IOSR Journal of Computer Engineering (IOSR-JCE)
e-ISSN: 2278-0661,p-ISSN: 2278-8727, Volume 19, Issue 5, Ser. 1 (Sept. – October. 2023), PP 11-20
www.iosrjournals.org

Theory And Classification Of *Sl* (2, ℂ) Modules

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Abstract
The present paper consists of Theory and classification of $sl(2,\mathbb{C})$ modules in that sense $sl(2,\mathbb{C})$ is considered as guide example for the study of different other Lie algebras. This work provides all that is fundamental, definitions, examples, propositions, lemma and related proofs to a better understanding of what is a Lie algebra, and in particular $sl(2,\mathbb{C})$ representations and modules. A literature review about fundamentals of Lie algebra and then representations through various textbooks is developed and the methodology is purely algebraic with a use the maximal weight theory. Based on Weyl's theorem on the reducibility of a Lie algebra representation, we proved that all finite dimensional representations of a semisimple Lie algebra are completely reducible and the paper takes end with the classification of all finite $sl(2,\mathbb{C})$ modules. The representation theory of $sl(2,\mathbb{C})$is very important because it is used as the model for the study of other Lie algebras. The present work brought an overview of how to construct representations, in particular those of $sl(2,\mathbb{C})$.
***Key Word**: Vector space, field, algebra, Lie algebra, simplicity, semisimplicity, representation, weight, module.*

Date of Submission: 01-09-2023 Date of Acceptance: 11-09-2023

I. Introduction

Lie algebras were discovered by Marius Sophus Lie (1842−1899) while he was attempting to classify certain 'smooth' subgroups of general linear groups. The groups he considered are now called Lie groups. In fact, Lie defined his groups as being analytic rather than just smooth. One of Poincarée's conjecture was that Lie groups (finite-parameter Lie groups, that is) could be equivalently defined as smooth groups, and it took almost another 50 years before this conjecture was proved by the American mathematician Deane Montgomery to be correct. Questions about the groups could be treated using Lie algebras that were introduced to study the concept of infinitesimal transformations. The term Lie algebra was introduced by Hermann Weyl in the 1930s and in older texts, the name infinitesimal group is used. [14]

Problem Statement

The two by two matrices with complex entries and trace zero $sl(2,\mathbb{C})$ do not form an associative ring but they have another algebraic structure, a so-called Lie algebra structure. Given two matrices A, B in $sl(2,\mathbb{C})$we can consider the Lie product $AB - BA$, which also has trace zero. This is called the Lie bracket of the two matrices. This Lie algebra, and similar ones occur in differential geometry, physics and the theory of differential equations, and they often act on some complex space $\mathbb{C}^n$. Even if the study of the general case $sl(n,\mathbb{C})$Lie algebra of all traceless n ×n matrices with complex entries has been developed [5], the representation theory of $sl(2,\mathbb{C})$is very important because it is used as the model for the study of other Lie algebras. The present work will bring an overview of how to construct representations, in particular those of $sl(2,\mathbb{C})$

Objectives

The aims of this work are the following:

i. To give a description of the Lie algebra $sl(2,\mathbb{C})$ and its finite-dimensional representations.
ii. To show that any constructed finite dimensional $sl(2,\mathbb{C})$representation is irreducible
iii. To classify irreducible $sl(2,\mathbb{C})$modules

Research methodology

In order to achieve the goals of this work, we first make a literature review about fundamentals of Lie algebra and then representations. Secondly, we construct $sl(2,\mathbb{C})$ representations as a complex semisimple Lie algebra.The methodology is purely algebraic and will use maximal weight theory.

II. Literature review

As mentioned in introduction, Lie algebras are due to Sophus Lie. They are used to study geometric objects such as Lie groups (finite groups in which multiplication and maps are infinitely differentiable) and differentiable manifolds. As an example, we can mention a group of translations and a group of rotations. Each group has an associated Lie algebra. In this section, fundamental concepts of Lie algebras are treated.

Definition and first examples of Lie algebras

Definition 2.1 *A Lie algebra over a field* $\mathbb{F}$ *is an* $\mathbb{F}-$*vector space* L *together with a bilinear map*

$$[.,.] : L \times L \to L$$

called the Lie bracket of L, which is skew symmetric for all x, y in L and satisfies the Jacobi identity, i.e

$[y,x] = -[x,y]$ (2.1)

$[x,[y,z]] + [y,[x,z] + [z,[x,y]] = 0, \forall x,y,z \in L$ (2.2)

In the following all vector spaces are considered over the field $\mathbb{F}$ ($\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$) and all maps are linear with respect to $\mathbb{F}$.

Example 2.1 Let L be the real vector space $\mathbb{R}^3$ equipped with $[x,y] = x \times y$ (cross product of vectors) $\forall x,y,z \in L$. Then L is a Lie algebra.

Proof. *Let*

$$f : L \times L \to L$$
$$(x,y) \to f(x,y) = x \times y$$

It is known that $x \times y = -y \times x$ *and that f is bilinear. For the cross product of vectors in the (oriented) three dimension Euclidean space, there exists an orthonormal basis* (e_1, e_2, e_3) *such that*
$e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2$, *and the cross product of a vector with itself is zero.*

$$\forall x,y,z \in \mathbb{R}^3; x = \sum_{i=1}^{3} x^i e_i,\ y = \sum_{i=1}^{3} y^j e_j \ \text{ and } z = \sum_{i=1}^{3} z^k e_k;$$

The Jacobi identity

$$x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0 \quad (2.3)$$

is equivalent to

$$\sum_{i=1}^{3} x^i e_i \times \left(\sum_{i=1}^{3} y^j e_j \times \sum_{i=1}^{3} z^k e_k\right) + \sum_{i=1}^{3} y^j e_j \times \left(\sum_{i=1}^{3} z^k e_k \times \sum_{i=1}^{3} x^i e_i\right) + \sum_{i=1}^{3} z^k e_k \times \left(\sum_{i=1}^{3} x^i e_i \times \sum_{i=1}^{3} y^j e_j\right) = 0$$

i.e:

$$\sum_{i,j,k=1}^{3} x^i y^j z^k \{e_i \times (e_j \times e_k) + e_j \times (e_k \times e_i) + e_k \times (e_i \times e_j)\} = 0 \quad (2.4)$$

Since the cross product is bilinear, we have to check whether the relation (2.2) holds. In fact, this relation is verified $\forall x^i, y^j, z^k\ iff$

$$e_i \times (e_j \times e_k) + e_j \times (e_k \times e_i) + e_k \times (e_i \times e_j) = 0 \quad (2.5)$$

Let us take e_i, e_j, e_k *and suppose that they are all different.*

Then $e_i \times e_j = \pm e_k$ *and so* $e_k \times (e_i \times e_j) = 0$, *and in fact all other ones are zero.*

Suppose now that all e_i, e_j, e_k *are equal, then by the same argument all* $e_i \times (e_i \times e_i)$ *are zero. Suppose now two of them are equal. For instance* $i = j$, *then one term vanishes and one is left with* $e_i \times (e_j \times e_k) + e_j \times (e_k \times e_i)$. *Using that* $e_j \times e_k = -e_k \times e_j$, *one sees that this is zero, and is done.*
The Jacobi identity is satisfied and therefore, the vector space $\mathbb{R}^3$ *together with the cross product is a Lie algebra.*
■

Example 2.2 If V is a finite dimensional vector space over $\mathbb{F}$, $End\ V$(the set of linear transformations $V \to V$) endowed with $[x,y] = xy - yx$ is a Lie algebra over $\mathbb{F}$. It is denoted by $gl(V)$ and is called general linear algebra because it is closely associated to the general linear group $GL(V)$, the group of all invertible endomorphisms of V.

Proof. *Since $gl(V)$ is an associative algebra over $\mathbb{F}$, the bilinearity property is already satisfied and by definition of the bracket: $[A,B] = AB - BA$ for any A and B in $gl(V)$, the anticommutativity property holds.*
Let us check the Jacobi identity.
$\forall A, B, C \in gl(V)$;

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = [A, BC - CB] + [B, CA - AC] + [C, AB - BA]$$
$$= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C$$
$$= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC$$
$$= 0$$

Therefore, $gl(V)$ is a Lie algebra. ■

Using matrices instead of linear transformations, we fix a basis for V and then identify $gl(V)$ to $gl(n,\mathbb{F})$, a vector space whose basis consists of matrices e_{ij} for $1 \le i,j \le n$ having 1 in the position (i,j) and 0 elsewhere. Since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows that

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj} \quad (2.6)$$

where

$$\delta_{ij} = \begin{cases} 1 \ if \ i = j \\ 0 \ if \ i \neq j \end{cases}$$

stands for Kronecker symbol.

Example 2.3 The set of all traceless endomorphisms of V, denoted by $sl(V)$ is a subalgebra of $gl(V)$**.** It is named ***Special linear algebra*** because it is connected to $SL(V)$, the special linear group of endomorphisms of determinant equal to one.

Proposition 2.1 $sl(V)$ is a subalgebra of $gl(V)$.

Proof. *One knows that the trace is a linear map.*

i.e For any A and B in $gl(V)$ and a scalar α:

$$tr(A + B) = tr(A) + tr(B) \quad (2.7)$$

$$tr(\alpha A) = \alpha tr(A) \quad (2.8)$$

so $sl(V)$ is a vector space.

In addition, for any $A = (a_j^i)1 \le i,j \le n$ and $B = (b_j^i)1 \le i,j \le n$;

$$tr(AB) = \sum_{j=1}^{n}\sum_{k=1}^{n} a_j^k\, b_k^j = \sum_{j=1}^{n}\sum_{k=1}^{n} a_k^j\, b_j^k = \sum_{k=1}^{n}\sum_{j=1}^{n} a_k^j\, b_j^k = tr(BA)$$

$$tr(AB) = tr(BA) \quad (2.9)$$

That is

$$tr[A,B] = tr(AB - BA) = tr(AB) - tr(BA) = 0 \quad (2.10)$$

which shows that $sl(V)$ is stabilized by the bracket; so it is an algebra and then a subalgebra of $gl(V)$. ■

For the same raisons, $sl(n,\mathbb{F})$ is a subalgebra of $gl(n,\mathbb{F})$ and in particular for $n = 2$ and $\mathbb{F} = \mathbb{C}$, $sl(2,\mathbb{C})$ is a subalgebra of $gl(2,\mathbb{C})$. A basis of $sl(n,\mathbb{C})$ is given by $e_{ij}\ (i \neq j)$ and $e_{ii} - e_{i+1,i+1}$. For $sl(2,\mathbb{C})$ it consists of X, Y, H where $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The commutation relations are such that:

$$[X,Y] = H, [X,H] = -2X, [Y,H] = 2Y$$

and the Jacobi identity is satisfied. i.e

$$[[X,Y],H] + [[Y,H],X] + [[H,X],Y] = 0$$

Thus $sl(2,\mathbb{C})$ is a subalgebra.

Example 2.4 This is an example of an abelian Lie algebra. Let V be an $\mathbb{F}$-vector space. Define the Lie bracket on V by $[x,y] = 0\ \forall x,y \in V$. To show that V is a Lie algebra we can proceed as follows: Given that $[x,y] = 0\ \forall x,y \in V$,

$$(i)[x,y] = -[y,x] = 0$$

$$(ii)[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

The Jacobi identity also is satisfied because of (i) and we conclude that V is a Lie algebra.

Definition 2.2 Given a Lie algebra L, an ideal I of L is a vector subspace of L such that $[x,y] \in I$ for all $x \in L$,and $y \in I$.

Definition 2.3 Given a Lie algebra L, a subspace K of L is called a subalgebra if K itself is a Lie algebra with respect to the induced bracket from L.

An ideal is always a subalgebra. By the following example, we realise that the converse is not always true.

Example 2.5 *One-dimensional abelian subalgebra of* $gl(2,\mathbb{F})$ *does not need necessarily to be an ideal.*

Taking $U \in gl(2,F), U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, fU = \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix} \in \mathbb{F}U,\ W = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ *with* $a,b \in F\backslash\{0\}$

$$[W,fU] = \begin{pmatrix} 0 & -2af \\ 2bf & 0 \end{pmatrix} \notin \mathbb{F}U$$

$\mathbb{F}U$ is an abelian subalgebra but not an ideal of $gl(2,\mathbb{F})$.

Definition 2.4 The center of a Lie algebra L is

$$Z(L) = \{x \in L : [x,y] = 0\ \forall y \in L\}$$

If $Z(L) = L$ then [x, y] = 0 for all $x \in L$ and $y \in L$. Then L is an abelian Lie algebra.

Definition 2.5 *Let L and* L' *be Lie algebras. A linear map*

$$\rho : L \to L'$$

is a Lie algebra homomorphism if

$$\rho([x,y]) = [\rho(x),\rho(y)]\ \forall x,y \in L$$

If the homomorphism ρ is bijective, i.e isomorphism, then L and L' *are said to isomorphic Lie algebras.*

Example 2.6 *The adjoint homomorphism is a good example of a Lie algebra homomorphism. Let L be a Lie algebra. Define* $ad : L \to gl(L)$ *by* $ad_x = [x,.]$; *that is,* $(ad_x)(y) = [x,y]$ *for* $y \in L$. *Then* ad *is a Lie algebra homomorphism.*

Proof. *If L is a Lie algebra then* $gl(L)$ *is a Lie algebra. The adjoint representation is such that:*

$$ad : g \to gl(L)$$

$$x \to adx(y) = [x,y]$$

is a Lie algebra homomorphism if it is linear and if it preserves the Lie bracket.

(i)For any scalars α *and* β, *and* $x,y \in L$

$$ad_{\alpha x}(\beta y) = [\alpha x,\beta y] = \alpha x\beta y - \beta y\alpha x = \alpha\beta[x,y]$$

The bracket is bilinear.

(ii)We need also to check if

$$ad_{[x,y]}(z) = [ad_x,ad_y](z)\ \forall z \in L.$$

By definition

$$ad_{[x,y]}(z) = [[x,y],z]$$

and

$$[ad_x, ad_y](z) = ad_x ad_y(z) - ad_y ad_x(z)$$

By Jacobi identity,

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]]$$

Therefore

$$ad[x,y](z) = [adx, ady](z)\ \forall z \in L.$$

The adjoint ad, which is linear and preserves the bracket, is a Lie algebra homomorphism. ■

Proposition 2.2 *Let $\rho: L \to L'$ be a Lie algebra homomorphism. Then $Ker\rho$ is an ideal of L and $Im\,\rho$ is a subalgebra of L' .*

Proof. *(a)*

$$Ker\rho = \{x \in L : \rho(x) = 0\}$$

$Ker\rho$ is a vector subspace because ρ is a vector space homomorphism. i.e

$$\forall x,y \in L, \rho[x,y] = [\rho(x),\rho(y)]. If\ x \in Ker\rho\ and\ y \in L,$$

then

$$\rho[x,y] = [\rho(x),\rho(y)] = [0,\rho(y)] = 0$$

because the bracket is bilinear. Thus,

$$\forall x \in Ker\rho, \forall\, y \in L, [x,y] \in Ker\rho$$

which means that $Ker\rho$ is an ideal

$$Ker\rho = Z(L) \subset L$$

reason why it is an Ideal of L.

(b)$Im\rho$ is a subalgebra of L' .

By definition, $Im\rho$ is a subalgebra if

$$\forall x', y' \in Im\rho, [x',y'] \in Im\rho \qquad (2.12)$$

If $x', y' \in Im\rho$; then $\exists x, y \in L$ such that $x' = \rho(x)$ and $y' = \rho(y)$.

$$[x',y'] = [\rho(x),\rho(y)] = \rho[x,y] \in Im\rho,$$

since ρ is a homomorphism.

The study of the structure of a given Lie algebra is done through its ideals.

Properties used to classify Lie Algebras

a. Simple Lie algebra

Definition 2.6 A Lie algebra L is called simple if it has no ideals except itself and $\{0\}$ and if it is not abelian. If a Lie algebra L is simple,

$$Z(L) = 0$$

and L non abelian means that

$$[L,L] \neq 0$$

Example 2.7 $sl(2,\mathbb{C})$is simple.

Proof. The Lie algebra $sl(2,\mathbb{C})$ admits for basis X, Y, H such that

$$[X,Y] = H;\ [H,X] = 2X;\ [H,Y] = -2Y$$

$sl(2,\mathbb{C})$is not abelian since

$$[sl(2,\mathbb{C}), sl(2,\mathbb{C})] \neq 0$$

What we have to verify is the existence of 1 or 2 dimensional ideals in $sl(2,\mathbb{C})$.

Case1: One-dimensional ideal

If ***One-dimensional ideal*** exists in $sl(2,\mathbb{C})$, it is generated by a nonzero vector

$$Z = aH + bX + cY$$

and for any

$$U \in sl(2,\mathbb{C}), \quad [U,Z] = \lambda Z$$

Let us take $U = H$

$$[aH + bX + cY, H] = -2bX + 2cY$$

$aH + bX + cY$ and $-2bX + 2cY$ are linearly dependent if and only

if $a = 0$ and $\begin{vmatrix} b & -2b \\ c & 2c \end{vmatrix} = 0$

$$\Leftrightarrow \begin{cases} a = 0 \\ 4bc = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a = 0 \\ b = 0 \ or \ c = 0 \end{cases}$$

Let us assume that $c = 0$ and $b \neq 0$. Then $Z = bX$, Z is a multiple of X.
Let us take $Z = X$ and $U = Y$, $[X,Y] = H$ is not a multiple of X.
The case $b = 0$ and $c \neq 0$ leads to the same results which implies that there is no one-dimensional ideal in $sl(2,\mathbb{C})$.

Case2 Two-dimensional ideal

If a two-dimensional ideal $\boldsymbol{g}$ *exists in* $sl(2,\mathbb{C})$ *, let* (U_1, U_2) *be a basis of* $\boldsymbol{g}$. *Then* $[U_1, U_2] \in \boldsymbol{g}$. *Let* $U_3 \in sl(2,\mathbb{C})$, $U_3 \notin \boldsymbol{g}$. *Then* (U_1, U_2, U_3) *is a basis of* $sl(2,\mathbb{C})$. *Since* $\boldsymbol{g}$ *is supposed to be an ideal*, $[U_1, U_3] \in g$ *and* $[U_2, U_3] \in \boldsymbol{g}$.
Let us take 2 arbitrary vectors $aU_1 + bU_2 + cU_3$ *and* $\alpha U_1 + \beta U_2 + \gamma U_3$
Their commutation is such that

$$[aU_1 + bU_2 + cU_3, \alpha U_1 + \beta U_2 + \gamma U_3]$$

$$= \underbrace{(a\beta - b\alpha)[U_1, U_2] + (a\gamma - c\alpha) + [U_1, U_3](b\gamma - c\beta)[U_2, U_3]}_{N}$$

$N \in \boldsymbol{g}$ *because* $\boldsymbol{g}$ *is an ideal. Since the Lie bracket of two arbitrary elements is in* $\boldsymbol{g}$, *this means that* $[sl(2,\mathbb{C}), sl(2,\mathbb{C})] = \boldsymbol{g} \neq sl(2,\mathbb{C})$ *which is a contradiction since* X, Y, H *is a basis of* $sl(2,\mathbb{C})$ *satisfying* $[X,Y] = H$; $[X,H] = -2X$; $[Y,H] = 2Y$

There is no one or two dimensional ideal in $sl(2,\mathbb{C})$. *Therefore* $sl(2,\mathbb{C})$ *is simple.* ■

b. Solvable Lie algebra

The definition of a solvable Lie algebra L is based on knowledge of its derived algebras ($L' = [L,L]$). The derived series of L is the series with terms: $L(0) = L, L(1) = L'$ and $L(k) = [L(k-1), L(k-1)]$ $for\ k \geq 2$. We say that L is solvable if $L(n) = 0$ for some $n \geq 1$. Giving examples, abelian Lie algebras are solvable because abelian implies solvability. In this way $sl(2,\mathbb{C})$ is not solvable.

Definition 2.7 A non-abelian Lie algebra L is said to be semisimple if it has no non-zero solvable ideals. [2]

Example 2.8 $sl(2,\mathbb{C})$ is an example of a semisimple Lie algebra while $gl(2,\mathbb{C}) = sl(2,\mathbb{C}) \oplus \mathbb{R}I$ is an example of a non-semisimple Lie algebra since it has $\mathbb{R}I$ as a non zero solvable ideal. I is a unit matrix of order 2.

Definition 2.8

The map

$$\kappa : L \times L \to \mathbb{F}$$
$$(x,y) \to \kappa(x,y) = tr\big((adx)(ady)\big) for\ all\ x, y\ in\ L$$

is called the Killing form.

The Killing form satisfies the following properties: For all $x, y, z \in L$ and $\alpha, \beta, \in \mathbb{F}$

(1) $\kappa(\alpha x+\beta y,z)=\alpha\kappa(x,z)+\beta\kappa(y,z)\ \kappa(x,\alpha y+\beta z)=\alpha\kappa(x,y)+\beta\kappa(x,z)$ *(Bilinearity)*
(2) $\kappa(x,y)=\kappa(y,x)$ *(Symmetry)*
(3) $\kappa([x,y],z)\ =\kappa(x,[y,z])$ *(Associativity)*
(4) $\kappa((ady)x),z)+\kappa(x,(ady)z)=0$ *(Invariance of the Killing form under the adjoint action)*

Theorem 2.1 *(Cartan's criterion of semisimplicity). A Lie algebra is semisimple iff the Killing form is non-degenerate*.[11]

Example 2.9 *The Killing form defined on* $sl(2,\mathbb{C})$ *is obtained as follows:*

$$adX=\begin{pmatrix}0&-2&0\\0&0&1\\0&0&0\end{pmatrix}, adH=\begin{pmatrix}2&0&0\\0&0&0\\0&0&-2\end{pmatrix} adX=\begin{pmatrix}0&0&0\\-1&0&0\\0&2&0\end{pmatrix}$$

The matrix elements of the Killing form are given by

$$\kappa(X,Y)\ =\ \kappa(Y,X)\ =\ 4$$
$$\kappa(H,H)\ =\ 0$$
$$\kappa(H,X)\ =\ \kappa(X,H)\ =\ \kappa(H,Y)\ =\ \kappa Y,H\ =\ \kappa(X,X)\ =\ \kappa(Y,Y)\ =\ 0$$

so the Killing form will be represented by the matrix

$$K\ =\begin{pmatrix}0&0&4&0\\0&8&0&0\\4&0&0&0\\0&0&0&0\end{pmatrix}$$

and it is nondegenerate since $det(K)\neq 0$ Thus $sl(2,C)$ is semisimple.

c. Nilpotent Lie algebra

A Lie algebra L such that $L^0\ =\ L,L^1\ =\ [L,L],L^2\ =\ [L,L^1\ ],\ldots,L^i\ =\ [L,L^{i-1}\ ]$ is nilpotent if $L^n=0$ for some $n\in\ \mathbb{N}$. Consequently, all abelian Lie algebras are nilpotent .

The reverse is not necessarily true. Taking the three-dimensional Heisenberg Lie algebra, denoted as $h(3)$, which has a basis $\{m,n,p\}$ and Lie brackets defined by

$$[m,n]\ =\ z,[m,p]\ =\ 0,[n,p]\ =\ 0$$

Proposition 2.3 *Let L be a Lie algebra*
a) If L is nilpotent, then all subalgebra and homomorphic images of L are nilpotent
b) If $L/Z(L)$ *is nilpotent, then L is nilpotent*
c) If L is nilpotent and non zero, then $Z(L)\ \neq 0$ [1]

Jordan-Chevalley decomposition

Definition 2.10 *Let V be a finite dimensional complex vector space and* $f\ :\ V\ \rightarrow\ V$ *a complex linear map. Then a Jordan decomposition of* f *is an expression* $f\ =\ f_1\ +\ f_2$*, where* $f_1,f_2{:}\,V\longrightarrow V$ *are complex linear maps such that* f_1 *is diagonalizable and* f_2 *is nilpotent and such that* $f_1 o f_2=f_2 o f_1$.[8]

Jordan-Chevalley decomposition expresses a linear operator f as a direct sum of its commuting semisimple and nilpotent parts. Let us consider a such f_1 and f_2 in $End(V\ )$ with V a finite dimensional vector space. f_1 is semisimple if the roots of its minimal polynomial over $\mathbb{F}$ are all distinct. i.e the operator f_1 is semisimple if it is diagonalizable over algebraically closed field $\mathbb{F}$. An operator f_2 is nilpotent if f_2^n= 0 for some n. In matrix form, the Jordan decomposition of an endomorphism f is given by:

$$f=\underbrace{\begin{pmatrix}a&0&\cdots&0\\0&a&\ddots&\vdots\\\vdots&\ddots&\ddots&0\\0&0&\cdots&a\end{pmatrix}}_{f_1}+\underbrace{\begin{pmatrix}0&1&\cdots&0\\0&0&\ddots&\vdots\\0&\ddots&\ddots&1\\0&0&\cdots&0\end{pmatrix}}_{f_2}$$

III. Lie algebra and finite dimensional representations

Module

Let L be a Lie algebra over the field $\mathbb{F}$. A vector space V endowed with an operation $L \times V \rightarrow V$ (denoted$(x, v) \rightarrow x.v \text{ or just } xv$) is called an $L-$ module if the following properties are satisfied:
1. $(ax + by).v = a(x.v) + b(y.v)$
2. $x.(av + bw) = a(x.v) + b.(x.w)$
3. $[x, y].v = xy.v - yx.v\ (x, y \in L;\ v, w \in V \text{ and } a, b \text{ are scalars})$ [1]

For example; if $\varphi : L \rightarrow gl(V)$ is a representation of L over V, then V is an $L-$module by the action $x.v = \varphi(x)(v)$

An $L-$module V is said to be $irreducible$ if its submodules are only itself and the trivial submodule 0. It is $completely\ reducible$ if it is a direct sum of irreducible L-submodules.
It is noticed that a zero-dimension vector space is not considered as an irreducible L-module and a 1-dimensional vector space on which acts L is irreducible $L-$module.

Theorem 3.1 Weyl's Theorem

If $\rho : L \rightarrow gl(V)$ is a (finite dimensional) representation of a semisimple Lie algebra. Then ρ is completely reducible. [1]

3.2 Irreducible representations of $sl(2, \mathbb{C})$

The 3 dimensional $sl(2, \mathbb{C})$ has for basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Satisfying:

$$[X, Y] = H, [X, H] = -2X \text{ and } [Y, H] = 2Y$$

Let V be an arbitrary L-module. Given that H is semisimple, its action on V is diagonalizable and V can be written as a direct sum of eigenspaces

$$V_\lambda = \{v | H.v = \lambda v;\ \lambda \in \mathbb{C}\}$$

$For\ V_\lambda \neq 0$, λ is a weight of H and V_λ is called a weight space. A *weight* of a representation is a generalization of the notion of an eigenvalue, and the corresponding eigenspace is called a *weight space*.

Lemma 3.1 If $v \in V_\lambda$ then $X.v \in V_{\lambda+2}$ and $Y.v \in V_{\lambda-2}$

Proof. $H.(X.v) = X.H(v) + [H, X].v = \lambda X(v) + 2X(v) = (\lambda + 2)X(v) \in V_{\lambda+2}$

$$and\ for\ Y;$$

$$H.Y.(v) = Y.H(v) + [H, Y](v) = \lambda Y(v) - 2Y(v) = (\lambda - 2)Y(v) \in V_{\lambda-2} \blacksquare$$

3.3 Classification of all finite-dimensional modules of $sl(2, \mathbb{C})$

In the previous sections, it has been shown that $sl(2, \mathbb{C})$is semi simple; and according to the Weyl's theorem, all finite-dimensional $sl(2, \mathbb{C})$ modules are reducible. Let $\lambda \in \{0, 1, 2, 3, \ldots\}$ and let $\mathbb{C}[z_1, z_2]$ be the polynomial ring over $\mathbb{C}$ in z_1 and z_2.

$$V_\lambda := Span(z_1^\lambda, z_1^{\lambda-1} z_2, \ldots, z_1 z_2^{\lambda-1}, z_2^\lambda)$$

is a $\mathbb{C}$ -vector space of homogeneous polynomial functions of degree λ. The action on V_λ is described as follows:

$$\begin{cases} X.z_1 = z_2 \\ X.z_2 = 0 \end{cases}, \begin{cases} Y.z_1 = 0 \\ Y.z_2 = \end{cases}, \begin{cases} H.z_1 = z_1 \\ H.z_2 = -z_2 \end{cases}$$

Using X, Y and H as derivation operators, we have the following scenario:

$$X.z_1^\lambda = \lambda z_1^{\lambda-1} z_2$$
$$X.\left(z_1^{\lambda-1} z_2\right) = (\lambda - 1) z_1^{\lambda-2} z_2^2$$
$$X.\ (z_1^{\lambda-2} z_2^2) = (\lambda - 2) z_1^{\lambda-3} z_2^3$$
$$\vdots$$

$$X.(z_1 z_2^{\lambda-1}) = z_2^{\lambda}$$

$$X.z_2^{\lambda} = 0$$

$$Y.z_2^{\lambda} = \lambda z_1 z_2^{\lambda-1}$$
$$Y.\left(z_1 z_2^{\lambda-1}\right) = (\lambda-1) z_1^2 z_2^{\lambda-2}$$
$$Y.\left(z_1^2 z_2^{\lambda-2}\right) = (\lambda-2) z_1^3 z_2^{\lambda-3}$$
$$\vdots$$
$$Y.(z_1^{\lambda-1} z_2) = z_1^{\lambda}$$
$$Y.(z_1^{\lambda}) = 0$$

$$H.z_1^{\lambda} = \lambda z_1^{\lambda-1}\ z_2$$
$$H.\left(z_1^{\lambda-1}\ z_2\right) = (\lambda-2) z_1^{\lambda-1} z_2$$
$$H.\ (z_1^{\lambda-2} z_2^2) = (\lambda-4) z_1^{\lambda-2} z_2^2$$
$$\vdots$$
$$H.(z_1 z_2^{\lambda-1}) = (2-\lambda) z_1 z_2^{\lambda}$$

$$H.z_2^{\lambda} = -\lambda z_2^{\lambda}$$

Corollary 3.2 All eigenvalues of H are integers and each one occurs along with its negative.

Proposition 3.1 Let V be a finite dimensional $sl(2,\mathbb{C})$ −module then there exists an eigenvector $w \in V$ for H such that $X.w = 0$.

Proof. Since we work over an algebraically closed field $\mathbb{C}$ the linear map, $H : V \to V$has at least one eigenvalue and hence at least one eigenvector. Let $H.v = \lambda v$ and consider the vectors

$$v;\ X.v;\ X^2.v;\ \ldots,$$

By the **Proposition 3.1**; if they are non zero, they constitute an infinite sequence of eigenvectors of H with different eigenvalues. However V is finite dimensional so these cannot all be non-zero, hence there exists a $k \geq 0$ such that $X^2.v \neq 0$ and $X^{k+1}.v = 0$. Setting $w = X^k.v$ we have

$$H.w = (\lambda + 2k)w$$

and

$$X.w = 0 \blacksquare$$

Proposition 3.2 For all $\lambda \in \{0,1,2,3,\ldots\}$, the module V_λ is irreducible.

Proof. Assume $0 \subset W \subseteq V_\lambda$ is an invariant non-zero subspace under the action of $sl(2,\mathbb{C})$. The endomorphism of W induced by the action of H has an eigenvalue λ with a corresponding non-zero eigenvector $w \in W$; i.e $Hw = \lambda w$. Since H has 1-dimensional eigenspaces spanned by the monomials

$$\left\{z_1^{\lambda}, z_1^{\lambda-1}\ z_2, \ldots, z_1 z_2^{\lambda-1}, z_2^{\lambda}\right\}$$

The vector w is a scalar multiple of one of these. The subspace W contains all such monomials since successive applications of X and Y map one to some non-zero scalar multiple of every other one. Thus

$$W = V_\lambda$$

which prove that V_λ is irreducible. ■

Briefly, let $L = sl(2,\mathbb{C})$and V an irreducible L −module of dimension $\lambda + 1$. Consider a maximal vector $v_0 \in V_\lambda$ with assumption that $v_{-1} = 0$ and $v_i = (1/i!) Y^i\ v_0 (i \geq 0)$.

Then H has eigenvalues

$$\{\lambda, \lambda-2, \lambda-4, \cdots, -\lambda\};$$

V has for basis

$$\{v_0, v_1, v_2. \cdots, v_n\}$$

and the following lemma holds.

Lemma 3.2 *(a)* $Hv_i = (\lambda - 2i) v_i$
(b) $Yv_i = (i + 1) v_{i+1}$
(c) $Xv_i = (\lambda - i + 1) v_{i-1} (i \geq 0)$

Theorem 3.3 *Let V an irreducible module for L. Relative to H, V is a direct sum of weight spaces $V_\mu, \mu = \lambda, \lambda - 2, \cdots, -(\lambda - 2), -\lambda$ where $\lambda + 1 = dimV$ and $dimV_\mu = 1$ for each μ $(if\ V_\mu \neq 0)$.*

(a) V has (up to nonzero scalar multiples) a unique maximal vector, whose weight (called the highest weight of V)is λ.
(b) *The action of L on V is given explicitly by the formulas (a), (b) and (c) in the above lemma, if the basis is chosen in the prescribed fashion. In particular, there exists at most one irreducible L-module (up to isomorphism)of each possible dimension $\lambda + 1, \lambda \geq 0$* [1]

IV. Conclusion

The purpose of this work was to make a description of the Lie algebra $sl(2, \mathbb{C})$and its finite dimensional representations. From proposed definitions, propositions, theorems and corresponding proofs, the present work established what is fundamental for a better understanding of a Lie algebra. We defined a module and proved that all finite dimensional $sl(2, \mathbb{C})$modules are irreducible. Thus, we have classified all modules of $sl(2, \mathbb{C})$.

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