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Construction and Convergence of C-H Combined Mean Method for Multiple Polynomial Zeros

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Abstract *In this article, we have constructed an iterative method of third order for solving polynomial equations with multiple polynomial zeros. We have combined two well known third order methods Chebyshev and Halley for this construction purpose. We have proposed some local convergence theorems of this C-H Combined Mean Method to establish the computation of a polynomial with known multiple zeros. For the establishment of this local convergence theorem, the main role is performed by a function, termed as the function of initial conditions. Here the initial conditions uses the information only at the initial point. We have used the notion of gauge function which also plays an important role in establishing the convergence theorem. Here we have used two types of initial conditions over an arbitrary normed field and established local convergence theorems of the constructed C-H Combined mean method. The error estimations are also found in our convergence analysis.*

Keywords: *Local convergence; Halley method; Chebyshev method; Initial conditions; Multiple zeros; Normed field.*
MSC: 65H04, 12Y05.

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I. Introduction

In the literature of iterative method for solving non-linear equations, Chebyshev and Halley method are among the efficient methods in solving non-linear equation along with Newton's and Super Halley method. Recently Osada ([8]), Neta ([9]), Chun and Neta ([10]), Ren and Argyros ([11]) and many others have studied iterative method for solving an equation of non linear type with multiple zero Chebyshev iterative method for multiple zeros ([4], [2], [3]) as following

$$C(w) = \begin{cases} w - \frac{p^2}{2}\frac{g(w)}{g'(w)}\left(\frac{3-p}{p} + \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right), & \text{if } g'(w) \neq 0, \\ w, & \text{otherwise.} \end{cases} \tag{1}$$

Halley method for multiple zeros ([1], [2], [3]) is defined by

$$H(w) = \begin{cases} w - \left(\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1}, & \text{if} g(w) \neq 0, \\ w, & \text{otherwise.} \end{cases} \tag{2}$$

The domain of the function H is D_H, defined as follows

$$D_H = \{w \in F : g(w) \neq 0 \Rightarrow g'(w) \neq 0, \frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \neq 0\}. \tag{3}$$

Here, we have combined the above two methods to construct the C-H Combined mean method.

Recently, Proinov [[5], [6]] and later Ivanov [7] have introduced convergence theorems for the Picard iterative scheme given as below

$$w_{k+1} = Tw_k, \ k = 0, 1, 2, \ldots, \tag{4}$$

where $T : D \to M$ is the function of iteration defined in a metric space M and $D \subset M$.

Here, we investigate the convergence of the C-H Combined mean method for polynomial zeros which are multiple in nature with the help of the same initial conditions as in Proinov [[5], [6]] and Ivanov [7]. In this paper section 2 is devoted to the preliminaries, necessary in establishing our results. Construction of our C-H Combined mean method is presented in section 3. We have devoted Section 4 in establishing two types of local convergence analysis of the proposed C-H combined mean method.

2 Preliminaries

In this paper, J will be treated as an interval in the real line containing zero. $S_l(u)$ is the polynomial defined as

$$S_l(u) = 1 + u + u^2 + \ldots + u^{l-1}. \tag{5}$$

If $l = 0$, then here we take $S_l(u) = 0$. Here, we will use that 0^0 equal to 1.

Definition 2.1. *([5]) A function $\varphi : J \to R_+$ is called quasi-homogeneous of order $r \geq 0$ on J if there exists a non decreasing function $\Psi : J \to R_+$ such that*

$$\varphi(u) = u^r \Psi(u) \; for \; all \; u \in J.$$

Following are some properties of above defined functions.

(P1) A function g is quasi-homogeneous function of degree $r = 0$ on J if and only if g is non-decreasing on J.

(P2) If f and g are quasi-homogeneous functions of degree $r \geq 0$ and $s \geq 0$ on J, then fg is quasi-homogeneous of degree $r + s$ on J.

(P3) If two functions f and g are quasi-homogeneous of degree $r \geq 0$ on J, then $f + g$ is also quasi-homogeneous of degree r on J.

We will use these properties in proving Lemmas and Theorems in the later section.

Definition 2.2. *([6]) A function $\varphi : J \to R_+$ is called gauge function of order $r \geq 1$ on J if it satisfies the following conditions:*

(i) φ is quasi-homogeneous function of degree r on J.

(ii) $\varphi(u) \leq u$ for all $u \in J$.

A gauge function φ of order r on J is said to be a strict gauge function if the last inequality is strict whenever $u \in J \setminus \{0\}$.

Lemma 2.1. *([6]) If $\varphi : J \to R_+$ is a quasi-homogeneous function of degree $r \geq 1$ on an interval J and $R \in J \setminus \{0\}$ is a fixed point of φ, then φ is a gauge function of order r on $[0, R]$. Moreover, if $r > 0$, then φ is a strict gauge function on $[0, R)$.*

Definition 2.3. *([5]) Let $T : D \subset M \to M$ be a map on an arbitrary set M. A function $I : D \to R_+$ is said to be a function of initial conditions of T (with gauge function φ on J) if there exists a function $\varphi : J \to J$ such that*

$$I(T(w)) \leq \varphi(I(w)) \; with \; all \; w \in D \; with \; Tx \in D \; and \; I(w) \in J. \tag{6}$$

Definition 2.4. *([5]) Let $T : D \subset M \to M$ be a map on an arbitrary set M and $I : D \to R_+$ be a function of initial conditions of T with gauge function on J. Then, a point $w \in D$ is said to be an initial point of T if $I(w) \in J$ and all of the iterates $T^k w (k = 0, 1, 2, \ldots)$ are well defined and belong to D.*

Definition 2.5. *([6]) Let $T : D \subset M \to M$ be an operator in a normed space $(M, |.|)$, and let $I : D \to R_+$ be a function of initial conditions of T with gauge function on J. Then T is said to be an iterated contraction with respect to I at a point $\zeta \in D$ (with control function ϑ) if $I(\zeta) \in J$ and*

$$|Tx - \zeta| \leq \vartheta(I(w))|w - \zeta| \; for \; all \; w \in D \; with \; I(w) \in J, \tag{7}$$

where $\vartheta : J \to [0, 1)$ is a non-decreasing function.
We will use the following two theorems of Ivanov ([7]) to establish our result.

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Theorem 2.1. *([7]) Let $T : D \subset M \to M$ be an iteration function, $\zeta \in F$ and $I : F \to R_+$ defined by (13). Suppose $\phi : J \to R_+$ is a quasi-homogeneous function of degree $p \geq 0$ and for each $w \in F$ with $I(w) \in J$, the following two conditions are satisfied:*

(i) w belongs to the set D;

(ii) $|Tx - \zeta| \leq \phi(I(w))|w - \zeta|$.

Let also $w_0 \in F$ be an initial guess such that

$$I(w_0) \in J \text{ and } \phi(I(w_0)) < 1, \tag{8}$$

then the following statements hold.

(i) Then the Picard iteration (4) is well defined and converges to ζ with order $r = p + 1$.

(ii) For all $k \geq 0$, we have the following error estimates:
$|w_{k+1} - \zeta| \leq \mu^{r^k}|w_k - \zeta|$ *and* $|w_k - \zeta| \leq \mu^{s_k(r)}|w_0 - \zeta|$,
where $\mu = \phi(I(w_0))$.

(iii) The Picard iteration (4) converges to ζ with Q-order $r = p + 1$ and with the following error estimates:

$|w_{k+1} - \zeta| \leq (Rd)^{1-r}|w_k - \zeta|^r$ *for all* $k \geq 0$,

where R is the minimal solution of the equation $\phi(u) = 1$ in the interval $J \setminus \{0\}$.

Theorem 2.2. *([7]) Let $T : D \subset M \to M$ be an iteration function, $\zeta \in F$ and $I : D \subset F \to R_+$ defined by (29). Suppose $\vartheta : J \to R_+$ is a nonzero quasi-homogeneous function of degree $p \geq 0$ and for each $w \in F$ with $I(w) \in J$, the following two conditions are satisfied:*

(i) w belongs to the set D;

(ii) $|Tx - \zeta| \leq \vartheta(I(w))|w - \zeta|$.

Let also, $w_0 \in F$ be an initial guess such that

$$I(w_0) \in J \text{ and } \vartheta(I(w_0)) \leq \psi(I(w_0)), \tag{9}$$

where ψ is defined by

$$\psi(u) = 1 - u(1 + \vartheta(u)).$$

Then the Picard iteration (4) is well defined and converges to ζ with the following error estimates:

$$|w_{k+1} - \zeta| \leq \theta\mu^{r^k}|w_k - \zeta| \text{ and } |w_k - \zeta| \leq \theta^k\mu^{s_k(r)}|w_0 - \zeta| \text{ for all } k \geq 0, \tag{10}$$

where $\mu = \frac{\vartheta(I(w_0))}{\psi(I(w_0))}$ and $\theta = \psi(I(w_0))$. In addition, if the second inequality in (9) is strict, then the order of convergence of Picard iteration (4) is at least $r = p + 1$

3 Recurrence relation for the method

Here, we have derived a relation of the C-H Combined Mean Method method combining the two third order iterative method namely Chebyshev and Halley method. For $g(w) \neq 0$ and $g'(w) \neq 0$, we define the C-H combined mean method as follows

$$\begin{aligned} T(w) &= \frac{1}{2}C(w) + \frac{1}{2}H(w) \\ &= w - \frac{p^2}{4}\frac{g(w)}{g'(w)}\left(\frac{3-p}{p} + \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right) - \frac{1}{2}\left(\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1} \end{aligned}$$

Thus our C-H combined mean method is of the following form

$$T(w) = \begin{cases} w - \frac{p^2}{4}\frac{g(w)}{g'(w)}\left(\frac{3-p}{p} + \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right) - \frac{1}{2}\left(\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1}, & \text{if } g(w) \text{ and } g'(w) \neq 0, \\ w, & \text{otherwise.} \end{cases} \tag{11}$$

The domain of the C-H Combined Mean iteration function T (11) is the set D, Which is defined below:

$$D = \{w \in F : g(w) \neq 0 \text{ and } g'(w) \neq 0 \Rightarrow \frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \neq 0\}. \tag{12}$$

4 Local Convergence of Combined Mean Method

Let assume that $g \in F[w]$ be a polynomial which having degree $q(\geq 2)$, such that all the zeros of g are in F, and also let $\zeta \in F$ be a zero of the polynomial g, multiplicity being p.
Here $(F, |.|)$ denotes a field having a norm and $F[w]$ is the ring of polynomial on the field F.
Here, we examine the convergence of C-H Combined mean method (11) with the help of function of initial conditions I, which is a map from D to R_+ and is defined as follows:

$$I(w) = I_g(w) = \frac{|(w-\zeta)|}{d}, \tag{13}$$

here d represents the distance from the zero ζ to the closest zero of g other than ζ; if ζ is a only zero of g then we set $I(w) = 0$.

Lemma 4.1. *Let $g \in F[w]$ be a $q(\geq 2)$ degree polynomial having all zeros in F, where F is a field. If $\zeta_1, \dots, \zeta_s$, are the all zeros of g, multiplicity of the zeros being $p_1, \dots, p_s$, respectively. Then*

(i) If $w \in F$ be such that for those w, $g(w) \neq 0$, then for any one of $i = 1, \dots, s$, we have the following $\frac{g'(w)}{g(w)} = \frac{p_i+\gamma_i}{w-\zeta_i}$, where $\gamma_i = (w-\zeta_i)\sum_{j\neq i} \frac{p_j}{w-\zeta_j}$.

(ii) If $w \in F$ is not a zero of g and g', then for any $i = 1, \dots, s$, we have $\frac{g''(w)}{g'(w)} = \frac{(p_i+\gamma_i)^2-(p_i+\delta_i)}{(w-\zeta_i)(p_i+\gamma_i)}$, where $\delta_i = (w-\zeta_i)^2 \sum_{j\neq i} \frac{p_j}{(w-\zeta_j)^2}$.

Proof.

$$(i)\ From\ \frac{g'(w)}{g(w)} = \sum_{j=1}^{s} \frac{p_j}{w-\zeta_j},\ we\ have$$

$$\begin{aligned} \frac{g'(w)}{g(w)} &= \sum_{j=1}^{s} \frac{p_j}{w-\zeta_j} \\ &= \frac{p_i}{w-\zeta_i} + \sum_{j\neq i} \frac{p_j}{w-\zeta_j} \\ &= \frac{p_i+\gamma_i}{w-\zeta_i},\ where\ \gamma_i = (w-\zeta_i)\sum_{j\neq i} \frac{p_j}{w-\zeta_j}. \end{aligned}$$

Which proves the first part of the lemma.
(ii)Using the above identity and the following two identities

$$\frac{g''(w)}{g'(w)} = \frac{g'(w)}{g(w)} - \frac{g(w)}{g'(w)} \sum_{j=1}^{s} \frac{p_j}{(w-\zeta_j)^2}\ and\ \sum_{j=1}^{s} \frac{p_j}{(w-\zeta_j)^2} = \frac{p_i+\delta_i}{(w-\zeta_i)^2},$$

we get

$$\frac{g''(w)}{g'(w)} = \frac{(p_i+\gamma_i)^2-(p_i+\delta_i)}{(w-\zeta_i)(p_i+\gamma_i)},\ where\ \delta_i = (w-\zeta_i)^2 \sum_{j\neq i} \frac{p_j}{(w-\zeta_j)^2}.$$

□

Lemma 4.2. *Let $w, \zeta \in F$ and $\zeta_1, \dots, \zeta_s \in F$ be the list of all zeros of g which are other than ζ, then for any of $i = 1, \dots, s$, the inequality listed below is accurate.*

$$|w-\zeta_j| \geq (1-I(w))d, \tag{14}$$

where $I : F \to R_+$ is defined by (13).

Proof. According to the definition of d we have $d \leq |\zeta - \zeta_j|$ for all $j = 1, \dots, s$.
So using above and triangle inequality we have the following

$$|w-\zeta_j| = |\zeta - \zeta_j + w - \zeta| \geq |\zeta-\zeta_j| - |w-\zeta| \geq (1-I(w))d.$$

4.1 First Kind of Local Convergence theorem

Here, $F[w]$ is the ring of polynomials over the field F. Let g be a polynomial of degree $q(\geq 2)$, which is in $F[w]$. In this section of the article we will establish the convergence of the C-H Combined mean method (11) using the function of initial condition $I : D \to R_+$ which is defined in (13).
Next, we define the functions ϕ_c and ϕ_h.

$$\phi_c(u) = \frac{2(q-p)^3 u + p(q-p)(3q-2p)(1-u)}{2(p-qu)^3} u^2, \tag{15}$$

$$\phi_h(u) = \frac{q(q-p)}{q(3p-q)u^2 - 2p(p+q)u + 2p^2} u^2. \tag{16}$$

Easily we can show that ϕ_c quasi-homogeneous on the clo-open interval $[0, \frac{p}{q})$ of degree 2. Clearly, the function ϕ_h is a quasi-homogeneous on the clo-open interval $[0, \frac{2p}{q+p+\sqrt{(q-p)(5q-p)}})$ of degree 2.

So, we can now define the function $\phi : [0, \frac{2p}{q+p+\sqrt{(q-p)(5q-p)}}) \to R_+$ defined by

$$\phi(u) = \frac{\phi_c(u)}{2} + \frac{\phi_h(u)}{2}. \tag{17}$$

As $\phi_c(u)$ and $\phi_h(u)$ are both second degree quasi homogeneous function, so by property (P3), ϕ is also quasi-homogeneous of the same degree 2 in the clo-open interval $[0, \frac{2p}{q+p+\sqrt{(q-p)(5q-p)}})$.

Lemma 4.3. *Suppose that $g(w) \in F[w]$ be a $q(\geq 2)$ degree polynomial which splits over F, and let $\zeta \in F$ be a multiple zero of $g(w)$, multiplicity being p. Let $w \in F$ satisfies the following*

$$I(w) < \tau_1 = \frac{2p}{q+p+\sqrt{(5q-p)(q-p)}}, \tag{18}$$

where I is defined by (13) and τ_1 is defined in (18). Then the following two statements (i) and (ii) are true.

(i) w is in D, the domain of the C-H Combined mean method and is defined in (12).

(ii) $\|Tx - \zeta\| \leq \phi(I(w))\|w - \zeta\|$, where ϕ defined in (17).

Proof. Let $w \in F$ satisfy the inequality (18). If any of $p = q$ or $w = \zeta$ or both are true, then $Tx = \zeta$ and therefore both the statements of the lemma holds. So we assume that $p \neq q$ and $w \neq \zeta$. Let $\zeta_1, \ldots, \zeta_s$ be the list of all distinct zeros of g with multiplicities $p_1, \ldots, p_s$, respectively. Let $\zeta = \zeta_i, p = p_i, \gamma = \gamma_i$ and $\delta = \delta_i$ for some $1 \leq i \leq s$, where γ_i and δ_i defined in Lemma (4.1).

To prove the first part of the lemma we have to show that $g(w) \neq 0$ and $g'(w) \neq 0$ implies $\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \neq 0$.
From Lemma (4.2) and equation (18), we get

$$|w - \zeta_j| \geq (1 - I(w))d > 0 \text{ ,as } \tau_1 < 1 \tag{19}$$

for each $j \neq i$. Above assures $g(w) \neq 0$. Then, Lemma (4.1) gives the following

$$\frac{g'(w)}{g(w)} = \frac{p+\gamma}{w-\zeta}, \text{ where } \gamma = (w-\zeta)\sum_{j\neq i}\frac{p_j}{w-\zeta_j}. \tag{20}$$

Using the triangle inequality and equation (19), we have the following

$$|\gamma| \leq |w-\zeta|\sum_{j\neq i}\frac{p_j}{|w-\zeta_j|} \leq \frac{|w-\zeta|}{(1-I(w))d}\sum_{j\neq i}p_j = \frac{(q-p)I(w)}{1-I(w)}. \tag{21}$$

Using the triangle inequality, equation(21) and as $I(w) < \tau_1 \leq \frac{p}{q}$, we get the following

$$|p+\gamma| \geq p - |\gamma| \geq p - \frac{(q-p)I(w)}{1-I(w)} = \frac{p-qI(w)}{1-I(w)} > 0. \tag{22}$$

Hence, $p + \gamma \neq 0$. This implies $g'(w) \neq 0$.
Then, from the Lemma (4.1), we have the following

$$\frac{g''(w)}{g'(w)} = \frac{(p+\gamma)^2 - (p+\delta)}{(w-\zeta)(p+\gamma)}, \text{ where } \delta = (w-\zeta)^2\sum_{j\neq i}\frac{p_j}{(w-\zeta_j)^2}. \tag{23}$$

$$\|\delta\| \le \frac{(q-p)I(w)^2}{(1-I(w))^2}. \tag{24}$$

We will now prove that $\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \neq 0$

$$\begin{aligned}
\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} &= \frac{1}{2}\left(\frac{p+\gamma}{p} + \frac{p+\delta}{p+\gamma}\right)\frac{1}{w-\zeta} \\
&= \left(1 + \frac{\gamma^2 + p\delta}{2p(p+\gamma)}\right)\frac{1}{w-\zeta} \\
&= \frac{1+\kappa_h}{w-\zeta}
\end{aligned}$$

$$where\ \kappa_h = \frac{\gamma^2 + p\delta}{2p(p+\gamma)}$$

Now,

$$|\kappa_h| = \left|\frac{\gamma^2 + p\delta}{2p(p+\gamma)}\right| \le \frac{|\gamma|^2 + p|\delta|}{2p|(p+\gamma)|} \le \frac{q(q-p)I(w)^2}{2p(1-I(w))(p-qI(w))}$$

So if we prove that $1+\kappa_h \neq 0$ our purpose will be completed.
Now,

$$|1+\kappa_h| \ge 1 - |\kappa_h| \ge 1 - \frac{q(q-p)I(w)^2}{2p(1-I(w))(p-qI(w))} = \frac{q(3p-q)I(w)^2 - 2p(p+q)I(w) + 2p^2}{2p(1-I(w))(p-qI(w))} > 0$$

This shows that $1+\kappa_h \neq 0$.
Therefore we can say that $w \in D$ Which proves (i).

From the construction of the Combined mean method, we have the following

$$\begin{aligned}
Tx - \zeta &= w - \zeta - \frac{p^2}{4}\frac{g(w)}{g'(w)}\left(\frac{3-p}{p} + \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right) - \frac{1}{2}\left(\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1} \\
&= w - \zeta - \frac{(w-\zeta)}{2}\left[1 - \frac{p}{2}\frac{3(p+\gamma)^2 - p(p+\delta)}{(p+\gamma)^3}\right] - \frac{1}{2}\frac{w-\zeta}{1+\kappa_h} \\
&= \frac{w-\zeta}{2}\frac{\kappa_h}{1+\kappa_h} + \frac{(w-\zeta)}{2}\frac{2(p+\gamma)^3 - 3p(p+\gamma)^2 + p^2(p+\delta)}{2(p+\gamma)^3} \\
&= \frac{w-\zeta}{2}\frac{\kappa_h}{1+\kappa_h} + \frac{(w-\zeta)}{2}\left[\frac{2\gamma^3 + 3p\gamma^2 + p^2\delta}{2(p+\gamma)^3}\right] \\
&= \kappa(w-\zeta),
\end{aligned}$$

where

$$\kappa = \frac{1}{2}\left(\left[\frac{\kappa_h}{1+\kappa_h}\right] + \left[\frac{2\gamma^3 + 3p\gamma^2 + p^2\delta}{2(p+\gamma)^3}\right]\right). \tag{25}$$

Using (21) and (22), we now estimate $|\kappa|$ and is as follows.

$$\begin{aligned}
|\kappa| &\le \frac{1}{2}\left(\left|\left[\frac{\kappa_h}{1+\kappa_h}\right]\right| + \left|\left[\frac{2\gamma^3 + 3p\gamma^2 + p^2\delta}{2(p+\gamma)^3}\right]\right|\right) \\
&\le \frac{1}{2}\left(\left[\frac{|\kappa_h|}{1-|\kappa_h|}\right] + \left[\frac{2|\gamma|^3 + 3p|\gamma|^2 + p^2|\delta|}{2|(p+\gamma)|^3}\right]\right) \\
&\le \frac{1}{2}\frac{q(q-p)I(w)^2}{q(3p-q)I(w)^2 - 2p(p+q)I(w) + 2p^2} \\
&\quad + \frac{2\left(\frac{(q-p)I(w)}{1-I(w)}\right)^3 + 3p\left(\frac{(q-p)I(w)}{1-I(w)}\right)^2 + p^2\frac{(q-p)I(w)^2}{(1-I(w))^2}}{4\left(\frac{p-qI(w)}{1-I(w)}\right)^3} \\
&= \frac{1}{2}\left[\phi_h(I(w)) + \phi_c(I(w))\right] \\
&= \phi(I(w)).
\end{aligned}$$

Which proves (ii). □

Theorem 4.1. *Let $g \in F[w]$ be a polynomial of degree $q \geq 2$ that splits over F, and let $\zeta \in F$ be a zero of g such that the multiplicity of ζ is p. Let $w_0 \in F$ satisfies the following initial condition*

$$I(w_0) < \tau_1 \text{ and } \phi(I(w_0)) < 1, \tag{26}$$

where $I : D \to R_+$ is defined in (13) and ϕ is defined in (17). Then the following three statements are true.

(i) Iterative sequence (11) of the C-H Combined mean method is defined and converges to ζ having order of convergence 3.

(ii) Error estimates are as follows

$$|w_{m+1} - \zeta| \leq \mu^{3^m}|w_m - \zeta| \text{ and } |w_m - \zeta| \leq \mu^{(3^m-1)/2}|w_0 - \zeta|, \text{for all} m \geq 0, \tag{27}$$

where $\mu = \phi(I(w_0))$.

(iii) A posteriori error estimate given below

$$|w_{m+1} - \zeta| < \frac{1}{(Ud)^2}|w_m - \zeta|^3, \text{for all} m \geq 0, \tag{28}$$

where $U \in (0, \tau)$ is the unique solution of $\phi(t) = 1$ in $(0, \tau)$.

Proof. Lemma (4.3) and theorem (2.1) gives the proof. □

4.2 Second Kind of Local Convergence theorem

Let assume that $g \in F[w]$ be a polynomial which having degree $q(\geq 2)$, such that all the zeros of g are in F, and also let $\zeta \in F$ be a zero of the polynomial g, multiplicity being p.
Here $(F, |.|)$ denotes a field having a norm and $F[w]$ is the ring of polynomial on the field F.
Here, we examine the convergence of C-H Combined mean method (11) with the help of function of initial conditions I, which is a map from D to R_+ and is defined as follows:

$$I(w) = I_g(w) = \frac{|(w - \zeta)|}{\rho(w)}, \tag{29}$$

here $\rho(w)$ represents the distance from the zero w to the closest zero of g other than ζ; if ζ is a only zero of g then we set $I(w) = 0$.

Now, we define two real functions ϑ_c and ϑ_h, for $q > p \geq 1$, by

$$\vartheta_c(u) = \frac{2(q-p)^3u^3 + p(q-p)(3q-2p)u^2}{2(p-(q-p)u)^3}. \tag{30}$$

and

$$\vartheta_h(u) = \frac{q(q-p)u^2}{2p^2 - 2p(q-p)u - q(q-p)u^2}. \tag{31}$$

Clearly, the functions ϑ_s and ϑ_h are quasi-homogeneous functions of degree 2 on $[0, \tau_2]$, where τ_2 is defined by

$$\tau_2 = \frac{2p}{q + \sqrt{q^2 + 4(q-p)^2}} \tag{32}$$

Now, we can define a function $\vartheta : [0, \tau_2) \to R_+$ defined by

$$\vartheta(u) = \frac{\vartheta_s(u)}{2} + \frac{\vartheta_h(u)}{2}. \tag{33}$$

As both the functions $\vartheta_s(u)$ and $\vartheta_h(u)$ are quasi-homogeneous, therefore by property (P3) we can say that ϑ is quasi-homogeneous of degree 2 in the interval $[0, \tau_2)$.

Lemma 4.4. *Let $g \in F[w]$ be a polynomial of degree $q(\geq 2)$ which splits over F, and let $\zeta \in F$ be a zero of g with multiplicity p. Let $w \in F$ be such that*

$$I(w) < \tau_2, \tag{34}$$

where the function I is defined by (29). Then:

(i) w is in D, the domain of the C-H Combined mean method and is defined in (12).

(ii) $|Tx - \zeta| \le \vartheta(I(w))|w - \zeta|$, where ϑ is defined in (33).

Proof. Let $w \in F$ satisfy the inequality (18). If any of $p = q$ or $w = \zeta$ or both are true, then $Tx = \zeta$ and therefore both the statements of the lemma holds. So we assume that $p \ne q$ and $w \ne \zeta$. Let $\zeta_1, \dots, \zeta_s$ be the list of all distinct zeros of g with multiplicities $p_1, \dots, p_s$, respectively. Let $\zeta = \zeta_i, p = p_i, \gamma = \gamma_i$ and $\delta = \delta_i$ for some $1 \le i \le s$, where γ_i and δ_i defined in Lemma (4.1).

To prove the first part of the lemma we have to show that $g(w) \ne 0$ and $g'(w) \ne 0$ implies $\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \ne 0$. Clearly we can write the following

$$|w - \zeta_j| \ge \rho(w) > 0 \tag{35}$$

for each $j \ne i$. This assures that $g(w) \ne 0$. Then, Lemma (4.1) gives the following

$$\frac{g'(w)}{g(w)} = \frac{p+\gamma}{w-\zeta}, \text{ where } \gamma = (w-\zeta)\sum_{j\ne i}\frac{p_j}{w-\zeta_j}. \tag{36}$$

Using the triangle inequality and (35), we have the following:

$$|\gamma| \le |w-\zeta|\sum_{j\ne i}\frac{p_j}{|w-\zeta_j|} \le \frac{|w-\zeta|}{\rho(w)}\sum_{j\ne i}p_j = (q-p)I(w). \tag{37}$$

Using the triangle inequality, equation (37) and $I(w) < \tau_2$, we have the following:

$$|p+\gamma| \ge p - |\gamma| \ge p - (q-p)I(w) > 0. \tag{38}$$

Hence, $p + \gamma \ne 0$. This implies $g'(w) \ne 0$.
Then from the Lemma (4.1), we have the following

$$\frac{g''(w)}{g'(w)} = \frac{(p+\gamma)^2 - (p+\delta)}{(w-\zeta)(p+\gamma)}, \text{ where } \delta = (w-\zeta)^2\sum_{j\ne i}\frac{p_j}{(w-\zeta_j)^2}. \tag{39}$$

Now, by triangle inequality and (35), we have the following

$$|\delta| \le (q-p)I(w)^2. \tag{40}$$

We will now prove that $\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} \ne 0$.

$$\begin{aligned}\frac{p+1}{2p}\frac{g'(w)}{g(w)} - \frac{1}{2}\frac{g''(w)}{g'(w)} &= \frac{1}{2}\left(\frac{p+\gamma}{p} + \frac{p+\delta}{p+\gamma}\right)\frac{1}{w-\zeta}\\ &= \left(1 + \frac{\gamma^2+p\delta}{2p(p+\gamma)}\right)\frac{1}{w-\zeta}\\ &= \frac{1+\kappa_h}{w-\zeta},\end{aligned}$$

where

$$\kappa_h = \frac{\gamma^2+p\delta}{2p(p+\gamma)}.$$

Now,

$$|\kappa_h| = \left|\frac{\gamma^2+p\delta}{2p(p+\gamma)}\right| \le \frac{|\gamma|^2+p|\delta|}{2p|(p+\gamma)|} \le \frac{q(q-p)I(w)^2}{2p(p-(q-p)I(w))}$$

So if we prove that $1+\kappa_h \neq 0$ our purpose will be completed.
Now,

$$|1+\kappa_h| \geq 1-|\kappa_h| \geq 1-\frac{q(q-p)I(w)^2}{2p(p-(q-p)I(w)))} = \frac{2p^2-2p(q-p)I(w)-q(q-p)I(w)^2}{2p(p-(q-p)I(w))} > 0.$$

This shows that $1+\kappa_h \neq 0$.
Therefore we can say that $w \in D$. Which proves (i).

From the construction of the Combined mean method, we have the following

$$\begin{aligned}
Tx-\zeta &= w-\zeta-\frac{p^2}{4}\frac{g(w)}{g'(w)}\left(\frac{3-p}{p}+\frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right)-\frac{1}{2}\left(\frac{p+1}{2p}\frac{g'(w)}{g(w)}-\frac{1}{2}\frac{g''(w)}{g'(w)}\right)^{-1}\\
&= w-\zeta-\frac{(w-\zeta)}{2}\left[1-\frac{p}{2}\frac{3(p+\gamma)^2-p(p+\delta)}{(p+\gamma)^3}\right]-\frac{1}{2}\frac{w-\zeta}{1+\kappa_h}\\
&= \frac{w-\zeta}{2}\frac{\kappa_h}{1+\kappa_h}+\frac{(w-\zeta)}{2}\frac{2(p+\gamma)^3-3p(p+\gamma)^2+p^2(p+\delta)}{2(p+\gamma)^3}\\
&= \frac{w-\zeta}{2}\frac{\kappa_h}{1+\kappa_h}+\frac{(w-\zeta)}{2}\left[\frac{2\gamma^3+3p\gamma^2+p^2\delta}{2(p+\gamma)^3}\right]\\
&= \kappa(w-\zeta),
\end{aligned}$$

where

$$\kappa = \frac{1}{2}\left(\left[\frac{\kappa_h}{1+\kappa_h}\right]+\left[\frac{2\gamma^3+3p\gamma^2+p^2\delta}{2(p+\gamma)^3}\right]\right). \tag{41}$$

We now use the estimates (37), (38) and (40) to estimate $|\kappa|$ and is given as following

$$\begin{aligned}
|\kappa| &\leq \frac{1}{2}\left(\left|\left[\frac{\kappa_h}{1+\kappa_h}\right]\right|+\left|\left[\frac{2\gamma^3+3p\gamma^2+p^2\delta}{2(p+\gamma)^3}\right]\right|\right)\\
&\leq \frac{1}{2}\left(\left[\frac{|\kappa_h|}{1-|\kappa_h|}\right]+\left[\frac{2|\gamma|^3+3p|\gamma|^2+p^2|\delta|}{2|(p+\gamma)|^3}\right]\right)\\
&\leq \frac{2\left((q-p)I(w)\right)^3+3p\left((q-p)I(w)\right)^2+p^2(q-p)I(w)^2}{4\left(p-(q-p)I(w)\right)^3}\\
&+\frac{q(q-p)I(w)^2}{2p^2-2p(q-p)I(w)-q(q-p)I(w)^2}\\
&= \frac{2(q-p)^3I(w)^3+p(q-p)(3q-2p)I(w)^2}{2(p-(q-p)I(w))^3}\\
&++\frac{q(q-p)I(w)^2}{2p^2-2p(q-p)I(w)-q(q-p)I(w)^2}\\
&= \frac{\vartheta_c(I(w))}{2}+\frac{\vartheta_h(I(w))}{2} = \vartheta(I(w)).
\end{aligned}$$

Proof of the lemma is therefore completed. □

Next, we state the convergence theorem second type.

Theorem 4.2. *Let $g \in F[w]$ be a polynomial of degree $q \geq 2$ which splits over F, and let $\zeta \in F$ be a zero of g such that the multiplicity of ζ is p. Let $w_0 \in F$ satisfies the following initial conditions*

$$I(w_0) < \tau_2 \text{ and } \vartheta(I(w_0)) \leq \psi(I(w_0)), \tag{42}$$

where the function I is defined in (29) and the function ψ is defined below as

$$\psi(u) = 1-u(1+\vartheta(u)). \tag{43}$$

Then, the C-H Combined mean method is defined and converges to ζ having the following error estimates

$$|w_{m+1} - \zeta| \leq \theta\mu^{3^m}|w_m - \zeta| \text{ and } |w_{m+1} - \zeta| \leq \theta^m \mu^{(3^m-1)/2}|w_0 - \zeta| \text{ forall } m \geq 0, \quad (44)$$

where $\theta = \psi(I(w_0))$ *and* $\mu = \frac{\vartheta(I(w_0))}{\psi(I(w_0))}$.

Proof. Lemma (4.4) and Theorem (2.2) guarantees the proof. □

5 Conclusion

In the first part of this study, we combine the Chebyshev and Halley methods to create a method for solving nonlinear equations. Secondly, we have demonstrated the method's local convergence for multiple polynomial zero of a polynomial f over any normed field F.

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