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The extension of the Riemann's zeta function

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Abstract : In mathematics, the search for exact formulas giving all the prime numbers, certain families of prime numbers or the n-th prime number has generally proved to be vain, which has led to contenting oneself with approximate formulas [8]. The purpose of this article is to give a new proof of the Riemann hypothesis [4] by y introducing $\hat{S}$ a new extension of the of the Riemann zeta function
Keywords: *Prime Number, number theory, distribution of prime numbers, the law of prime numbers, the Gamma function, the Mertens function, quantum mechanics, black Holes, holomorphic function, Hilbert-Polya's conjecture, the Riemann hypothesis*

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In memory of the great professor, the physicist and mathematician, Moshé Flato.

I- INTRODUCTION, RECALL, NOTATIONS AND DEFINITIONS

Prime numbers [See 4, 5, 6, 7, 8] are used especially in information technology, such as public-key cryptography which relies on factoring large numbers into their prime factors. And in abstract algebra, prime elements and prime ideals give a generalization of prime numbers.

In mathematics, the search for exact formulas giving all the prime numbers, certain families of prime numbers or the n-th prime number has generally proved to be vain, which has led to contenting oneself with approximate formulas [8].

Recall that Mills' Theorem [8]: "There exists a real number A, Mills' constant, such that, for any integer n > 0, the integer part of A^{3^n} is a prime number" was demonstrated in 1947 by mathematician William H. Mills [11], assuming the Riemann hypothesis [4, 5, 6,7] is true. Mills' Theorem [8] is also of little use for generating prime numbers.

The purpose of this article is to to give a new proof of the Riemann hypothesis [4]. by y introducing $\hat{S}$ a new extension of the of the Riemann zeta function

Theorem : The real part of every nontrivial zero of the Riemann zeta function is 1/2.

The link between the function ζ and the prime numbers had already been established by Leonhard Euler with the formula [5], valid for $\Re(s)>1$:

$$\zeta(s)=\prod_{p\in P}\frac{1}{1-p^{-s}}=\frac{1}{(1-\frac{1}{2^s})(1-\frac{1}{3^s})(1-\frac{1}{5^s})\ldots}$$

where the infinite product is extended to the set P of prime numbers. This formula is sometimes called the Eulerian product.

And since the Dirichlet eta function can be defined by $\eta(s)=(1-2^{1-s})\zeta(s)$ where :

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

We have in particular : $\zeta(z)=\frac{1}{1-2^{1-z}}\frac{\sum_{n=1}^{\infty}(-1)^{n-1}}{n^z}$ for $0<\Re(z)<1$

Let $s=x+iy$, with $0<\Re(s)<1$

$$\zeta(s)\zeta(s)=\prod_{p\in P}\frac{1}{1-p^{-s}}\frac{1}{1-p^{-s}}=\prod_{p\in P}\frac{1}{(1-e^{-xln(p)}\cos(yln(p)))^2+(e^{-xln(p)}\sin(yln(p)))^2}$$

But : $\prod_{p\in P}\frac{1}{(1-e^{-xln(p)}\cos(yln(p)))^2+(e^{-xln(p)}\sin(yln(p)))^2}\geq\prod_{p\in P}\frac{1}{(1+e^{-xln(p)})^2+(e^{-xln(p)})^2}$

If $\zeta(s)=0$, then $\prod_{p\in P}\frac{1}{(1+e^{-xln(p)})^2+(e^{-xln(p)})^2}=0$ and since the non-trivial zeros of $\zeta(s)=0$ are symmetric with respect to the line $X=\frac{1}{2}$ because the zeta function satisfies the functional equation [4, 6] :

$$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$$

then $x=\frac{1}{2}+\alpha$, and if $s'=\frac{1}{2}-\alpha+iy$, then $\zeta(s')=0$

But the function $\frac{1}{(1+e^{-tln(p)})^2+(e^{-tln(p)})^2}$ is increasing in $[0,1]$, so $\prod_{p\in P}\frac{1}{(1+e^{-tln(p)})^2+(e^{-tln(p)})^2}=0$

$\forall\ t \in [\frac{1}{2} - \alpha, \frac{1}{2} + \alpha]$

As $\prod_{p\in P} \frac{1}{(1+e^{-z\ln(p)})^2 + (e^{-z\ln(p)})^2}$ is holomorphic : because :

$\prod_{p\in P} \frac{1}{(1+e^{-z\ln(p)})^2 + (e^{-z\ln(p)})^2} = \prod_{p\in P} \frac{1}{1-A/p^z} \frac{1}{1-B/p^z}$ with $A = i-1$ and $B = -i-1$, and both $\prod_{p\in P} \frac{1}{1-A/p^z}$ and $\prod_{p\in P} \frac{1}{1-B/p^z}$ are holomorphic in $\{z \in \mathbb{C} \setminus \{1\}, \Re(z) \geq \frac{1}{2}\}$

as we have :

$\prod_{p\in P} \frac{1}{1-A/p^z} = \prod_{p\in P} 1 + f_p(z)$ with $f_p(z) = \frac{1}{(p^z/A) - 1}$.

$|f_p(z)| \leq \frac{1}{|p^z/A| - 1} = \frac{1}{(p^{\Re(z)}/\sqrt{2}) - 1} \leq \frac{k}{p^{\frac{1}{2}}}$, where k is a positive real constant.

So $\left|\sum_{p\in P, p=N}^{\infty} f_p(z)\right| \leq k \left|\sum_{p=N}^{\infty} \frac{1}{n^{\frac{1}{2}}}\right| = k \left|\zeta_N(\frac{1}{2})\right|$

But(see Lemma 1 [6]) : $\zeta_N(\frac{1}{2}) = o_N(1)$

We deduce that the series $\sum_p |f_p|$ converges normally on any compact of $\{z \in \mathbb{C} \setminus \{1\}, \Re(z) \geq \frac{1}{2}\}$ and consequently $\prod_{p\in P} \frac{1}{1-A/p^z}$ is holomorphic in $\{z \in \mathbb{C} \setminus \{1\}, \Re(z) \geq \frac{1}{2}\}$

In the same way $\prod_{p\in P} \frac{1}{1-B/p^z}$ is holomorphic in $\{z \in \mathbb{C} \setminus \{1\}, \Re(z) \geq \frac{1}{2}\}$

If $\alpha \neq 0$, then the holomorphic function $\prod_{p\in P} \frac{1}{(1+e^{-z\ln(p)})^2 + (e^{-z\ln(p)})^2}$

will be null (because null on $]\frac{1}{2},\frac{1}{2}+\alpha\,]$), and it follows that $\prod_{p\in P}\frac{1}{1-A/p^z}$ or $\prod_{p\in P}\frac{1}{1-B/p^z}$ is null in $\{z\in\mathbb{C}\setminus\{1\},\Re(z)\geq\frac{1}{2}\}$. Let's show that this is impossible:

If $\prod_{p\in P}\frac{1}{1-A/p^z}=\prod_{p\in P}1+f_p(z)=0$ with $f_p(z)=\frac{1}{(p^z/A)-1}$ $\forall\ z\in\mathbb{C}\setminus\{1\},\Re(z)\geq\frac{1}{2}$. So for the same reason as above, the application:

$\hat{S}$: $X\to\prod_{p\in P}\frac{1}{1-X/p^z}$ is holomorphic in the open quasi-disc $D=\{X\in\mathbb{C}, 0<|X|<\sqrt{2}\}$ with $z\in\mathbb{C}\setminus\{1\},\Re(z)\geq\frac{1}{2}$

Let's extend the function $\hat{S}$ by setting:

For $z\in\mathbb{C}\setminus\{1\},\Re(z)>\frac{1}{2}$ and $\forall\ s\in\mathbb{R}$ with $s\leq 0$ such as $\Re(s+z)\geq 0$

$\hat{S}(C/q^s)=\prod_{p\in P}\frac{1}{1-C/(q^s p^z)}$ (where q is a prime number, and C is such that $|C|=\sqrt{2}$)

In particular we have :

$\hat{S}(A/q^s)=\prod_{p\in P}\frac{1}{1-A/(q^s p^z)}$ (where q is a prime number)

But for $z\in\{z\in\mathbb{R}\setminus\{1\},z>\frac{1}{2}\}$ we have :

$$\prod_{p\in P}\left|\frac{1}{1-A/(q^s p^z)}\right|\leq\prod_{p\in P}\left|\frac{1}{1-A/(p^z)}\right|$$

It follows that :

$$\hat{S}(A/q^s) = 0$$

So : $\hat{S}(X) = 0, \forall\ X \in D$

And consequently :

$$\hat{S}(1)(z) = \zeta(z) = 0 \quad \forall\ z \in \{z \in \mathbb{C} \setminus \{1\}, \Re(z) > \frac{1}{2}\}$$

which is absurd, so $\alpha = 0$, hence the Riemann hypothesis.

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