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# Multivariate Weighted Poisson Distribution: An Application To The Evaluation Of The Effect Of The Number Of Doses Of Intermittent Treatment In Pregnant Women On The Multiplicity Of Plasmodium Falciparum Infections.

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***Abstract:***
***Background****: Most of the classical distributions of count data available in the literature have a univariate weighted Poisson version. This family of weighted Poisson distributions is therefore very useful for dealing with all forms of dispersion as a function of the data. The bivariate case has also been studied by Elion et al, and Nganga et al.*
***Methods****: In this paper, we propose the generalization of weighted Poisson distributions to n-variables by constructing its multivariate probability density via the product of conditional distributions. We also present the structure of its variance-covariance matrix and the estimation of the model parameters. After proposing some special cases of discrete multivariate distributions, an application is made to a trivariate weighted Poisson model with real data on the multiplicity of malaria infections.*
*About the application after choice of the model, one found that the outcome variables are effectively dependent to each other, and the correlations are positive and negative. Therefore, the effect of the dose of Sulfadoxine Pyrimethamine (SP) taken express a better causality by modeling together all the multiplicity of infection from the three compartments.*
***Conclusion:*** *It is shown that the proposed multivariate weighted Poisson distribution allows classical discrete distributions to be combined and even manipulated more easily than some existing models.*
***Key Word****: Weighted Poisson, dispersion index, marginal distribution, generalized linear model.*

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## I. Introduction

Weighted Poisson distributions are known to offer a flexible selection of the standard family of probability distributions useful for modelling random discrete data. From Fisher [1], Rao[2], to today, it has been shown that the way in which data from real phenomena is collected affects the distributions followed by that discrete data. We therefore need to carefully choose the weighted function to deal with the dispersion of the data when verifying and confirming the model. It is worth remembering that the ratio of variance to mean, commonly known as the Fisher dispersion index (*DI*), is the key parameter used to define underdispersion (*DI* < *1*), equidispersion (*DI* = *1*) and overdispersion (*DI* > *1*) in the case of a univariate distribution. And that the weighted discrete random variable $Y^w$ follows a weighted Poisson distribution(*WPD)* if its probability mass function (*pmf*) is given by[3],

$$P(Y^w = y) = p_w(y;\theta) = \frac{w(y)p(y;\theta)}{E_w[w(Y)]}.$$

Where $p(y;\theta) = \frac{\theta^y}{y!} e^{-\theta}$ is the *pmf* of the Poisson distribution with parameter $\theta > 0$ followed by the random variable $Y \in \mathbb{N}$. Note that *w(y)* is the non-negative weight function and that $E_\theta$ denotes the mathematical expectation with respect to $p(y;\ \theta)$, i.e.,

$$E_\theta[w(Y\,)] \ = \sum_{y=0}^{\infty} w(y\,)p(y\,;\,\theta) \ < \ \infty.$$

This weighted distribution is characterized by certain properties, such as its mathematical expectation and its variance, which are given by Kokonendji, Mizère and Balakrishnan respectively[4].

$$E_\theta[Y^w] = \ \theta\left(1 + \frac{d}{d\theta} lnE_\theta[w(Y\,)]\right); \qquad (0)$$
$$Var(Y^w) = \ E_\theta[Y^w] + \theta^2 \frac{d^2}{d\theta^2} \ln E_\theta[w(Y\,)].$$

Due to their wide applicability to real phenomena in different areas of life, a fact that could be attributed to the aforementioned interpretation, several univariate or bivariate weighted Poisson derivative distributions have been widely investigated in the literature (e.g., Efron[5] ; Cameron and Johansson[6] ; Ridout and Besbeas[7] ; Castillo and Pérez-Casany[8], [9] ; Kokonendji, Mizère and Balakrishnan[4]; Bohm and Zech[10] ; Balakrishnan, Koutras , and Milienos[11]; Elion et al[12]; Nganga et al[13]). In general, discrete data source phenomena of interest in biomedical involve the examination of more than one measure at a time, of relationships between measures and comparisons between them. However, certain multivariate Poisson distributions (*MPD*) have been introduced to meet this need. These distributions deal with the case where all the explained variables follow the univariate Poisson distribution. Most of the proposed *MPD* allow positive correlation (see, for example, Kawamura[14]; and Kokonendji and Puig[15]). Starting from the limit of the multivariate Poisson distribution proposed by Kokonendji and Puig, via the framework defined by Berkhout and Plug[16], there is recently a *MPD* constructed by crossing the conditional univariate Poisson distribution to allow zero and negative correlation with the *pmf* defined as (see, Mizélé Kitoti, Bidounga and Mizère[17]):

$$P(Y_1 = \ y_1, \dots, Y_k = \ y_k) = \prod_{i=1}^{k} \frac{[\theta_i(y_1, \dots, y_{i-1})]^{y_i}\ e^{-\theta_i(y_1,\dots,y_{i-1})}}{y_i!};\ \forall\ \mathrm{i} \ = \ 2, \dots, \mathrm{k};\ \forall(y_1, \dots, y_{i-1}) \in \ \mathbb{N}^k. \quad (1)$$

However, some counting events may combine two or more discrete variables, some of which are not necessarily standard Poisson variables, but rather over-dispersed (under-dispersed) counts. The equality of the mean and variance for each $Y_i$ in the *MPD* makes it very restrictive, and so more flexible models are often sought. Therefore, in this paper, we mainly focus on the construction of a new multivariate weighted Poisson distribution (*MWPD*) using the same framework as Elion[12] and Mizélé[17] in order to extend their work and introduce a generalized multivariate model based on discrete weighted explained variables.

Indeed, to our knowledge, no work of this kind has been suggested in the literature to date. In section 2, we construct the mass function and estimate the classical parameters and the structure of its variance-covariance matrix. Section 3 is devoted to the study of the multivariate Fischer dispersion index and to the presentation of some examples of special cases of multivariate classical laws. Section 4 presents the application to malaria data. In fact, among pregnant women, the multiplicity of infection of the *P. Falciparum* seems to be linked to the dosage of intermittent treatment in a univariate manner in peripheral blood, umbilical cord blood and placental blood. We therefore plan to examine the situation in a trivariate manner by pooling the three compartments. section 5 concludes by presenting some keys results.

## II. Multivariate weighted Poisson distribution

**Constructing the model.**

Starting from Berkhout's framework[16], we find a rational way to generalize the concept to weighted Poisson variables by approaching it from a probabilistic point of view to obtain the results below.

**Lemma 2.1.1.** Let $\left(Y_1^{w_1}, \dots, Y_s^{w_s}\right) \in \ \mathbb{N}^s,\ \forall$ s > 1, be a *s*-dimensional random vector of weighted Poisson variables depending on the canonical parameters $(\theta_1, \dots, \theta_s) \in \ \mathbb{R}^{s*+}$ and having $\forall$ i $\in$ $\{1, \dots, s\}$, $w_i(y_i\,)$ a weighted function. For all variables $Y_i^{w_i}$ we can define a conditional distribution whose probability density function is expressed as follows,

$$P(\ Y_i^{w_i} = y_i/Y_1^{w_1}, Y_2^{w_2}, \dots, Y_{i-1}^{w_{i-1}}\ ) = \frac{w_i(y_i)}{E_{\theta_i}[w_i(Y_i\,)]} \frac{\left[\theta_i\left(Y_1^{w_1}, Y_2^{w_2}, \dots, Y_{i-1}^{w_{i-1}}\right)\right]^{y_i}\ e^{-\theta_i\left(Y_1^{w_1}, Y_2^{w_2}, \dots, Y_{i-1}^{w_{i-1}}\right)}}{y_i!}. \qquad (2)$$

In fact, $\theta_i$ is taken as the dependency parameter of $Y_i^{w_i}$ over all given variables $Y_1^{w_1}, Y_2^{w_2}, \ldots, Y_{i-1}^{w_{i-1}}$. The step-by-step modality shift from $y_2$ to $y_s$ in the expression (2) leads to the following *pmf*,

$$P\left( Y_i^{w_i} = y_i / Y_1^{w_1} = y_1, \ldots, Y_{i-1}^{w_{i-1}} = y_{i-1} \right) = \frac{w_i(y_i)}{E_{\theta_i}[w_i(Y_i)]} \frac{[\theta_i(y_1, \ldots, y_{i-1})]^{y_i} e^{-\theta_i(y_1,\ldots,y_{i-1})}}{y_i!};$$

$\forall\, i = 2, \ldots, s$ ; $\forall\, yi \in \mathbb{N}$.

Obviously, the above expression can be simplified,

$$P\left( Y_i^{w_i} = y_i / Y_1^{w_1} = y_1, \ldots, Y_{i-1}^{w_{i-1}} = y_{i-1} \right) = \frac{w_i(y_i)}{E_{\theta_i}[w_i(Y_i)]} \frac{[\theta_i]^{y_i} e^{-\theta_i}}{y_i!};\ \forall\, i = 2, \ldots, s;\ \forall\, yi \in \mathbb{N}. \quad (3)$$

By inference on the weighted Poisson variable (see for example Nganga et al[13]), this conditional distribution is characterized by the mean and variance such that,

$$E_{\theta_i}\left[Y_i^{w_i} / Y_1^{w_1}, Y_2^{w_2}, \ldots, Y_{i-1}^{w_{i-1}}\right] = \theta_i \left(1 + \frac{d}{d\theta_i} lnE_{\theta_i}[w(Y_i)]\right); \quad (4)$$

$$Var\left(Y_i^{w_i} / Y_1^{w_1}, Y_2^{w_2}, \ldots, Y_{i-1}^{w_{i-1}}\right) = E_{\theta_i}\left[Y_i^{w_i}\right] + \theta_i^2 \frac{d^2}{d\theta_i^2} lnE_{\theta_i}[w(Y_i)].$$

Similarly, its probability generating function (*pgf*) and its moment generating function (*mgf*) are respectively summarized by the formulae,

$$G_{Y_i^{w_i}}(t) = \frac{e^{\theta_i(t-1)} E_{t\theta_i}[w_i(Y_i)]}{E_{\theta_i}[w_i(Y_i)]},$$

and

$$M_{Y_i^{w_i}}(t) = \frac{e^{\theta_i(e^t-1)} E_{e^t\theta_i}[w_i(Y_i)]}{E_{\theta_i}[w_i(Y_i)]}, -1 < t \leq 1.$$

**Theorem 2.1.1.** ∀ s > 1, the combination of weighted counts $(Y_i^{w_i})$, i = 2,…, s, having a conditional distribution as shown above (see Lemma 1), produces a joint distribution expressed as follows:

$$P\left( Y_1^{w_1} = y_1, Y_2^{w_2} = y_2, \ldots, Y_i^{w_i} = y_i \right) = \prod_{i=1}^{s} \frac{w_i(y_i)}{E_{\theta_i}[w_i(Y_i)]} \frac{[\theta_i]^{y_i} e^{-\theta_i}}{y_i!}; \quad (5)$$

With by convention for $i = 1$, $\theta_1(y_0) = \theta_1$.

The conditional distributions built in Lemma 1 are univariate weighted Poisson distributions with values between 0 and 1. So, the finite product of these distributions in ℕ gives a maximum sum of one. This clearly proves that this is a probability distribution.

**Definition 2.1.1.** The *s*-dimensional random vector of dependent weighted Poisson variables $Y^w = (Y_1^{w_1}, \ldots, Y_s^{w_s})$ follows a multivariate weighted Poisson distribution if its joint distribution can be given as the function (5). With $w_i(y_i)$ its weighted function and $E_{\theta_i}$ its mean with respect to the distribution of $Y_i$ as the normalization constant in $\theta_i$.

**Generalized log-linear model.**

The linear model combines a linear variance function and a logarithmic linear relationship between the means and the covariates. Undoubtedly, for any permutation, one can derive from the joint distribution a marginal weighted Poisson distribution and $s - 1$ conditional weighted Poison distributions $\text{WPD}(Y_i^{w_i}; \theta_i)$, which belong to the family of regular exponential distributions and are maximum entropy distributions about the sufficient statistics. In addition, $\text{WPD}(Y_i^{w_i}; \theta_i)$, can be used in generalized linear models with covariates $x^T = (x_1, x_2, \ldots, x_n)$ in log-linear models where the variance function is assumed to be linear. Obviously, the linearity of the variance function is verified for the Poisson distribution, but for a conditional weighted Poison distribution $\text{WPD}(Y_i^{w_i}; \theta_i)$, the variance function is almost linear over a wide range of mean parameters. Consequently, its mathematical expectation can be formulated as follows,

$$E_{\theta_i}\left[Y_i^{w_i} / Y_1^{w_1}, Y_2^{w_2}, \ldots, Y_{i-1}^{w_{i-1}}\right] = \mu_i = \theta_i(1 + \alpha_i). \quad (6)$$

With $\alpha_i = \frac{d}{d\theta_i} lnE_{\theta_i}[w(Y_i)] \in \mathbb{R}^+$.

Of course, this parameter $\mu_i$ which is a function of $\theta_i$ can in turning vary according to the covariates of the regressor $x$ expressing the heterogeneity observed between individuals, while remaining the key to the interaction between all the $Y_i^{w_i}$. The generalized log-linear model is therefore highlighted by the Poisson family link function below,

$$\forall\, i \in \{2; 3; \dots; k\};\ \ln(\mu_i) = \ln(\theta_i) + \ln(1 + \alpha_i),$$

$$\ln(\mu_i) = x^T\beta_i + \sum_{r=1}^{i-1} \eta_{ir} y_r\,. \tag{7}$$

When $\eta_{ir}$ is zero, the expression (7) becomes the linear model according to the marginal distribution. This representation of the model appears to be the same as that of the multivariate Poisson distribution by the simple fact that $M\mathrm{WPD}(Y^w;\theta)$ is its prominent and plausible generalization. This is why $\mu_i$ and $\theta_i$ are good generators of the same estimators of the model parameters ($\beta_i$ and $\eta_{ir}$). And the log-likelihood function according to the joint distribution with respect to the parameter $\theta_i$ is,

$$\ln \mathcal{L}(y;\theta) = \sum_{i=1}^{s} \left( \ln[w_i(y_i)] - \ln\left(E_{\theta_i}[w_i(Y_i)]\right) + y_i \ln\theta_i - \ln(y_i!) - \theta_i \right). \tag{8}$$

$\forall\, s \in \mathbb{N} - \{0; 1\};\ \theta = (\theta_1, \theta_2, \dots, \theta_s)$ and $y = (y_1, y_2, \dots, y_s)$.

**Structure of the variance-covariance matrix.**

The development of this structure is easier when the situation is examined between two variables. Let us therefore concentrate on estimating the characteristics of the joint distribution around the *i*-th, *i+1*-th and *i+2*-th variables of the random vector $Y^w = (Y_1^{w_1}, \dots, Y_s^{w_s})$. The expression (7) allows us to deduce the relationships below,

$$\ln(\mu_{i+1}) = x^T\beta_{i+1} + \sum_{r=1}^{i-1} \eta_{ir} y_r + \eta_{ii} y_i\,; \tag{9}$$

$$\ln(\mu_{i+2}) = x^T\beta_i + \sum_{r=1}^{i-1} \eta_{ir} y_r + \eta_{ii} y_i + \eta_{(i+1)i} y_{i+1}\,. \tag{10}$$

These equations, with a judicious parametrization, can be used to manipulate the conditional expectation through the moment generating method from a probabilistic point of view to find the properties below.

**Proposition 2.1.1.** For two consecutive weighted random variables $Y_i^{w_i}$ and $Y_{i+1}^{w_{i+1}}$ we can define the following characteristics:

1. $E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right] = \mu_{i+1} e^{\theta_i(e^{\eta_{ii}}-1)} \frac{E_{e^{\eta_{ii}}\theta_i}[w_i(Y_i)]}{E_{\theta_i}[w_i(Y_i)]}$ ;
2. $Var\left(Y_{i+1}^{w_{i+1}}\right) = E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right] + \left(E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]\right)^2 \left( e^{\theta_i(e^{\eta_{ii}}-1)^2} E_{\theta_i}[w_i(Y_i)] \frac{E_{e^{2\eta_{ii}}\theta_i}[w(Y_i)]}{(E_{e^{\eta_{ii}}\theta_i}[w(Y_i)])^2} - 1 \right)$;
3. $Cov\left(Y_i^{w_i}, Y_{i+1}^{w_{i+1}}\right) = E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right] \left( \theta_i e^{\eta_{ii}} + \frac{d}{d\eta_{ii}} \ln E_{e^{\eta_{ii}}\theta_i}[w_i(Y_i)] - E_{\theta_i}\left[Y_i^{w_i}\right] \right)$.

**Corollary 2.1.1.** As for the non-consecutive variables $Y_i^{w_i}$ and $Y_{i+1}^{w_{i+1}}$, these same characteristics are found to be as follows:

1. $E_{\theta_{i+2}}\left[Y_{i+2}^{w_{i+2}}\right] = \mu_{i+2} e^{\theta_i(e^{\eta_{ii}}-1)} \frac{E_{e^{\eta_{ii}}\theta_i}[w_i(Y_i)]}{E_{\theta_i}[w_i(Y_i)]}$ ;
2. $Var\left(Y_{i+2}^{w_{i+2}}\right) = E_{\theta_{i+2}}\left[Y_{i+2}^{w_{i+2}}\right] + \left(E_{\theta_{i+2}}\left[Y_{i+2}^{w_{i+2}}\right]\right)^2 \left( e^{\theta_i(e^{\eta_{ii}}-1)^2} E_{\theta_i}[w_i(Y_i)] \frac{E_{e^{2\eta_{ii}}\theta_i}[w(Y_i)]}{(E_{e^{\eta_{ii}}\theta_i}[w(Y_i)])^2} - 1 \right)$;
3. $Cov\left(Y_i^{w_i}, Y_{i+2}^{w_{i+2}}\right) = E_{\theta_{i+2}}\left[Y_{i+2}^{w_{i+2}}\right] \left( \theta_i e^{\eta_{ii}} + \frac{d}{d\eta_{ii}} \ln E_{e^{\eta_{ii}}\theta_i}[w_i(Y_i)] - E_{\theta_i}\left[Y_i^{w_i}\right] \right)$. $\forall\, i = 1; 2; \dots; s-2$

The proof of Proposition 2.2.1 and its Corollary 2.2.1 is shown by posing $C_i = \sum_{r=1}^{i-1} \eta_{ir} y_r$ and $h_i = \ln(1 + \alpha_i)$
then using the demonstration from Nganga et al[13].

**Proposition 2.1.2.** For any value of $i \in \{1; 2; \dots; s\}$, if $\theta_i >$ e (basic value of the exponential) and if

$\frac{\ln y_i - \ln \theta_i}{\theta_i} < \eta_{ii} < 0$, then we have the inequality $\theta_i\, e^{\eta_{ii}} + \frac{d}{d\eta_{ii}} \ln E_{e^{\eta_{ii}}\theta_i}[w_i(Y_i)] - E_{\theta_i}[Y_i^{w_i}] < 0$.

The proof of this Proposition 2.2.2 is given in Appendix A.

This clearly shows that the sign of the covariance between the variables $Y_i^{w_i}$ depends on the sign of $\eta_{ii}$. When $\eta_{ii}$ is zero for all the $i \in \{1;\ 2;\ldots;\ s\}$, the $Y_i^{w_i}$ are two by two independent. Thus, if $\forall\, r \in \{1;\ 2;\ldots;\ i-1\}$ ; $\eta_{ir} = 0$, there is no conditional distribution and any $Y_i^{w_i}$ becomes a simple weighted Poisson variable. From these covariance and variance values, we construct the variance-covariance matrix, which can contain negative, zero and positive values depending on the correlation of the pairwise weighted variables. We show that this joint density is made up of a flexible correlation structure.

**Remark.** For any $i = 1;\ 2;\ldots;\ s$, if the weighting ratio, i.e., the weight function over the normalization constant, is equal to 1, then we find the basic multivariate Poisson distribution proposed by Mizélé et al[17].

**Estimation of the parameters $\beta_i$ and $\eta_i$ from the model.**

The log-likelihood equation expressed in (8) is a useful bilinear function for estimating the parameters $\beta_i$ and $\eta_{ir}$ mentioned in our assumption of modelling. We indeed use one of the most common frameworks, namely maximum likelihood estimation, to find the modelling elements (parameter values) that maximize the likelihood function; let $\hat{\beta}_i$ and $\hat{\eta}_{ir}$ respectively be likelihood estimators of $\beta_i$ and $\eta_{ir}$. Let us investigate the estimate of $\hat{\beta}_i$ by calculating the partial derivatives of the log-likelihood function $\ln \mathcal{L}$.

$$\frac{\partial \ln \mathcal{L}}{\partial \beta_i} = x^T y_i - x^T e^{x^T\beta_i + c_i} - \frac{\partial}{\partial \beta_i} \ln E_{\theta_i}[w_i(Y_i)]$$
$$= x^T y_i - x^T e^{x^T\beta_i + c_i} - \frac{\partial \theta_i}{\partial \beta_i} \times \frac{\partial}{\partial \theta_i} \ln E_{\theta_i}[w_i(Y_i)]$$

We know that $\frac{\partial \theta_i}{\partial \beta_i} = x^T e^{x^T\beta_i + c_i}$; so, we find,

$$\frac{\partial \ln \mathcal{L}}{\partial \beta_i} = x^T y_i - x^T e^{x^T\beta_i + c_i}(1 + \alpha_i)$$
$$= x^T(y_i - \theta_i(1 + \alpha_i))$$
$$= x^T(y_i - \mu_i)$$

Its secondary derivative can be found as follows,

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \beta_i \partial {\beta_i}^T} = -x^T \left[ x e^{x{\beta_i}^T + c_i}(1 + \alpha_i) + e^{x^T\beta_i + c_i} \frac{\partial (1 + \alpha_i)}{\partial {\beta_i}^T} \right]$$

$$= -x^T \left[ x\mu_i + x e^{x^T\beta_i + c_i} \frac{\partial \theta_i}{\partial {\beta_i}^T} \times \frac{\partial \alpha_i}{\partial \theta_i} \right]$$

$$= -x^T x \left[ \mu_i + e^{2(x{\beta_i}^T + c_i)} \frac{\partial^2}{\partial {\theta_i}^2} \ln E_{\theta_i}[w_i(Y_i)] \right]$$

$$= -\|x\|^2 \left[ E_{\theta_i}\left[Y_i^{w_i}/Y_1^{w_1}, Y_2^{w_2}, \ldots, Y_{i-1}^{w_{i-1}}\right] + {\theta_i}^2 \frac{\partial^2}{\partial {\theta_i}^2} \ln E_{\theta_i}[w_i(Y_i)] \right]$$

$$= -\|x\|^2 Var\,(Y_i^{w_i}/Y_1^{w_1}, Y_2^{w_2}, \ldots, Y_{i-1}^{w_{i-1}})$$

Where $\|x\|$ is a scalar product of the explanatory vector $x$. Similarly, the first and second differentials of $\ln \mathcal{L}$ with respect to the parameter $\eta_{ir}$ are obtained as follows,

$$\frac{\partial \ln \mathcal{L}}{\partial \eta_{ir}} = -\frac{\partial}{\partial \eta_{ir}} \ln E_{\theta_i}[w_i(Y_i)] + \frac{\partial}{\partial \eta_{ir}} y_i \ln \theta_i - \frac{\partial \theta_i}{\partial \eta_{ir}} = \frac{\partial \theta_i}{\partial \eta_{ir}} \left[ -\frac{\partial}{\partial \theta_i} \ln E_{\theta_i}[w_i(Y_i)] + \frac{y_i}{\theta_i} - 1 \right]$$

However, $\frac{\partial \theta_i}{\partial \eta_{ir}} = \theta_i \sum_{r=1}^{i-1} y_r$, so one finds,

$$\frac{\partial \ln \mathcal{L}}{\partial \eta_{ir}} = \sum_{r=1}^{i-1} y_r (y_i - \theta_i(1 + \alpha_i)) = (y_i - \mu_i) \sum_{r=1}^{i-1} y_r.$$

So, we have its second partial derivative,

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \eta_{ir} \partial \eta_{ir}} = -\sum_{r=1}^{i-1} y_r \left[\theta_i (1 + \alpha_i) \sum_{r=1}^{i-1} y_r\right] = -\mu_i \left(\sum_{r=1}^{i-1} y_r\right)^2.$$

Intuitively, we have also got this,

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \eta_{ir} \partial \beta_i} = -\sum_{r=1}^{i-1} y_r \left[\frac{\partial \mu_i}{\partial \beta_i}\right] = -\mu_i \sum_{r=1}^{i-1} x^T y_r.$$

All these results are summarized in the proposition below.

**Proposition 2.1.3.** For the estimation of the model parameters as defined in the generalized linear model (15), we obtain the following results,

1. $\frac{\partial \ln \mathcal{L}}{\partial \beta_i} = x^T (y_i - \mu_i)$, with $\frac{\partial^2 \ln \mathcal{L}}{\partial \beta_i \partial {\beta_i}^T} = -\|x\|^2 Var\,(/Y_1^{w_1}, Y_2^{w_2}, \dots, Y_{i-1}^{w_{i-1}})$.
2. $\frac{\partial \ln \mathcal{L}}{\partial \eta_{ir}} = (y_i - \mu_i) \sum_{r=1}^{i-1} y_r$, with $\frac{\partial^2 \ln \mathcal{L}}{\partial \eta_{ir} \partial \eta_{ir}} = -\mu_i \left(\sum_{r=1}^{i-1} y_r\right)^2$.
3. $\frac{\partial^2 \ln \mathcal{L}}{\partial \eta_{ir} \partial \beta_i} = -\mu_i \sum_{r=1}^{i-1} x^T y_r$.

## III. Multivariate dispersion index and some examples of classic distributions

**Multivariate dispersion index.**

In the multivariate framework, Fischer's dispersion index seems complex to estimate. Numerous suggestions with certain limitations have been made based on matrix and scalar methods for measuring dispersion (see, for example, Jørgensen and Kokonendji[18], Karlis and Xekalaki[19], Alerts et al[20], Reyment[21], van Valen[22]). However, an alternative new index caught our attention among Kokonendji and Puig's proposals[15], for its flexibility and ability to deal with the dispersion of marginal distributions and correlation coefficients between variables through the covariance matrix. There are no restrictions on the dimension $s$, the sample size $n$ and the *rank* of the dispersion matrix. It is useful and easily adaptable for synthesizing the dispersion of the weighted joint distribution from the marginal and conditional distributions that belong to the natural exponential family. Let $Y^w$ be the discrete non-degenerate weighted random vector on $\mathbb{N}^s$ with a variance-covariance matrix $COV\ Y^w$, a correlation coefficient matrix $RY^w$ and an expectation matrix $EY^w$. We assume that $det, diag,$ and $DI$ are respectively the determinant, the diagonal, and the dispersion index of the univariate distribution. In this framework, we have three parameters defined as follows.

**The *generalized variance* (*sGV*)**, which compares the variability of two multivariate discrete models represented by mathematical expectation matrices with the same dimension. And the appropriate measure seems to be the ratio of the variances of the weighted vector $Y^w$and the variance of its Poisson vector $Y$. But this is simplified to the following expression,

$$sGV(Y^w) = \frac{\det(COV\ Y^w)}{\det(diag\ EY^w)};$$

with $\det(COV\ Y^w) = \det(RY^w) \times \prod_{i=1}^{s} Var\left(Y_i^{w_i}\right)$, and $\det(RY^w) \in ]0,1]$.

We then have a multivariate (over, equi or under) dispersion measure of $Y^w$depending on whether $(sGV(Y^w) > 1,\ sGV(Y^w) = 1,\ sGV(Y^w) < 1)$, respectively.

**The *marginal multiple dispersion index* (*MDI*)** is the parameter that considers only the dispersion information from the marginal distribution. For the multivariate weighted Poisson, the univariate conditional distributions constructed can be taken one at a time to be marginally weighted and have the following index,

$$\text{MDI}(Y^w) = \sum_{i=1}^{s} \frac{\left(E_{\theta_i}[Y_i^{w_i}]\right)^2}{(EY^w)^T EY^w} DI(Y_i^{w_i}).$$

We retain the main definition of the ***generalized dispersion index (GDI)*** of $Y^w$ as,

$$GDI(Y^w) = \frac{\sqrt{EY^w}^T COV\ Y^w \sqrt{EY^w}}{\left(\sqrt{EY^w}\right)^T \sqrt{EY^w}}.$$

Thus, the dispersion corresponding to the variables of $Y^w$ can be defined such that $Y^w$ is over- (equi- or under-) dispersed, if and only if $GDI(Y^w)$ is greater than 1 (equal to 1 or less than 1), respectively.

**Some examples of classic distributions.**

In this subsection, we firstly deal with the case of a series of discrete variables that follow the same probability distribution but whose canonical parameters may be different. Next, we present the case of data describing a multivariate configuration where some are under-dispersed and others over-dispersed. Let us also note that the weighting function $w_i(y_i) = w_i(y_i;\theta_i;\delta_i)$ may depend on the parameter $\delta_i$ which derives from the data registration procedure and on the canonical parameter $\theta_i$. In fact, we use Proposition 2.2.1 to deduce certain characteristics of the proposed distributions. And the $\mu_{i+1}$ is indeed the one taken from equation (9).

**The multivariate binomial distribution.** After a good parametrization as defined by Elion et al[12], the univariate binomial distribution is indeed a weighted Poisson distribution whose multivariate version has the mass function:

$$p(y_1,\dots,y_s;\theta_1,\dots,\theta_s;k_1,\dots,k_s) = \prod_{i=1}^{s} \frac{\frac{k_i!}{(k_i-y_i)!}}{(\theta_i+1)^{k_i}e^{-\theta_i}} \frac{\theta_i^{y_i}}{y_i!} e^{-\theta_i}, \forall\,(y_i,k_i)\in\mathbb{N}^2, \theta_i > 0. \quad (11)$$

Where its weight function in the univariate framework is,

$$w_i(y_i;k_i) = \begin{cases} \frac{k_i!}{(k_i-y_i)!}; & y_i = 0,1,2,\dots,k_i \\ 0; & y_i = k_i+1, k_i+2, k_i+3,\dots \end{cases}$$

And the associated normalization constant is,

$$E_{\theta_i}[w_i(Y_i)] = (\theta_i+1)^{k_i}e^{-\theta_i}.$$

Thus, for consecutive discrete variables $Y_i^{w_i}$ and $Y_{i+1}^{w_{i+1}}$, one has:

1. $E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right] = \mu_{i+1}\left(\frac{1+\theta_i e^{\eta_{ii}}}{\theta_i+1}\right)^{k_i}$ ;
2. $Var\left(Y_{i+1}^{w_{i+1}}\right) = E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]\left(1+\mu_{i+1}\frac{(1+\theta_i e^{2\eta_{ii}})^{k_i}}{(1+\theta_i e^{\eta_{ii}})^{k_i}} - E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]\right)$;
3. $Cov\left(Y_i^{w_i}, Y_{i+1}^{w_{i+1}}\right) = \mu_{i+1}\theta_i\, k_i(e^{2\eta_{ii}}-1)\frac{(1+\theta_i e^{\eta_{ii}})^{k_i-1}}{(1+\theta_i)^{k_i+1}}$.

**The multivariate negative binomial distribution I.** Its joint distribution is:

$$p(y_1,\dots,y_s;\theta_1,\dots,\theta_s;\delta_1,\dots,\delta_s) = \prod_{i=1}^{s} \frac{\frac{\Gamma\left(\theta_i{\delta_i}^{-1}+y_i\right)}{\Gamma\left(\theta_i{\delta_i}^{-1}\right)}\left(\frac{\delta_i}{\theta_i(\delta_i+1)}\right)^{y_i}}{(1+\delta_i)^{\theta_i{\delta_i}^{-1}}e^{-\theta_i}} \frac{\theta_i^{y_i}}{y_i!} e^{-\theta_i}, \qquad \forall\, y_i \in \mathbb{N}, \theta_i > 0, \delta_i > 0. \,(12)$$

Where its weight function in the univariate framework is[12],

$$w_i(y_i;\theta_i;\delta_i) = \frac{\Gamma(\theta_i{\delta_i}^{-1}+y_i)}{\Gamma(\theta_i{\delta_i}^{-1})}\left(\frac{\delta_i}{\theta_i(\delta_i+1)}\right)^{y_i}.$$

And the associated normalization constant is,

$$E_{\theta_i}[w_i(Y_i)] = (1+\delta_i)^{\theta_i{\delta_i}^{-1}}e^{-\theta_i}.$$

Thus, for consecutive discrete variables $Y_i^{w_i}$ and $Y_{i+1}^{w_{i+1}}$, one has the proprieties below:

1. $E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right] = \mu_{i+1}(1+\delta_i)^{\theta_i{\delta_i}^{-1}(e^{\eta_{ii}}-1)}$ ;
2. $Var\left(Y_{i+1}^{w_{i+1}}\right) = \left(E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]\right)^2\left(e^{\theta_i}(1+\delta_i)^{\theta_i{\delta_i}^{-1}(e^{\eta_{ii}}-1)^2} + \frac{1}{E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]} - 1\right)$;
3. $Cov\left(Y_i^{w_i}, Y_{i+1}^{w_{i+1}}\right) = E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]E_{\theta_i}\left[Y_i^{w_i}\right]\left(\frac{\theta_i{\delta_i}^{-1}e^{\eta_{ii}}\ln(1+\delta_i)}{E_{\theta_i}\left[Y_i^{w_i}\right]} - 1\right)$.

**The multivariate translated Poisson distribution.** This distribution is obviously a weighted Poisson distribution[12] whose multivariate version has the mass function:

$$p(y_1,\dots,y_s;\theta_1,\dots,\theta_s;\delta_1,\dots,\delta_s)=\prod_{i=1}^{s}\frac{\frac{y_i!}{(y_i-\delta_i)!}}{\theta_i^{\delta_i}}\frac{\theta_i^{y_i}}{y_i!}e^{-\theta_i}.\ \forall\, y_i\in\mathbb{N},\theta_i>0,\delta_i\in\mathbb{N}^*. \quad (13)$$

Where its weighting function in the univariate framework is,

$$w_i(y_i;\delta_i)=\begin{cases}\frac{y_i!}{(y_i-\delta_i)!}; & y_i=\delta_i+1,\delta_i+2,\delta_i+3,\dots\\ 0; & y_i=0,1,2,\dots,\delta_i.\end{cases}$$

And the associated normalization constant is,

$$E_{\theta_i}[w_i(Y_i)]=\theta_i^{\delta_i}.$$

Thus, for consecutive discrete variables $Y_i^{w_i}$ and $Y_{i+1}^{w_{i+1}}$, one has some characteristics below:

1. $E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]=\mu_{i+1}e^{\theta_i(e^{\eta_{ii}}-1)+\delta_i\eta_{ii}}$ ;
2. $Var\left(Y_{i+1}^{w_{i+1}}\right)=E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]\left(1+\mu_{i+1}e^{\theta_i e^{2\eta_{ii}}-\theta_i e^{\eta_{ii}}+\delta_i\eta_{ii}}\right)$;
3. $Cov\left(Y_i^{w_i},Y_{i+1}^{w_{i+1}}\right)=\mu_{i+1}(\theta_i e^{\eta_{ii}}+\eta_{ii}-\theta_i-\delta_i)e^{\theta_i(e^{\eta_{ii}}-1)+\delta_i\eta_{ii}}$.

**The multivariate zero-weighted Poisson distribution.** Like the one above, this distribution is also a weighted Poisson distribution[12] whose multivariate version has the mass function:

$$p(y_1,\dots,y_s;\theta_1,\dots,\theta_s;\delta_1,\dots,\delta_s)=\prod_{i=1}^{s}\frac{1}{1-\sum_{y_i=0}^{\delta_i}\frac{\theta_i^{y_i}}{y_i!}e^{-\theta_i}}\frac{\theta_i^{y_i}}{y_i!}e^{-\theta_i},\qquad \forall\, y_i\in\mathbb{N},\theta_i>0,\delta_i\in\mathbb{N}^*. \quad (14)$$

Where its weighting function in the univariate framework is,

$$w_i(y_i;\delta_i)=\begin{cases}1; & y_i=\delta_i+1,\delta_i+2,\delta_i+3,\dots\\ 0; & y_i=0,1,2,\dots,\delta_i.\end{cases}$$

And the associated normalization constant is,

$$E_{\theta_i}[w_i(Y_i)]=1-\sum_{y_i=0}^{\delta_i}\frac{\theta_i^{y_i}}{y_i!}e^{-\theta_i}.$$

From this expression we can derive several other expressions, such as:

$$E_{\theta_i e^{\eta_{ii}}}[w_i(Y_i)]=1-\sum_{y_i=0}^{\delta_i}\frac{(\theta_i e^{\eta_{ii}})^{y_i}}{y_i!}e^{-\theta_i e^{\eta_{ii}}}.$$

$$E_{\theta_i e^{2\eta_{ii}}}[w_i(Y_i)]=1-\sum_{y_i=0}^{\delta_i}\frac{(\theta_i e^{2\eta_{ii}})^{y_i}}{y_i!}e^{-\theta_i e^{2\eta_{ii}}}.$$

$$\frac{d}{d\eta_{ii}}\ln E_{e^{\eta_{ii}}\theta_i}[w_i(Y_i)]=\frac{\sum_{y_i=0}^{\delta_i}\frac{(\theta_i)^{y_i}(y_i-\theta_i e^{\eta_{ii}})}{y_i!}e^{y_i\eta_{ii}-\theta_i e^{\eta_{ii}}}}{E_{\theta_i e^{\eta_{ii}}}[w_i(Y_i)]}.$$

Therefore, for consecutive discrete variables $Y_i^{w_i}$ and $Y_{i+1}^{w_{i+1}}$, it is easier to carry out the expression of the $E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]$, $Var\left(Y_{i+1}^{w_{i+1}}\right)$ and $Cov\left(Y_i^{w_i},Y_{i+1}^{w_{i+1}}\right)$. So, we can find out:

$$E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]=\mu_{i+1}e^{\theta_i(e^{\eta_{ii}}-1)}\frac{1-\sum_{y_i=0}^{\delta_i}\frac{(\theta_i e^{\eta_{ii}})^{y_i}}{y_i!}e^{-\theta_i e^{\eta_{ii}}}}{1-\sum_{y_i=0}^{\delta_i}\frac{\theta_i^{y_i}}{y_i!}e^{-\theta_i}}.$$

**The multivariate COM-Poisson distribution.** It is used to manage the dispersion in the same model of data, some of which are under-dispersed or equi-dispersed and others over-dispersed. This distribution is very useful and is a weighted Poisson distribution[23] whose multivariate version has the joint distribution:

$$p(y_1,\dots,y_s;\theta_1,\dots,\theta_s;\delta_1,\dots,\delta_s)=\prod_{i=1}^{s}\frac{(y_i!)^{\delta_i-1}}{e^{\theta_i}Z(\theta_i,\delta_i)}\frac{\theta_i^{y_i}}{y_i!}e^{-\theta_i},\forall\, y_i\in\mathbb{N},\theta_i>0,\delta_i\in\mathbb{N}^*. \quad (15)$$

Where its weighting function in the univariate framework is,

$$w_i(y_i;\delta_i) = (y_i!)^{\delta_i - 1}.$$

And the associated normalization constant is,

$$E_{\theta_i}[w_i(Y_i)] = e^{\theta_i}Z(\theta_i,\delta_i).$$

With $Z(\theta_i,\delta_i) = \sum_{y_i=0}^{\infty}\frac{\theta_i^{y_i}}{(y_i!)^{\delta_i}}$. Thus, for consecutive discrete variables $Y_i^{w_i}$ and $Y_{i+1}^{w_{i+1}}$, the characteristics below can be defined:

1. $E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right] = \mu_{i+1}\frac{Z(\theta_i e^{\eta_{ii}},\delta_i)}{Z(\theta_i,\delta_i)}$ ;
2. $Var\left(Y_{i+1}^{w_{i+1}}\right) = \left(E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]\right)^2\left(\frac{Z(\theta_i,\delta_i)}{\mu_{i+1}Z(\theta_i e^{\eta_{ii}},\delta_i)} + \frac{Z(\theta_i,\delta_i)Z(\theta_i e^{2\eta_{ii}},\delta_i)}{\left[Z(\theta_i e^{\eta_{ii}},\delta_i)\right]^2} - 1\right)$;
3. $Cov\left(Y_i^{w_i},Y_{i+1}^{w_{i+1}}\right) = E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]E_{\theta_i}\left[Y_i^{w_i}\right]\left(\frac{1}{E_{\theta_i}\left[Y_i^{w_i}\right]}\frac{d}{d\eta_{ii}}\ln Z(\theta_i e^{\eta_{ii}},\delta_i) - 1\right)$.

**The multivariate Poisson-Lindley distribution.** This distribution, although not well known, is very useful in that it facilitates the deal of the dispersion of data within the same framework, some of which are under-dispersed, or equi-dispersed, and others over-dispersed[24]. In this work we claim that this distribution is a weighted Poisson distribution whose multivariate joint distribution can be written as:

$$p(y_1,\ldots,y_s;\theta_1,\ldots,\theta_s) = \prod_{i=1}^{s}\frac{(y_i+\theta_i+2)y_i!}{(\theta_i+1)^{y_i+3}\theta_i^{y_i-2}e^{-\theta_i}}\frac{\theta_i^{y_i}}{y_i!}e^{-\theta_i}, \qquad \forall\, y_i \in \mathbb{N}, \theta_i > 0. \tag{16}$$

Simply remove the product to obtain the univariate version whose weight function is,

$$w_i(y_i;\theta_i) = \frac{(y_i+\theta_i+2)y_i!}{(\theta_i+1)^{y_i+2}\theta_i^{y_i-2}}.$$

And the associated normalization constant is,

$$E_{\theta_i}[w_i(Y_i)] = (\theta_i+1)e^{-\theta_i}.$$

So, to build the structure of the variance-covariance matrix with consecutive discrete variables $Y_i^{w_i}$ and $Y_{i+1}^{w_{i+1}}$, we have:

1. $E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right] = \mu_{i+1}\frac{\theta_i e^{\eta_{ii}}+1}{\theta_i+1}$ ;
2. $Var\left(Y_{i+1}^{w_{i+1}}\right) = E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]\left(1+\mu_{i+1}\frac{\theta_i e^{2\eta_{ii}}+1}{\theta_i e^{\eta_{ii}}+1}\right)\left(e^{-4\theta_i e^{\eta_{ii}}} - 1\right)$;
3. $Cov\left(Y_i^{w_i},Y_{i+1}^{w_{i+1}}\right) = E_{\theta_{i+1}}\left[Y_{i+1}^{w_{i+1}}\right]\left(\theta_i e^{\eta_{ii}} - \theta_i^{\,2}e^{2\eta_{ii}}e^{-\theta_i e^{\eta_{ii}}} - E_{\theta_i}\left[Y_i^{w_i}\right]\right)$.

## IV. Application to biomedical data

In this section, we describe the calculations between patient characteristics and the results of the models associated with each variable of interest on the one hand, and with the trivariate weighted Poisson model on the other.

**The data.**

The data used come from a cross-sectional study carried out between March 2014 and April 2015 in the south of Brazzaville, Republic of Congo, on the molecular characterization of malaria infection due to Plasmodium *Falciparum* and the search for risk factors in pregnant women and newborns. Eight years ago, on the recommendation of the WHO, the government introduced intermittent preventive treatment based on sulfadoxine-pyrimethamine (IPTp-SP) for pregnant women. However, high resistance to this drug had been reported in another study carried out years earlier in the same area of the city. Hence the interest in launching this molecular surveillance, as the parasite can be detected in several ways, depending on the detection marker and the type of allele. Thus, for consenting women who met the study's inclusion criteria, sociodemographic and clinical data were collected, including our study variables: number of IPTp-SP doses taken by the pregnant woman ($x_1$), number of pregnancies got ($x_2$) and the woman's age ($x_3$). In addition, blood samples from the three compartments (peripheral blood, placental blood, and cord blood) were examined using RT-PCR targeting the merozoite surface protein genes of P. *Falciparum* (*MSP1* and *MSP2*). As the parasite is a haplotype, the presence of each different allele symbolizes a different infection. And the multiplicity of infections (MI) is defined as the number of different infections in an individual (See Massamba et al[25] for more explanation).

The hypothesis to be tested here is that as the number of IPTp-SP doses increases, the multiplicity of infections (MI) decreases in the three compartments. And that the probability of having a high multiplicity of infections at the same time in the three compartments would be associated with the decrease in the number of IPTp-SP doses. Data were analyzed using the R Studio Core Team (2022) (R Foundation for Statistical Computing, Vienna, Austria. URL https://www.Rproject.org/).

**Results.**

At the start of the analyses, we described all the variables on the $N = 172$ patients, in particular central trend, and dispersion parameters (Table 1). This gave us an overall idea of the average dose of 2.09 SP taken by the women. Half of them were in their fifth pregnancy 5 [1-9] and half were over 38 [15-43] years old.

Regarding the dependent variables, the mean of peripheral MI $\bar{y}_1 = 1.59$ seems greater than its variance $\hat{\sigma}_1^2 = 0.816$. This is confirmed by the dispersion index $DI(Y_1^{w_1}) = 0.512$, which suggests that the variable $Y_1^{w_1}$ follows an under-dispersed distribution (Table 1). By examining these modalities, which vary between 1 and 6, we can conclude that $p_{1j}(y_{1j}, \theta_1)$ is a zero-truncated Poisson distribution. The $\chi^2$ value from goodness-of-fit test on the observed numbers $n_{1j}$ and the theoretical numbers $N \times \hat{p}_{1j}$ confirms our choice (Table 2).

For placental MI, the mean $\bar{y}_2 = 0.73$ is few lower than the variance $\hat{\sigma}_2^2 = 0.819$ with a dispersion index $DI(Y_2^{w_2}) = 1.128$, which helps us to not choose the classical Poisson distribution (Table 1). Similarly, we set aside the negative binomial distribution due to the happening of events, which here gives a mean certainly low but close to the variance. Therefore, the Poisson-Lindley distribution $p_{2j}(y_{2j}, \theta_2)$ whose $\hat{\theta}_2$ was estimated by Shanker using the maximum likelihood method[24] to best describing the variable $Y_2^{w_2}$ (Table 2).

It turns out that the MI of the cord $Y_3^{w_3}$ shows the same phenomenon as $Y_2^{w_2}$, with a higher dispersion index $DI(Y_3^{w_3}) = 2.50$ (Table 1). Thus, the Poisson-Lindley distribution $p_{3j}(y_{3j}, \theta_3)$ is statistically the most appropriate for modelling of the variable $Y_3^{w_3}$ (Table 2).

Finally, the dispersion relative to these three weighted variables can be examined vectorially using the variance $sGV(Y_1^{w_1}, Y_2^{w_2}, Y_3^{w_3}) = 1.094$ which reveals an over-dispersion of the data. The multiple dispersion of the marginal distributions $MDI(Y_1^{w_1}, Y_2^{w_2}, Y_3^{w_3}) = 0.711$ at its side converges towards under-dispersion. But here the major parameter that allows us to decide is the generalized dispersion index $GDI(Y_1^{w_1}, Y_2^{w_2}, Y_3^{w_3}) = 1.156$ which confirms that this trivariate weighted Poisson distribution is indeed over-dispersed but very close to the classical multivariate Poisson distribution (Table 1).

**Table 1:** Descriptive analysis.

| Statistics | Min | Max | Median | Mean | Variance | *DI* | |
|---|---|---|---|---|---|---|---|
| Number of SP doses taken | 0 | 3 | 3 | 2.09 | 1.30 | | |
| Number of pregnancies got | 1 | 9 | 5 | 4.86 | 5.73 | | |
| Age | 15 | 43 | 38 | 35.92 | 39.65 | | |
| Peripheral MI ($Y_1^{w_1}$) | 1 | 6 | 1 | 1.59 | 0.816 | 0.512 | *sGV* = 1.094 |
| Placental MI ($Y_2^{w_2}$) | 0 | 5 | 0 | 0.73 | 0.819 | 1.128 | *MDI* = 0.711 |
| Umbilical cord MI ($Y_3^{w_3}$) | 0 | 5 | 0 | 0.72 | 0.907 | 1.268 | *GDI* = 1.156 |

**Weighted Poisson regression.**

This trivariate regression model is summarized by the equations (7), (9) and (10), each representing a generalized linear model for what we have below:

1. $\ln \mu_1 = \beta_{10} + \beta_{11}x_1 + \beta_{12}x_2 + \beta_{13}x_3;$
2. $\ln \mu_2 = \beta_{20} + \beta_{21}x_1 + \beta_{22}x_2 + \beta_{23}x_3 + \eta_{21}y_1;$
3. $\ln \mu_3 = \beta_{30} + \beta_{31}x_1 + \beta_{32}x_2 + \beta_{33}x_3 + \eta_{31}y_1 + \eta_{32}y_{2.}$

The idea is that having confirmed the absence of interactions between the explanatory variables, we want to estimate the $\beta_{ij}$ parameters of the model and then test the null hypothesis $H_{00} : \hat{\beta}_{ij} = 0; \forall\ (\mathrm{i,j}) = \{1,2,3\} \times \{0,1,2,3\}$ whose alternative hypothesis is $H_{10} : \hat{\beta}_{ij} \neq 0; \forall\ (\mathrm{i,j}) = \{1,2,3\} \times \{0,1,2,3\}$. In the same way the estimation of $\eta_{ij}$ leads to the verification of the null hypothesis $H_{01} : \hat{\eta}_{ij} = 0; \forall\ (\mathrm{i,j}) = \{2,3\} \times \{1,2\}$ which has the alternative hypothesis $H_{11} : \hat{\eta}_{ij} \neq 0; \forall\ (\mathrm{i,j}) = \{2,3\} \times \{1,2\}$.

**Table 2:** Distributions of the multiplicity of infections according to the three compartments and the Chi-square adequacy test.

| Multiplicity of infections | Peripheral | | | Placental | | | Umbilical cord | |
|---|---|---|---|---|---|---|---|---|
| | $n_{1j}$ | $N \times \hat{p}_{1j}$ | | $n_{2j}$ | $N \times \hat{p}_{2j}$ | | $n_{3j}$ | $N \times \hat{p}_{3j}$ |
| 0 | - | - | | 122 | 115.3558 | | 156 | 156.5525 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 77 | 73.9999 | | 33 | 38.5045 | | 15 | 22.6646 |
| 2 | 55 | 55.5821 | | 9 | 12.4485 | | 7 | 3.2463 |
| 3 | 21 | 27.8321 | | 5 | 3.9333 | | 3 | 0.4609 |
| 4 | 14 | 10.4524 | | 2 | 1.2215 | | 1 | 0.0065 |
| 5 | 3 | 3.1403 | | 1 | 0.3743 | | 1 | 0.0009 |
| 6 | 2 | 0.7863 | | - | - | | - | - |
| $\hat{\theta}_i$ | | 1.502219 | | | 2.766052 | | | 4.376413 |
| $\chi^2$ | | 2.065 | | | 1.7645 | | | 6.5925 |
| *ddl* | | 5 | | | 5 | | | 5 |
| *P.value* | | 0.8401 | | | 0.8807 | | | 0.2528 |

**Table 3:** Regression models of the multiplicity of infections.

| Coefficients | Model (A) | | | Model (B) | | | Model (C) | |
|---|---|---|---|---|---|---|---|---|
| | $\hat{\beta}_{1j}(SE)$ | $P\left(>\left\|\frac{\hat{\beta}_{1j}}{SE}\right\|\right)$ | | $\hat{\beta}_{2j}(SE)$ | $P\left(>\left\|\frac{\hat{\beta}_{2j}}{SE}\right\|\right)$ | | $\hat{\beta}_{3j}(SE)$ | $P\left(>\left\|\frac{\hat{\beta}_{3j}}{SE}\right\|\right)$ |
| Intercept | 1.63(0.31) | < 0.001 | | - 0.29(0.69) | 0.6768 | | -1.01(0.71) | 0.1555 |
| $x_1$ | -0.22(0.05) | < 0.001 | | - 0.35(0.09) | 0.0002 | | -0.31(0.11) | 0.0074 |
| $x_2$ | -0.04(0.03) | 0.1972 | | 0.03(0.04) | 0.4167 | | 0.05(0.04) | 0.2688 |
| $x_3$ | -0.02(0.01) | 0.0593 | | - 0.02(0.01) | 0.8927 | | 0.01(0.01) | 0.3437 |
| $y_1$ | - | - | | 0.27(0.10) | 0.009 | | -0.45(0.15) | 0.0027 |
| $y_2$ | - | - | | - | - | | 1.08(0.14) | < 0.001 |
| *Log-likelihood* | -218.0 | | | -170.1 | | | -128.13 | |
| *AIC* | 443.64 | | | 350.38 | | | 268.1 | |
| *BIC* | 456.0 | | | 366.2 | | | 287.3 | |

SE is the Standard Error; P is the P.value according to the fitted model.

Model (A) explains peripheral MI, which appears to be more correlated with the explanatory variables in the univariate analyses. However, causality was only observed with the number of SP doses taken $\hat{\beta}_{11} = -0.22$ and the association with the intercept $\hat{\beta}_{10} = 1.63$ (see Table 3). However, model (B) rejects the intercept $\hat{\beta}_{20} = -0.29$, by including the number of SP doses taken $\hat{\beta}_{21} = -0.35$, it also highlights its bivariate aspect with respect to placental MI via $\hat{\eta}_{21} = 0.27$. Model (C) combines the three dependents' variables with $\hat{\eta}_{31} = -0.45$ and $\hat{\eta}_{32} = 1.08$, and we can see the protective effect of taking more of dose of SP ($\hat{\beta}_{31} = -0.31$) over the multiplicity of infections in the three compartments. This model shows that the number of pregnancies got, and age of woman may be correlated but do not have a direct effect over the MI variables. The three models shown in the table above are not the only models run, but rather the most necessary ones that explain our dataset. The comparison of the AIC, BIC and Log-likelihood (Table 3) between models allows us to deduce that model (C) is the most adequate with adjustment on these data.

## V. Conclusion

In this work we have proposed the generalization of the weighted Poisson distribution in the multivariate framework. The elements constituting the variance-covariance matrix have been obtained with an explanation of how the covariance can take positive as well as negative values. The suggestion of a few classical distributions as multivariate weighted Poisson distributions was used to illustrate this notion of generalization to a large family of classical discrete distributions. The application to the trivariate Poisson model revealed, firstly, that it is preferable to study simultaneously the effect of taking a dose of SP over the multiplicity of infections in the three compartments. Secondly, it highlighted the possibility of modelling a random vector of discrete dependent variables that do not all follow the same probability law, without having to worry about managing the correlation between variables.

**Appendix A.** Proof of Proposition 2.2.2.

Let's assume that $\theta_i > e$, for all $i = 1,2,\dots,s$. If $\frac{\ln y_i - \ln \theta_i}{\theta_i} < \eta_{ii} < 0$. Then

$$\ln\left(\frac{y_i}{\theta_i}\right) < \eta_{ii}\theta_i \Rightarrow \left(\frac{y_i}{\theta_i}\right) < e^{\eta_{ii}\theta_i} \Rightarrow y_i - \theta_i e^{\eta_{ii}\theta_i} < 0. \quad (17)$$

From the other side,

$$\frac{d}{d\eta_{ii}}\ln E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)] = \sum_{i\geq 0}\frac{w_i(y_i)}{E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)]}\frac{d}{d\eta_{ii}}\left[\frac{e^{y_i\eta_{ii}}\theta_i^{\,y_i}e^{-e^{\eta_{ii}\theta_i}}}{y_i!}\right]$$

$$\frac{d}{d\eta_{ii}}\ln E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)] = \sum_{i\geq 0}\frac{w_i(y_i)}{E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)]}\frac{\theta_i^{\,y_i}}{y_i!}\frac{d}{d\eta_{ii}}\left[e^{y_i\eta_{ii}-e^{\eta_{ii}\theta_i}}\right]$$

$$\frac{d}{d\eta_{ii}}\ln E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)] = \sum_{i\geq 0}\frac{w_i(y_i)}{E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)]}\frac{\theta_i^{\,y_i}}{y_i!}\left[\left(y_i-\theta_i e^{\eta_{ii}\theta_i}\right)e^{\left(y_i\eta_{ii}-e^{\eta_{ii}\theta_i}\right)}\right].$$

If we consider the above inequation (17), we find that,

$$\frac{d}{d\eta_{ii}}\ln E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)] < 0.$$

We can proceed by adding the terms on either side of this inequality as follows.

$$\theta_i e^{\eta_{ii}} + \frac{d}{d\eta_{ii}}\ln E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)] < \theta_i e^{\eta_{ii}}.$$

However, $\theta_i e^{\eta_{ii}} < \theta_i$, thus

$$\theta_i e^{\eta_{ii}} + \frac{d}{d\eta_{ii}}\ln E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)] < \theta_i \Rightarrow \theta_i e^{\eta_{ii}} + \frac{d}{d\eta_{ii}}\ln E_{e^{\eta_{ii}\theta_i}}[w_i(Y_i)] - E_{\theta_i}\left[Y_i^{w_i}\right] < \theta_i - E_{\theta_i}\left[Y_i^{w_i}\right].$$

Since for $\theta_i > e$,

$$\frac{d}{d\theta_i}\ln E_{\theta_i}[w_i(Y_i)] = \sum_{i\geq 0}\frac{w_i(y_i)}{E_{\theta_i}[w_i(Y_i)]}\frac{\theta_i^{\,y_i}}{y_i!}e^{-\theta_i}(\ln\theta_i - 1) > 0.$$

Then according to the equation (0), $\theta_i - E_{\theta_i}\left[Y_i^{w_i}\right] < 0$. What needed to be demonstrated.

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