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Generalized β −Conformal Change Of Finsler Metric By An h −Vector

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ABSTRACT
Let M^n be an n-dimensional differentiable manifold and $F^n = (M^n, L)$be a Finsler space with a fundamental functionL(x, y). We consider a change of this metric by $L \to \bar{L} = f\{e^{\phi} L(x,y), \beta(x,y)\}$, where $\beta(x,y) = v_i(x,y)y^i$, v_i is an h-vector in $F^n = (M^n, L)$. We call this change a generalized β-conformal change by an h-vector. In this paper, we have determined the relations between the v-curvature tensor, v-Ricci tensor and v-scalar curvature with respect to the Cartan connection of Finsler spaces $F^n = (M^n, L)$and$\bar{F}^n = (M^n, \bar{L})$. We have also determined the conditions under which C-reducible, quasi C-reducible, semi C-reducible and S3-like Finsler spaces remains a Finsler space of the same kind under a transformed Finsler metric.

Keywords:-*Finsler space, (α, β) metric, Cartan connection, β-change , conformal change, ,* h-vector, generalized β-conformal change

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I. INTRODUCTION

Let M^n be an n-dimensional C^{∞}-differentiable manifold and $F^n = (M^n, L)$be a Finsler space equipped with a fundamental function L(x, y) $(y^i = \dot{x}^i)$ on M^n.Matsumoto [10] determined the properties of the Finsler space equipped with the metric.

$$'L(x,y) = L(x,y) + \beta(x,y) \qquad (1.1)$$

Where $\beta(x,y) = v_i(x)y^i$ is a differentiable one form on M^n. If $L(x,y)$ is a Riemannian metric then (1.1) is called a Rander's metric.Rander'smetric was introduced by G. Rander's ([16]) during the study of General Theory of Relativity and applied to the theory of Election microscope by R.S. Ingarden ([5]). The properties of Finsler spaces with Rander's metric have been studied by C. Shibata, H-Shimada, M.Azuma and H Yasuda ([17]) in detail. The geometrical properties of Finsler space with Rander's metric have also been studied by various authors ([10], [19]). If $L(x,y)$ is a Finsler metric, then **L$(x,y) = f(L,\beta)$will be called a β-change and the properties of Finsler space with a β-change has been studied by C. Shibata ([20]) in detail. S.H. Abed ([1]) has introduced the Finsler space with the metric$'L(x,y) = e^{\phi(x)}L(x,y) + v_j(x)y^j$ and named it a β-conformal change. Nabil L. Youseff, S.H. Abed & S.G. Elgend([26]) have introduced a change of Finsler metric called a generalized β-conformal change given by

$L(x,y) \to \bar{\bar{L}}(x,y) = f\left[e^{\phi(x)}L(x,y), \beta(x,y)\right]$andstudied the properties of Finsler spaces equipped with this metric. In all the above mentioned words, the function $v_i(x)$ are assumed to be a function of coordinates only. Izumi ([6]) introduced the concept of an h-vector defined by $v_i|j = 0$and satisfies $LC^h_{ij}v_h = Kh_{ij}$, where $|j$ denotes the v-covariant derivative with respect to Cartan connection in $F^n = (M^n, L)$, C^h_{ij} is Cartan' C-tensor, h_{ij} is the angular metric tensor, K $= \frac{L\,C^i v_i}{n-1}$, $C^i = g^{jh}C^i_{jh}$. Prasad ([15]) has obtained the relation between the Cartan's connection of Finsler spaces $F^n = (M^n, L)$ and $"F^n = (M^n, "L)$ where $"L(x,y) = L(x,y) + v_i(x,y)y^iandv_i(x,y)$ is an h-vector in $F^n = (M^n, L)$. Here $v_i(x,y)$ is a function of coordinates and directional arguments both satisfying. $L\dot{\partial}_j v_i = Kh_{ij}$, $\dot{\partial}_j = \partial/\partial y^j$, K $= \frac{L\,C^i v_i}{n-1}$ is a scalar function. Singh and Srivastava [21] has also studied the properties of Finsler space with this metric. Singh and Srivastava ([22]) and the present author ([25]) has studied the properties of Finsler space with the metric $'L = f(L,\beta)$, where $\beta(x,y) = v_i(x,y)y^i$ is a differentiable one form and $v_i(x,y)$ is an$h$-vector in $F^n = (M^n, L)$. Recently the present author ([23][24]) has introduced the change $'''L(x,y) = e^{\phi(x)}L(x,y) + v_i(x,y)y^i$ where $v_i(x,y)$ is an h-vector in $F^n = (M^n, L)$ and studied properties of some special Finsler spaces equipped with this metric

In this paper we shall introduced a change $\bar{L} = f[*L(x,y), \beta(x,y)]$ where $*L = e^{\phi}L$ which we call a generalized β-conformal change by an h-vector. Here $\beta(x,y) = v_i(x,y)y^i$, $v_i(x,y)$ is an h-vector in $F^n =$

(M^n, L) and f(*L, β) is a positively homogenous function of degree 1 in *L and β. This change is a generalization of all the changes which were introduced earlier.

The purpose of the present paper is to determine the conditions under which C-reducible, quasiC-reducible, semi C-reducible and S3-like Finsler spaces remains a Finsler space of the same kind under a transformed Finsler metric. We have also determined the relations between the v-curvature tensor,$w.r.t$Cartan connection of Finsler spaces $F^n = (M^n, L) and \bar{F}^n = (M^n, \bar{L})$

The terminology and notations are referred to well-known Matsumoto's book ([14]) unless otherwise stated.

II. THE FINSLER SPACE $\bar{F}^n = (M^n, \bar{L})$

Let M^n be an $n-$dimensional differentiable manifold and $F^n = (M^n, L)$be a Finsler space equipped with the fundamental function$L(x, y)$. We consider the change of Finsler structure defined by

$L(x,y) \to \bar{L}(x,y) = f\{e^{\phi(x)}L(x,y), \beta(x,y)\} = f(\text{*L}, \beta)$ (2.1)

and have another Finsler space $\bar{F}^n = (M^n, \bar{L})$, where$\bar{L} = f(\text{*L}, \beta)$,*L = $e^{\phi}L$, $\beta = v_i(x,y)y^i$, $v_i(x,y)$ is an h-vector in $F^n = (M^n, L)$ and f is a positively homogeneous function of degree one in*L and β. Throughout the paper the entities of the Finsler space $\bar{F}^n = (M^n, \bar{L})$ will be denoted by putting bar $(-)$ on the top of the corresponding entities of the Finsler space F^n. We define

$f_1 = \partial f/\partial\text{*L}$, $f_2 = \partial f/\partial\beta$, $f_{12} = \partial^2 f/\partial\text{*L}\partial\beta$ etc.

$\partial_i = \partial/\partial x^i$, $\dot{\partial}_i = \partial/\partial y^i$

Since $\bar{L} = f(\text{*L}, \beta)$ is a positively homogeneous function of degree one in *L andβ, hence we have

$f = e^{\phi}Lf_1 + \beta f_2$, $e^{\phi}Lf_{12} + \beta f_{22} = 0$,$e^{\phi}Lf_{11} + \beta f_{12} = 0$ (2.1)

Differentiating $\bar{L} = f(\text{*L}, \beta)$ with respect toy^j and using identities (2.1), we have

$\bar{l}_j = \frac{\partial \bar{L}}{\partial y^j} = e^{\phi} f_1 l_j + f_2 v_j$ (2.2)

Differentiating (2.2) with respect to y^k and using identities (2.1), we have the angular metric tensor $\bar{h}_{jk} = \bar{L}\dot{\partial}_k \bar{l}_j = \bar{g}_{jk} - \bar{l}_j \bar{l}_k$ given by

$$\bar{h}_{jk} = e^{\phi}\left(\frac{ff_1}{L}\right)h_{jk} + ff_{22}\left[v_j v_k - (\beta/L)\left(l_j v_k + l_k v_j\right) + (\beta^2/L^2)y_j y_k\right] + \frac{ff_2}{L}Kh_{jk}$$

or $\bar{h}_{jk} = q' h_{jk} + r_0 m_j m_k$, (2.3)

where $q' = (e^{\phi}q + Kr/L)$, $q = ff_1/L$, $m_j = v_j - (\beta/L)\, l_j$, $r_0 = ff_{22}$, $r = ff_2$

Equation (2.3) can be written as

$$\bar{g}_{jk} - \bar{l}_j \bar{l}_k = (e^{\phi}ff_1/L + Kff_2/L)(g_{jk} - l_j l_k) + ff_{22}[v_j v_k - (\beta/L)\left(l_j v_k + l_k v_j\right) + (\beta^2/L^2)y_j y_k]$$

or$\bar{g}_{jk} = q' g_{jk} + q_0 v_j v_k + e^{\phi}q_{-1}\left(v_j y_k + v_k y_j\right) + (e^{\phi}q_{-2} - Kr/L^3)y_j y_k$

or$\bar{g}_{jk} = q' g_{jk} + q_0 v_j v_k + e^{\phi}q_{-1}\left(v_j y_k + v_k y_j\right) + q'_{-2}y_j y_k$ (2.4)

where $q_0 = f_2^2 + ff_{22}$, $q_{-1} = \frac{f_1 f_2}{L} + \frac{ff_{12}}{L} = q\frac{f_2}{f} + r_{-1}$, $q_{-2} = r_{-2} + e^{\phi}q^2/f^2$,

$q'_{-2} = e^{\phi}q_{-2} - Kr/L^3$,$r_{-1} = ff_{12}/L$, $q_0 = r_0 + f_2^2$, $r_{-2} = f(e^{\phi}f_{11} - f_1/L)/L^2$ (2.5)

The reciprocal tensor $\bar{g}^{jk}$ of $\bar{g}_{jk}$ can be written as

$\bar{g}^{jk} = (1/q')g^{jk} - s'_0 v^k v^j - s'_{-1}(v^j y^k + v^k y^j) - s'_{-2}y^j y^k$ (2.6)

Where $v^j = g^{jk}v_k$, $v^2 = g^{jk}v_j v_k$, $\nu = \{v^2 - (\beta^2/L^2)\}$

$\tau' = f^2/L^2\,(q' + \nu r_0)$, $s'_0 = (f^2 r_0)/q'\tau' L^2$, $s'_{-1} = (f^2/q'\tau' L^2)(e^{\phi}q_{-1} + Kf_2^2/L)$

$s'_{-2} = q'_{-2}/e^{\phi}qq' - (s'_{-1}/e^{\phi}q)(\nu e^{\phi}q_{-1} - rK\beta/L^3)$ (2.7)

From the homogeneity, it follows that.

$r_0\beta + e^{\phi}r_{-1}L^2 = 0$, $r_{-1}\beta + r_{-2}L^2 = -q$, $q_0\beta + e^{\phi}q_{-1}L^2 = r$,

$r\beta + e^{\phi}qL^2 = f^2$, $q_{-1}\beta + q_{-2}L^2 = 0$ (2.8)

From the definition of m_i, it is evident that

(a) $m_i l^i = 0$ (b) $m_i v^i = m_i m^i = v^2 - \beta^2|L^2 = \nu$ where $m^i = g^{ij}m_j$,

(c) $h_{ij}m^i = h_{ij}v^i = m_j$ (d) $C^h_{ij}m_h = \frac{K}{L}h_{ij}$ (2.9)

We have the following identities

(a) $\dot{\partial}_j r = q_0 m_j + (r/L)l_j$

(b) $\dot{\partial}_j q = q_{-1}m_j$

(c) $\dot{\partial}_j q' = q'_{-1}m_j$, where $q'_{-1} = e^{\phi}q_{-1} + K/L\,q_0$

(d) $\dot{\partial}_j q_0 = q_{02}m_j$, where $q_{02} = \partial q_0/\partial\beta$

(e) $\dot{\partial}_j q_{-1} = -e^{-\phi}(\beta/L^2)q_{02}m_j - (q_{-1}/L)\,l_j$

(f) $\dot{\partial}_j q_{-2} = [e^{-\phi}(\beta^2/L^4)q_{02} - (q_{-1}/L^2)]m_j + q_{-1}(2\beta/L^3)l_j$

(g)$\dot{\partial}_j q'_{-2} = [(\beta^2/L^4)q_{02} - q'_{-1}/L^2]\,m_j + e^{\phi}q_{-1}(2\beta/L^3)l_j + (2Kr/L^4)l_j$ (2.10)

Differentiating (2.4) with respect to y^l and using (2.5), (2.8), (2.9) and (2.10) the $(h)hv$ torsion tensor of $\bar{F}^n$ is given by

$$\frac{\partial \bar{g}_{jk}}{\partial y^l} = 2\bar{C}_{jkl} = 2q^{'}C_{jkl} + q_{02}m_l v_j v_k + (q_0K)/L(h_{jl}v_k + h_{kl}v_j) + e^{\phi}q_{-1}(v_j g_{kl} + v_k g_{jl})$$
$$+(Ke^{\phi}q_{-1}/L)(h_{jl}y_k + h_{kl}y_j)$$
$$-e^{\phi}(v_j y_k + v_k y_j)[-e^{-\phi}(\beta/L^2)q_{02}m_l - (q_{-1}/L)\, l_l]$$
$$+q^{'}_{-2}(g_{jl}y_k + g_{kl}y_j) + y_j y_k[\{(\beta^2/L^4)q_{02} - (q^{'}_{-1}/L^2)\}m_l + e^{\phi}q_{-1}(2\beta/L^3)l_l + (2Kq/L^4)l_l\}]$$
$$= 2q^{'}C_{jkl} + q^{'}_{-1}h_{jk}m_l + e^{\phi}q_{-1}(h_{kl}v_j + h_{jl}v_k) + q_{02}m_j m_k m_l$$
$$-e^{\phi}q_{-1}(\beta/L)(h_{jl}l_k + h_{kl}l_j + 2l_j l_k l_l) + 2e^{\phi}(\beta/L)\, q_{-1}l_j l_k l_l$$
$$+(Kq_0/L)(h_{jl}v_k + h_{kl}v_j) + (e^{\phi}q_{-1}K/L)(h_{jl}y_k + h_{kl}y_j)$$
$$-(Kr/L^3)(h_{kl}y_j + h_{jl}y_k)$$
$$= 2q^{'}C_{jkl} + q^{'}_{-1}(h_{jk}m_l + h_{kl}m_j + h_{lj}m_k) + q_{02}m_j m_k m_l$$

or $\bar{C}_{jkl} = q^{'}C_{jkl} + q^{'}_{-1}(h_{jk}m_l + h_{kl}m_j + h_{lj}m_k)/2 + q_{02}m_j m_k m_l/2$ (2.11)

or $\bar{C}_{jkl} = q^{'}C_{jkl} + V_{jkl}$
(2.12)

where $V_{jkl} = q^{'}_{-1}(h_{jk}m_l + h_{kl}m_j + h_{lj}m_k)/2 + q_{02}m_j m_k m_l/2$ (2.13)

$q^{'}_{-1} = e^{\phi}q_{-1} + (K/L)q_0$ (2.14)

Contracting (2.11) by $\bar{g}^{lp}$ and using (2.9) we have

$\bar{C}^p_{jk} = C^p_{jk} + M^p_{jk}$, (2.15)

where

$$M^p_{jk} = \frac{1}{2}\left[\frac{m^p}{q^{'}} - \nu(s^{'}_0 v^p + s^{'}_{-1}y^p)\right][q_{02}m_j m_k + q^{'}_{-1}h_{jk}] + \left(\frac{q^{'}_{-1}}{2q^{'}}\right)(h^p_j m_k + h^p_k m_j) - (s^{'}_0 v^p + s^{'}_{-1}y^p)$$
$$(q^{'}C_{jk\beta} + q^{'}_{-1}m_j m_k)$$

Putting $k = p$ in M^p_{jk} we have

$$M^p_{jk} = \frac{1}{2}\frac{q_{02}}{q^{'}}m_j\nu - \frac{\nu^2}{2}s^{'}_0 m_i q_{02} - \frac{\nu}{2}s^{'}_0 q^{'}_{-1}m_j + \frac{q^{'}_{-1}}{2q^{'}}[m_j + (n-1)m_j] - q^{'}s^{'}_0 C_{j\beta\beta} - s^{'}_0 q^{'}_{-1}\nu m_j$$
$$= \left[\frac{(n+1)q^{'}_{-1}}{2q^{'}} - \frac{3}{2}s^{'}_0 q^{'}_{-1}\nu + \frac{q_{02}\nu}{2(q^{'}+\nu r_0)}\right]m_j - q^{'}s^{'}_0 C_{j\beta\beta} \quad (2.16)$$

Where and in the following the subscript β denotes contraction with respect to the h-vector v^k

$\therefore \bar{C}_j = C_j - q^{'}s^{'}_0 C_{j\beta\beta} + \mu m_j$, (2.17)

Where $\mu = \frac{(n+1)q_{-1}}{2q^{'}} - \frac{3}{2}s^{'}_0 q^{'}_{-1}\nu + \frac{q_{02}\nu}{2(q^{'}+\nu r_0)}$

$$\bar{C}^j = \bar{g}^{ij}\bar{C}_i = \left[\frac{1}{q^{'}}g^{ij} - s^{'}_0 v^i v^j - s^{'}_{-1}(v^i y^j + v^j y^i) - s^{'}_{-2}y^i y^j\right](C_i - q^{'}s^{'}_0 C_{i\beta\beta} + \mu m_i)$$
$$= \frac{\mu}{q^{'}}m^j - s^{'}_0 C^j_{\beta\beta} + \frac{1}{q^{'}}C^j - (s^{'}_0 v^j + s^{'}_{-1}y^j)(C_\beta - q^{'}s^{'}_0 C_{\beta\beta\beta} + \mu\nu)$$

or $\bar{C}^j = \frac{1}{q^{'}}C^j + N^j$ (2.18)

Where $N^j = \frac{\mu}{q^{'}}m^j - s^{'}_0 C^j_{\beta\beta} - (C_\beta + \mu\nu - q^{'}s^{'}_0 C_{\beta\beta\beta})(s^{'}_0 v^j + s^{'}_{-1}y^j)$ (2.19)

$\bar{C}^2 = \bar{C}^j\bar{C}_j = \frac{1}{q^{'}}C^2 + \psi$ (2.20)

$$\psi = \mu^2\nu\left(\frac{1}{q^{'}} - s^{'}_0\nu\right) + C_\beta\left[\frac{2\mu}{q^{'}} - s^{'}_0(C_\beta + 2\mu\nu)\right]$$
$$+s^{'}_0 C_{\beta\beta\beta}(-2\mu + 2q^{'}s^{'}_0 C_\beta)$$
$$+s^{'}_0 C_{\beta\beta\beta}(-2C^j + q^{'}s^{'2}_0 C_{\beta\beta\beta}v^j - 2\mu s^{'}_0\nu q^{'}v^j - q^{'}s^{'}_0 C^j_{\beta\beta}) \quad (2.21)$$

From equations (2.9),(2.11) and (2.15), the v-curvature tensor of $\bar{F}^n$ with respect to Cartan connection is written as ([8])

$$\bar{S}_{ijkl} = \bar{C}_{ilp}\bar{C}^p_{jk} - \bar{C}_{ikp}\bar{C}^p_{jl}$$

or $\bar{S}_{ijkl} = q^{'}S_{ijkl} + A_{(kl)}\{h_{il}K_{jk} + h_{jk}K_{il}\}$ (2.22)

where $K_{jk} = K_1 m_j m_k + K_2 h_{jk}$ (2.23),

$A_{kl}\{\dots\}$ denotes the interchange of indices k, l and subtraction,

$K_1 = \frac{q^2_{-1}}{4q^{'}}(1 - 2s^{'}_0\nu q^{'}) + \frac{\nu q_{02}q^{'}_{-1}}{4(q^{'}+r_0\nu)} + \frac{K}{L}\left\{\frac{q^{'}q_{02}}{2(q^{'}+\nu r_0)} - q^{'}q^{'}_{-1}s^{'}_0\right\}$ (2.24)

$K_2 = \frac{q^{'}_{-1}\nu}{8(q^{'}+\nu r_0)} + \frac{Kq^{'}_{-1}}{2L}(1 - s^{'}_0 q^{'}\nu) - \frac{K^2}{2L^2}q^{'2}s^{'}_0$ (2.25)

From equations (2.6), (2.22) and (2.23), the v-Ricci tensor of $\bar{F}^n$ is given by-

$$\bar{S}_{jl} = \bar{g}^{ik}\bar{S}_{ijkl} = S_{jl} - q^{'}s_0^{'}S_{ijkl}\,v^iv^k + \left[\frac{(3-n)K_1}{q^{'}} - s_0^{'}(vK_1 + 2K_2)\right]m_jm_l$$

$+[\{(4-2n)K_2 - K_1v\}/q^{'} + s_0^{'}v(K_1v + 2K_2)]h_{jl}$

$= S_{jl} - q^{'}s_0^{'}S_{ijkl}\,v^iv^k + \lambda_1h_{jl} + \lambda_2m_jm_l$ (2.26)

where

$\lambda_1 = \{(4-2n)K_2 - K_1v\}/q^{'} + s_0^{'}v(K_1v + 2K_2)$ (2.27)

$\lambda_2 = \frac{(3-n)K_1}{q^{'}} - s_0^{'}(vK_1 + 2K_2)$ (2.28)

From equations (2.6) and (2.26), the v-scalar curvature of $\bar{F}^n$ is given by-

$\bar{S} = \bar{g}^{il}\bar{S}_{jl} = \frac{1}{q^{'}}S - 2s_0^{'}S_{ik}v^iv^k + q^{'}s_0^{'2}S_{ijkl}\,v^iv^jv^kv^l + \{\lambda_1(n-1) + \lambda_2v\}/q^{'} - s_0^{'}v(\lambda_1 + \lambda_2v)$ (2.29)

Definition (2.1):- A Finsler space (M^n, L) with dimension $n \geq 3$ is said to be a quasi-C-reducible if the Cartan tensor C_{ijk} satisfies ([14])

$C_{ijk} = B_{ij}C_k + B_{jk}C_i + B_{ki}C_j$,

whereB_{ij} is a symmetric and indicatory tensor

We know that the $(h)hv$-torsion tensor of $\bar{F}^n$ is written as

$$\bar{C}_{ijk} = q^{'}C_{ijk} + \frac{q^{'}_{-1}}{2}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{q_{02}}{2}m_im_jm_k$$

From the above equations and equation (2.17), we have

$\bar{C}_{ijk} = q^{'}C_{ijk} + \frac{1}{6\mu}A_{(ijk)}\left[(3q^{'}_{-1}h_{ij} + q_{02}m_im_j)(\bar{C}_k - C_k + q^{'}s_0^{'}C_{k\beta\beta})\right]$

$= q^{'}C_{ijk} + \frac{1}{6\mu}A_{(ijk)}\{(3q^{'}_{-1}h_{ij} + q_{02}m_im_j)\bar{C}_k\} + \frac{1}{6\mu}A_{(ijk)}\{(3q^{'}_{-1}h_{ij} + q_{02}m_im_j)(q^{'}s_0^{'}C_{k\beta\beta} - C_k)\}$

Where $A_{(ijk)}$ (…….) denotes the cyclic interchange of indices i, j, k and summation.

Hence we have the following

LEMMA (2.1) :- The Cartan tensor $\bar{C}_{ijk}$ of the generalized β −conformal change by an h-vector can be written in the form

$\bar{C}_{ijk} = A_{(ijk)}(\bar{B}_{ij}\bar{C}_k) + q_{ijk}$,

where $\bar{B}_{ij} = \frac{1}{6\mu}(3q^{'}_{-1}h_{ij} + q_{02}m_im_j)$,

$$q_{ijk} = \frac{1}{6\mu}A_{(ijk)}\{2\mu q^{'}C_{ijk} + (3q^{'}_{-1}h_{ij} + q_{02}m_im_j)(q^{'}s_0^{'}C_{k\beta\beta} - C_k)\}$$

Since the tensor $\bar{B}_{ij}$ is symmetric and indicatory, using the above lemma, we have the following.

THEOREM (2.1) :- Finsler space $\bar{F}^n = (M^n, \bar{L})$ is quasi C-reducible if $q_{ijk} = 0$

COROLLARY (2.1) :- A Riemannian space (M^n, L) is transformed to a quasi C-reducible Finsler space $\bar{F}^n = (M^n, \bar{L})$ under a generalized β-conformal change by an h-vector.

Definition (2.2):- A Finsler space $F^n$of dimension $(n \geq 3)$is called semi C-reducible, if the $(h)hv$-torsion tensor C_{ijk} is written in the form([14])

$C_{ijk} = \frac{p}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j) + \frac{t}{C^2}C_iC_jC_k$,

where$p$ and t are scalar function such that $p + t = 1$

THEOREM (2.2) :- A Riemannian space is transformed to a semi C-reducible Finsler space by a generalized β-conformal change by an h-vector.

Proof :-The $(h)hv$ torsion tensor of$\bar{F}^n$ is written as

$$\bar{C}_{ijk} = \frac{1}{2}q^{'}_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \frac{1}{2}q_{02}m_im_jm_k$$

From the above equation and equation (2.17), we have

$\bar{C}_{ijk} = \frac{q^{'}_{-1}}{2q^{'}\mu}(\bar{h}_{ij}\bar{C}_k + \bar{h}_{jk}\bar{C}_i + \bar{h}_{ki}\bar{C}_j) + \frac{v(q^{'}q_{02} - 3q^{'}_{-1}r_0)}{2q^{'}\mu(q^{'} + vr_0)\bar{C}^2}\bar{C}_i\bar{C}_j\bar{C}_k$

$= \frac{p}{n+1}(\bar{h}_{ij}\bar{C}_k + \bar{h}_{jk}\bar{C}_i + \bar{h}_{ki}\bar{C}_j) + \frac{t}{\bar{C}^2}\bar{C}_i\bar{C}_j\bar{C}_k$

where $p = \frac{q^{'}_{-1}(n+1)}{2q^{'}\mu}$, $t = \frac{v(q^{'}q_{02} - 3q^{'}_{-1}r_0)}{2q^{'}\mu(q^{'} + vr_0)}$

Here $p + t = 1$

Hence $\bar{F}^n$ is a semi-C-reducible.

Definition (2.3):- A Finsler space $F^n = (M^n, L)$ of dimension $(n \geq 3)$ is called C-reducible if the $(h)hv$- torsion tensor C_{ijk} is of the form ([14])

$$C_{ijk} = \frac{1}{n+1}\left(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\right)$$

Let $W_{ijk} = C_{ijk} - \frac{1}{(n+1)}\left(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\right)$

W_{ijk} is symmetric and indicatory tensor. Also $W_{ijk} = 0$ iff the Finsler space $F^n = (M^n, L)$ is C-reducible

The $(h)hv$- torsion tensor of $\bar{F}^n$ can be written as

$\bar{C}_{ijk} = q' C_{ijk} + q'_{-1}\left(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j\right)/2 + q_{02}m_i m_j m_k/2$

From the above equation and equations (2.3) and (2.17), we have

$$\bar{C}_{ijk} - \frac{1}{n+1}\left(\bar{h}_{ij}\bar{C}_k + \bar{h}_{jk}\bar{C}_i + \bar{h}_{ki}\bar{C}_j\right)$$

$= q' C_{ijk} - q'\left[\frac{1}{n+1}\left(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j\right)\right] + a_{ijk}$

or$\bar{W}_{ijk} = q' W_{ijk} + a_{ijk}$

where $a_{ijk} = \frac{1}{(n+1)}A_{(ijk)}\{(\beta_1 h_{ij} + \beta_2 m_i m_j)m_k - r_0 m_i m_j C_k + (s'_0 q' r_0 m_i m_j + q'^2 s'_0 h_{ij})C_{k\beta\beta}\}$,

$\beta_1 = \frac{q'_{-1}}{2} - \frac{q'\mu}{n+1}, \quad \beta_2 = \frac{q_{02}}{6} - \frac{\mu r_0}{n+1}$

Hence we have the following theorem

THEOREM (2.3) :- The following statements are equivalent

(a) $F^n$is a C-reducible Finsler space

(b) $\bar{F}^n$is a C-reducible Finsler space

iff the tensor a_{ijk} vanishes.

Definition (2.4):- A Finsler space $F^n = (M^n, L)$ with $n > 3$ is called an S3-like Finsler space if the v-curvature tensor S_{ijkl} satisfies. ([14])

$$S_{ijkl} = \frac{\mathrm{S}}{(n-1)(n-2)}\{h_{ik}h_{jl} - h_{il}h_{jk}\}$$

Where S is the vertical scalar curvature

Let

$$\eta_{ijkl} = S_{ijkl} - \frac{\mathrm{S}}{(n-1)(n-2)}\{h_{ik}h_{jl} - h_{il}h_{jk}\}$$

$\eta_{ijkl} = 0$iff the space F^n is S3-like.

From equations (2.3), (2.22) and (2.29), we have

$$\bar{\eta}_{ijkl} = \bar{S}_{ijkl} - \frac{\bar{S}}{(n-1)(n-2)}\{\bar{h}_{ik}\bar{h}_{jl} - \bar{h}_{il}\bar{h}_{jk}\}$$

$$= q'\eta_{ijkl} + \xi_{ijkl}$$

where$\xi_{ijkl} = A_{(kl)}\left[h_{il}K_{jk} + h_{jk}K_{il} - \frac{q'^2\mathrm{H}}{(n-1)(n-2)}h_{ik}h_{jl} - \frac{r_0}{(n-1)(n-2)}(S + q'\mathrm{H})(h_{jl}m_i m_k + h_{ik}m_l m_j)\right]$

$$\mathrm{H} = q' s'^2_0 S_{ijkl}v^i v^j v^k v^l + (n-1)\lambda_1/q' + \lambda_2 v/q' - 2S_{jl}s'_0 v^j v^l - s'_0 v(\lambda_1 + \lambda_2 v)$$

Hence we have the following theorem

THEOREM (2.4) :- The following statements

(a) $F^n = (M^n, L)$is an S3-like Finsler space.

(b) $\bar{F}^n = (M^n, \bar{L})$is an S3-like Finsler space.

are equivalent iff the tensor ξ_{ijkl} vanishes.

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