X}),$ we have

$\psi\bar{X} - \sum\eta_x(\psi\bar{X})\xi_X = \psi(\bar{X} - \sum\eta_X(\bar{X})\xi_X),$

from which, we obtain $\sum\eta_x(\psi\bar{X})\xi_X = 0$ Virtue of (10.5) (a) Since $\xi_1, ..., \xi_r$ are linearly independent , we have $\eta_x(\psi\bar{X}) = 0,$ that is $\eta_x \circ \psi = 0$ (10.6) Using (10.2) and (10.4) for $G(\psi\bar{X}, \psi^2\bar{Y}),$ the equation (10.6) is easily verified.

It is obvious that ψ satisfies ψ^3-ψ =0. Because of (10.1) and (10.5) a), r vector fields $\xi_1,....,\xi_r$ are the mutualy orthogonal eigen vectors of a matrix (ψ) and their eigen values are all equal to 0. Since a matrix (Φ) is symmetric, the eigen values of the matrix (ψ) are all real. If we denote by ζ the eigen vector orthogonal to ξ_X (X=1,....,r) and by a its eigen value, then we have $\psi\zeta = \alpha\,\zeta$ therefore, we get $\psi^2\zeta$- $\alpha^2\zeta$. Accordingly, we see that the eigen values of (ψ) are 0, $\pm$1, where the multiplicity of 0 is equal to r and hence rank (ψ) = $m \to r$.

If we denote by $\bar{\nabla}$ a Riemannian connection, then the torsion tensor $\bar{N}$ for ψ may be expressed as follows [5],[9].

(10.7) $$\bar{N}(\bar{X},\bar{Y}) = (\bar{\nabla}_{\psi\bar{y}}\psi)\bar{X} - (\bar{\nabla}_{\bar{X}}\psi)\psi\bar{Y} - (\bar{\nabla}_{\psi\bar{X}}\psi)\bar{Y} + (\bar{\nabla}_{\bar{y}}\psi)\psi\bar{X}$$
$$+\sum_X \eta_X(\bar{X})\bar{\nabla}_{\bar{y}}\xi_X - \sum_X \eta_X(\bar{Y})\bar{\nabla}_{\bar{X}}\xi_X$$

when the torsion tensor for ψ vanishes, the almost r-para contact Riemannian manifold, or its structure is said to be normal.

XI. Conformally flat submanifolds:

Let M^m (m>3) be a Riemannian manifold covered by coordinate neighbourhoods (U,x^h) the indices h,i,j,k.... running over the range 1,2....,m. Let $g_{ij}, \nabla_h, K_{kji}^h, k_{ji}$ and R denote the metric tensor, the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M^m respectively. Let M^n ($n \geqq 3$) be a submanifold of M^m and be covered by a system of coordinate neighbourhoods (V,u^a) the indices a,b,c,.... running over the range 1,2,....,n. The immersion of M^n in M^m is locally given by $X^h = X^h(u^a)$. Let g_{ab}, ∇_b denote the metric tensor and the Riemannian connection of M^n induced from those of M^m. We have $g_{cb} = g_{ij}B_c^j B_b^i$ when $B_b^i = \frac{\partial X^i}{\partial u^b}$ Let K_{dcb}^a, K_{cb} and K denote the curvature tensor, the Ricci tensor and the Scalar curvature of M^n respectively.

We choose m-n orthogonal unit normal vectors C_x^h, (the indices x,y,z running over the range (n+1, n+2,...,m) in such a way that $\{B_a^h, C_x^h\}$ from a positively oriented frame of M^m along M^n . The equations of Gauss and Weingarten are given by.

(11.1) $$\nabla_c B_b^h = H_{cb}^x C_x^h, \quad \nabla_c C_x^h = -H_{cx}^a B_a^h,$$

where H_{cb}^x and $H_{bx}^c = H_{ba}^y g^{ac} g_{yx}$ and are the second fundamental tensors of M^n with respect to the normal C_x^h, g_{yx} being the metric tensor of the normal bundle. The equation of Gauss for M^n are

(11.2) $$K_{kijh}B_d^k B_c^j B_b^i B_a^h = K_{dcba} - A_{dcba},$$

where we set

(11.3) $$A_{dcba} = H_{cb}^x H_{dax} - H_{db}^x H_{cax}$$

Theorem A:

Let M^n (n >3) be a submanifold of a conformally flat Riemannian manifold M^m (m>3). Then M^n is conformally flat if and only if

(11.4) $$A_{dcba} - (g_{da}A_{cb} - g_{db}A_{ca} + A_{da}g_{cb} - A_{db}g_{ca})/(n-2)$$
$$+ A(g_{da}g_{cb} - g_{db}g_{ca})/(n-1)(n-2) = 0,$$

where A_{dcba} is given by (11.3) and

(11.5) $A_{da} = g^{cb} A_{dcba}$ $\quad A = g^{da}A_{da}$

Theorem B:

Let M^n (n >3) be a totally umbilical submanifold of a conformally flat Riemannian manifold M^m (m>3) then M^n is conformally flat.

XII. The Main Theorem and its Applications:

If M^m (m>3) is conformally flat, then the Weyl conformal curvature tensor $C_{kjih} = 0$ and we have

(12.1) $\nabla_j C_{ih} - \nabla_i C_{jh} = 0$

where $C_{ih} = - k_{ih}/(m-2) + Rg_{ih}/2(m-1)(m-2)$. we set

(12.2) $C_{cba} = \nabla_c C_{ba} - \nabla_b C_{ca}$

where C_{ba} is defined by a formula similar to the one for C_{ij} in (12.1)

Theorem 12.1:

Let M^n ($n \geqq 3$) be a submanifold of a Conformally flat Riemannian manifold M^m (m>3). Then

(12.3) $$C_{cba} = (\nabla_b A_{ca} - \nabla_c A_{ba})/(n-2) - \{(\nabla_b A) g_{ca} - (\nabla_c A) g_{ba}\}/2(n-1)(n-2) + \left(L_{bx} H^x_{ca} - L_{cx} H^x_{ba}\right),$$

where A_{ca} is given by (11.5) and

(12.4) $$L_{cx} = C_{ji} B^j_c C^i_x$$

Proof:

Since M^m is conformally flat, we have

(12.5) $K_{kjih} = g_{hj} C_{ki} - g_{hk} C_{ji} + C_{hj} g_{ki} - C_{hk} g_{ji}$

Transvecting (12.5) with $B^k_d B^j_c B^i_b B^h_a$ and using (11.2) we get

(12.6) $K_{dcba} = A_{dcba} + g_{ca} P_{db} - g_{da} P_{cb} + P_{ca} g_{db} - P_{da} g_{cb}$,

where we have set $P_{ca} = B^j_c B^h_a C_{jh}$. Transvecting (12.6) with g^{da} and the resulting equation with g^{cb} we get

(12.7) $K_{cb} = A_{cb} + (2-n) P_{cb} - P g_{cb}$, $K = A + 2(1-n)P$,

where $P = g^{cb} P_{cb}$ from (12.7) we get

$C_{cb} = P_{cb} - A_{cb}/(n-2) + A g_{cb}/2(n-1)(n-2)$

Hence

(12.8) $$C_{cba} = \nabla_c P_{ba} - \nabla_b P_{ca} - \{\nabla_c A_{ba} - \nabla_b A_{ca}\}/(n-2) + \{(\nabla_c A) g_{ba} - (\nabla_b A) g_{ca}\}/2(n-1)(n-2)$$

Now transvecting (12.1) with $B^j_c B^i_b B^h_a$ we obtain

(12.9) $$\nabla_c p_{ba} - \nabla_b p_{ca} = L_{bx} H^x_{ca} - L_{cx} H^x_{ba}$$

where L_{cx} is defined by (12.4) from (12.8) and (12.9), we obtain (12.3)

XIII. K-Contact Riemannian Manifold:

An n-dimensional K-contact Riemannian manifold M is a differentiable manifold with a contact metric structure (φ, ξ, η, g) such that ξ is a killing vector filed. Therefore, with respect to an arbitrary coordinate neighbourhoods of M, we have the following conditions:

$$\xi^\lambda \eta_\lambda = 1, \varphi^\lambda_\mu \xi^\mu = 0, \varphi^\lambda_\mu \eta_\lambda = 0, \varphi^\lambda_\mu \varphi^\mu_\nu = -\delta^\lambda_\nu + \eta_\nu \xi^\lambda, g_{\lambda\mu} \xi^\lambda = \eta_\mu, ^{2)}$$

where the matrix $\left(\varphi^\mu_\lambda\right)$ is of rank n-1. Hereafter, we write η instead of ξ. It is well-known that a K-contact Riemannian manifold is orientable and odd dimensional.

On a K-contact Riemannian manifold the following identities hold good .

(13.1) $$\nabla_\lambda \varphi^\lambda_\mu = (n-1)\eta_\mu, \quad \nabla_\lambda \varphi_{\mu\nu} + R_{\varepsilon\lambda\mu\nu} \eta^\varepsilon = 0,$$

(13.2) $$R_{\lambda\mu\nu\varepsilon} \eta^\lambda \eta^\varepsilon = g_{\mu\nu} - \eta_\mu \eta_\nu, \quad R_{\lambda\varepsilon} \eta^\varepsilon = (n-1)\eta_\lambda,$$

where ∇_λ is the covariant derivative with respect to the metric g and $R_{\varepsilon\nu\mu\lambda}$ and $R_{\mu\lambda}$ denote the Riemannian curvature tensor and the Ricci tensor respectively.

Next, the exterior differential du and co differential δu of p-form u are given by

$$(du)_{\mu\lambda_1 \ldots \lambda_p} = \nabla_\mu u_{\lambda_1 \ldots \lambda_p} - \sum_{i=1}^{P} \nabla_{\lambda i} u_{\lambda i} \ldots \overset{i}{\hat{\mu}} \ldots \lambda_p, \qquad P \geqq 1,$$

$$(du)_\lambda = \nabla_\lambda u, \qquad P = 0,$$

$$(\delta u)_{\lambda_2} \ldots_{\lambda_p} = -\nabla^\lambda u_{\lambda\lambda_2 \ldots \lambda_p}, \qquad P \geqq 1,$$

$$\delta u = 0, \qquad P = 0,$$

The Laplacian is given by $\Delta = d\delta + \delta d$. for a p-form u we have explicitly

$$(\Delta u)_{\lambda_1 \ldots\ldots \lambda_p} = -\nabla^{\lambda}\nabla_{\lambda} u_{\lambda_1 \ldots\ldots \lambda_\rho} + \sum_{i=1}^{P} R^{\sigma}_{\lambda_i} u_{\lambda_1 \ldots \overset{i}{\hat{\sigma}} \ldots \lambda_p} + \sum_{j<i}^{P} R_{\lambda_j \lambda_i}{}^{\rho\sigma} u_{\lambda_1 \ldots \overset{j}{\hat{\rho}} \ldots \overset{i}{\hat{\sigma}} \ldots \lambda_p}, \quad P \geqq 2,$$

$$(\Delta u)_{\lambda} = -\nabla^{\alpha}\nabla_{\alpha} u_{\lambda} + R^{\varepsilon}_{\lambda} u_{\varepsilon}, \qquad P = 1,$$

$$\Delta f = -\nabla^{\alpha}\nabla_{\alpha} f, \qquad P = 0,$$

XIV. Invariant Submanifolds in a k-contact Riemannian Manifold:

Theorem 14.1:

For an invariant submanifold M of a k-contact Riemannian manifold $\overline{M}$, if the vector field X on M is orthogonal to x, we have

$$\overline{\phi}\overline{R}(\overline{\xi}, N_A)X = -\overline{R}(\overline{\xi}, N_A)\overline{\phi}X.$$

Proof:

First, we calculate $\overline{\nabla}_{N_A}(\overline{\phi}^2 X)$ and find

$$\overline{\nabla}_{N_A}(\overline{\phi}^2 X) = (\overline{\nabla}_{N_A}\overline{\phi})\overline{\phi}X + \overline{\phi}(\overline{\nabla}_{N_A}(\overline{\phi}X)) = (\overline{\nabla}_{N_A}\overline{\phi})\overline{\phi}X + \overline{\phi}(\overline{\nabla}_{N_A}\overline{\phi})X + \overline{\phi}^2(\overline{\nabla}_{N_A}X)$$

Using

$$\overline{\phi}\overline{\xi} = 0, \overline{\eta}(\overline{\xi}) = 1, \overline{\phi}^2 = -I + \overline{\eta}\otimes\overline{\xi},$$

$$\overline{g}(\overline{\phi}\overline{X}, \overline{\phi}\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \overline{\eta}(\overline{X})\overline{\eta}(\overline{Y}), \overline{g}(\overline{\phi}\overline{X}, \overline{Y}) = d\overline{\eta}(\overline{X}, \overline{Y}), \overline{\eta}(\overline{X}) = \overline{g}(\overline{\xi}, \overline{X})$$

for any vector fields $\overline{X}$ and $\overline{Y}$ on $\overline{M}$.

$\overline{M}$ is called a k-contact Riemannian manifold, if $\overline{\xi}$ is a killing vector field. Then, we have $\overline{\nabla}_{\overline{X}}\overline{\xi} = \overline{\phi}\,\overline{X}$ and

$$\overline{R}(\overline{X}, \overline{\xi})\overline{Y} = (\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y},$$

We have

$$\overline{\nabla}_{N_A}(-X + \overline{g}(X, \overline{\xi})\overline{\xi}) = \overline{R}(N_A, \overline{\xi})\overline{\phi}X + \overline{\phi}\overline{R}(N_A, \overline{\xi})X - \overline{\nabla}_{N_A}X + \overline{g}(\overline{\nabla}_{N_A}X, \overline{\xi})\overline{\xi}$$

which implies that $0 = \overline{R}(N_A, \overline{\xi})\overline{\phi}X + \overline{\phi}\overline{R}(N_A, \overline{\xi})X + \overline{g}(\overline{\nabla}_{N_A}X, \overline{\xi})\overline{\xi}$

on the other hand, by the assumption, we have

$\overline{g}(\overline{\nabla}_{N_A}X, \overline{\xi}) = \overline{\nabla}_{N_A}(\overline{g}(X, \overline{\xi})) - \overline{g}(X, \overline{\phi}N_A) = 0.$ Consequently, we obtain

$$\overline{\phi}\overline{R}(\overline{\xi}, N_A)X = -\overline{R}(\overline{\xi}, N_A)\overline{\phi}X.$$

Theorem 14.2:

Any invariant submanifold M of a k-contact Riemannian manifold M is minimal.

Proof:

First, using $\overline{\nabla}_X Y = \nabla_X Y + \sum_A h_A(X, Y)N_A$ we calculate $\overline{\nabla}_X(\phi Y)$ and find

$$\overline{\nabla}_X(\phi Y) = \nabla_X(\phi Y) + \sum_A h_A(X, \phi Y)N_A = (\nabla_X\phi)Y + \phi(\nabla_X Y) + \sum_A h_A(X, \phi Y)N_A$$

And we have

$$\overline{\nabla}_X(\phi Y) = \nabla_X(\phi Y) = \overline{\nabla}_X(\phi Y) + \overline{\phi}(\overline{\nabla}_X Y) = (\overline{\nabla}_X\phi)Y + \overline{\phi}(\nabla_X Y + \sum_B h_B(X, Y)N_B)$$

$$= (\overline{\nabla}_X\overline{\phi})Y + \phi(\nabla_X Y) + \sum_B h_B(X, Y)\overline{\phi}N_B$$

By the definition of k-contact Riemannian manifold, we get

$$(\nabla_X\phi)Y+\sum_A h_A(X,\phi Y)N_A=\bar{R}(X,\bar{\xi})Y+\sum_B h_B(X,Y)\bar{\phi}N_B,$$ from which,

$$h_C(X,\phi Y)=\bar{g}(\bar{R}(X,\bar{\xi})Y,N_C)+\sum_B h_B(X,Y)\bar{g}(\bar{\phi}N_B,N_C)$$

Replacing *Y* by ϕY, we find,

$$h_C(X,\phi^2Y)=g\left(H_C X,\phi^2Y\right)=\bar{g}\left(\bar{R}\left(X,\bar{\xi}\right)\phi Y,N_C\right)+\sum_B g\left(H_B X,\phi Y\right)\bar{g}\left(\bar{\phi}N_B,N_C\right)$$

using, $H_A\xi=0$

we have

$$-g(H_C X,Y)=\bar{g}\left(\bar{R}\left(X,\bar{\xi}\right)\phi Y,N_C\right)-\sum_B g\left(\phi H_B X,Y\right)\bar{g}\left(\bar{\phi}N_B,N_C\right)$$

Here taking a ϕ -basis (ξ, E_1, ϕE_1, E_2, ϕE_2,....,E_m, ϕE_m) we have

$$-trH_c=\sum_{i=1}^{m}\bar{g}\left(\bar{R}\left(E_i,\bar{\xi}\right)\phi E_i,N_C\right)+\sum_{i=1}^{m}\bar{g}\left(\bar{R}\left(\phi E_i,\bar{\xi}\right)\phi^2E_i,N_C\right)-\sum_B\left(tr\phi H_B\right)\bar{g}\left(\bar{\phi}N_B,N_C\right)$$

$$=\sum_{i=1}^{m}\bar{g}\left(\bar{R}\left(E_i,\bar{\xi}\right)\phi E_i,N_C\right)-\sum_{i=1}^{m}\bar{g}\left(\bar{R}\left(\phi E_i,\bar{\xi}\right)E_i,N_C\right)-\sum_B\left(tr\phi H_B\right)\bar{g}\left(\bar{\phi}N_B,N_C\right)$$

However, since ϕ is skew-symmetric and H_A is symmetric $tr\phi H_B$ vanishes identically and hence, we get

$$-trH_C=\sum_{i=1}^{m}\left[\bar{g}(\bar{R}(E_i,\bar{\xi})\phi E_i,N_C)-\bar{g}(\bar{R}(\phi E_i,\bar{\xi})E_i,N_C)\right]$$

$$=\sum_{i=1}^{m}\left[\bar{g}(\bar{R}(E_i,\bar{\xi})\phi E_i,N_C)-\bar{g}(\bar{R}(\bar{\xi},\phi E_i)E_i,N_C)\right]$$

By virtue of the Bianchi's identity, we get

$$trH_C=\sum_{i=1}^{m}\bar{g}(\bar{R}(\phi E_i,E_i)\bar{\xi},N_C)$$

On the other hand, from theorem 14.1, we have

$$\bar{g}(\bar{R}(\phi E_i,E_i)\bar{\xi},N_C)=\bar{g}(\bar{R}(\bar{\xi},N_C)\phi E_i,E_i)=-\bar{g}(\bar{\phi}\bar{R}(\bar{\xi},N_C)E_i,E_i)$$

$$=\bar{g}(\bar{R}(\bar{\xi},N_C)E_i,\phi E_i)=\bar{g}(\bar{R}(E_i,\phi E_i)\bar{\xi},N_C)$$

Therefore we get $\bar{g}(\bar{R}(\phi E_i,E_i)\bar{\xi},N_C)=0$, Hence we obtain $trH_C=0$

XV. Invariant Submanifolds Immersed in an Almost Paracontact Riemannian Manifold:

An *n*-dimensional differentiable manifold *M* of class C^∞ is called an almsot paracontact Riemannian manifold [9], if their exist in *M* a tansor field $\varphi_\mu^{[\lambda 2]}$, a positive definite Riemannian metric $g_{\mu\lambda}$ and vector fields ξ^λ and η_λ satisfying.

(15.1) (a) $\eta_\alpha\xi^\alpha=1$, (b) $\varphi_\alpha^\lambda\varphi_\mu^\alpha=\delta_\mu^\lambda-\eta_\mu\xi^\lambda$

$\eta_\lambda=g_{\lambda\alpha}\xi^\alpha$, $g_{\beta\alpha}\varphi_\mu^\beta\varphi_\lambda^\alpha=g_{\mu\lambda}-\eta_\mu\eta_\lambda$,

The set $\left(\varphi_\mu^\lambda,\xi^\lambda,\eta_\lambda,g_{\mu\lambda}\right)$ is called an almost paracontact Riemannian structure.

In the manifold *M*, the following relations hold good [3].

(15.2) (a) $\varphi_\alpha^\lambda\xi^\alpha=0,\quad \eta_\alpha\varphi_\mu^\alpha=0,$

(b) $\varphi_{\mu\lambda}=\varphi_{\lambda\mu}\qquad(\varphi_{\lambda\mu}=g_{\lambda\alpha}\varphi_\mu^\alpha)$

We consider an m-dimensional Riemannian manifold *V* with local coordinates $\{Y^h\}$ immersed in the almost paracontact Riemannian manifold M with local co-ordinates $\{X^\lambda\}$ and denote the immersion by $X^\lambda=X^\lambda$

(Y^h). We put $B_i^\lambda = \partial X^\lambda / \partial Y^i$. The induced Riemannian metric is given by $g_{ji} = g_{\beta\alpha} B_j^\beta B_i^\alpha$. We denote by $N_x^\lambda n - m$ mutually orthogonal unit normals to *V*.

We assume that the submanifold *V* of *M* is φ invariant, then we have.

(15.3) $$\varphi_\alpha^\lambda B_i^\alpha = \varphi_i^t B_t^\lambda,$$

Where φ_i^t is a tensor field on *V*. It follows from (15.3) that $\varphi_{\beta\alpha} N_x^\beta B_i^\alpha = 0$ which implies that, $\varphi_\beta^\lambda N_x^\beta$ is normal to *V*. Thus, we put:

(15.4)

a

where γ_{xy} are functions on *V*. The vector ξ^λ can be expressed as follows:

(15.5) $$\xi^\lambda = \xi^t B_t^\lambda + \sum_x \alpha_X N_X^\lambda$$

where ξ^t and α_x are a vector field and functions on *V* respectively

Contracting (15.3) and (15.5) with $B_{j\lambda} (= g_{\lambda\alpha} B_j^\alpha)$ respectively and making use of (15.2) b), we get

(15.6) $$\varphi_{ji} = \varphi_{\beta\alpha} B_j^\beta B_i^\alpha = \varphi_{ij} \left(\varphi_{ji} = g_{it} \varphi_j^t\right),$$

(15.7) $$\xi^h = B_\alpha^h \xi^\alpha \left(B_\alpha^h = g^{ht} B_{t\alpha}\right)$$

from(15.4) and (15.5), we have

$$\gamma_{xy} = \varphi_{\beta\alpha} N_X^\beta N_y^\alpha = \gamma_{yx},\ \alpha_X = N_{X\beta} \xi^\beta \left(N_{X\beta} = g_{\beta\lambda} N_X^\lambda\right)$$

Contracting (15.3), (15.4) and (15.5) with φ_λ^μ respectively and using (15.1) b), (15.2)a), (15.4), (15.5), (15.7) and the above equations, we find

(15.8) (a) $$\varphi_t^h \varphi_i^t = \delta_i^h - \eta_i \xi^h (\eta_i = g_{it} \xi^t),$$

(b) $$\alpha_x \eta_i = 0$$

XVI. An Invariant Submanifold Immersed in an Almost Paracontact Riemannian Manifold with Vanishing Torsion Tensor:

Differentiating (15.3) and (15.5) covariantly along *V* respectively and making use of Gauss and Weignarten's equations

$$\nabla_j B_i^\lambda = \sum_X h_{jiX} N_X^\lambda$$

$$\nabla_j N_X^\lambda = -h_{jX}^t B_t^\lambda + \sum_y l_{jXy} N_y^\lambda \left(h_{jX}^t = g^{ti} h_{jiX}\right),$$

Where ∇_j denotes covariant differentiation with respect to g_{ji} h_{jiX} I_{jXY} are the so-called second and third fundamental tensors respectively and satisfy

$h_{jiX} = h_{ijX}$, $I_{jXY} = -I_{jyX}$,

we obtain.

$$(\nabla_j \varphi_\alpha^\lambda) B_i^\alpha = (\nabla_j \varphi_i^s) B_s^\lambda + \sum_Y (\varphi_i^s h_{jsy} - \sum_x (h_{jiX} \gamma_{XY}) N_y^\lambda,$$

(16.1) $$\left[\nabla_j \xi^\lambda = (\nabla_j \xi^S - \sum_x \alpha_X h_{jX}^S) B_S^\lambda + \sum_y (\nabla_j \alpha_y + \xi^S h_{jsy} + \sum_X (\alpha_X I_{jXy})\right] N_y^\lambda,$$

We now assume that the so-called torsion tensor $N_{\nu\mu}$ introduced by *I*. Satō [9] vanishes. Then we have

(16.2) $$N_{\nu\mu}^\lambda = \varphi_\nu^\alpha \left(\nabla_\alpha \varphi_\mu^\lambda - \nabla \mu \varphi_\alpha^\lambda\right) - \varphi_\mu^\alpha \left(\nabla_\alpha \varphi_\nu^\lambda - \nabla_\nu \varphi_\alpha^\lambda\right) + \eta_\mu \nabla_\nu \xi^\lambda - \eta_\nu \nabla_\mu \xi^\lambda = 0$$

Where ∇_μ denotes covariant differentiation with respect of $g_{\mu\lambda}$. Contracting (16.2) with $B_j^\nu B_i^\mu$ and using (16.3), (16.7), (16.8) a) and (16.1), we obtain

(16.3) $$\left\{\varphi_j^t(\nabla_i\varphi_i^S - \nabla_i\varphi_t^S) - \varphi_i^t(\nabla_t\varphi_j^S - \nabla_j\varphi_t^S) + \eta_i\nabla_j\xi^S - \eta_j\nabla_i\xi^S - \sum_X \alpha_X(\eta_i h_{jX}^S - \eta_j h_{iX}^S)\right\}B_S^\lambda + \sum_Y\left\{\eta_i(\nabla_j\alpha_Y + \sum_X\alpha_X l_{jXY}) - \eta_j(\nabla_i\alpha_Y + \sum_X\alpha_X l_{iXY})\right\}N_Y^\lambda = 0$$

first consider the case (I). In this case, from (16.3), we find

$$\varphi_j^t(\nabla_t\varphi_i^S - \nabla_i\varphi_t^S) - \varphi_i^t(\nabla_t\varphi_j^S - \nabla_j\varphi_t^S) = 0$$

that is, the Nijenhuis tensor of φ_i^h vanishes

we next consider the case (II). In this case, from (16.3), we find

$$\varphi_j^t(\nabla_t\varphi_i^S - \nabla_i\varphi_t^S) - \varphi_i^t(\nabla_t\varphi_j^S - \nabla_j\varphi_t^S) + \eta_i\nabla_j\xi^S - \eta_j\nabla_i\xi^S = 0$$

that is the torsion tensor of *V* vanishes.

XVII. Invariant Submanifold Immersed in a Paracontact Riemannian Manifold

An almost Paracontact Riemannian manifold *M* with structure $(\varphi_\mu^\lambda, \xi^\lambda, \eta_\lambda, g_{\mu\lambda})$ is called a paracontract Riemannian manifold [10] if the following relation holds good.

$2\varphi_{\mu\lambda} = \nabla_\mu \eta_\lambda + \nabla_\lambda \eta_\mu$

We assume that *M* is a Paracontact Riemannian manifold Contracting the above equation with $B_j^\mu B_i^\lambda$ using. (15.3) and (16.1), we can find

$$2\varphi_{ji} = (\nabla_j\eta_i + \nabla_i\eta_j) - 2\sum_X \alpha_X h_{jiX}$$

Hence, we observe that

ξ^λ is normal to *V*. In this case, *V* admits an almost Product Riemannian structure (φ_i^h, g_{ji}) whenever φ_i^h is non-trivial.

We get

$$\varphi_{ji} = -\sum_X \alpha_X h_{jiX} \text{ and using,}$$

ξ^λ is normal to *V*. In this case, *V* admits an almost Paracontact Riemannian structure $(\varphi_i^h, \xi^h, \eta_i, g_{ji})$

We get

$2\varphi_{ji} = \nabla_j \eta_i + \nabla_i \eta_j$

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