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On Riemannian Banach Submanifolds

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Abstract
In this paper for a Banach space F and $\forall\, x \in F$, we prove that all orthogonal topological complement $F_x^{\perp}$ to F are isomorphic to a unique Banach space G. Also, the derivative equations of a Riemannian Banach submanifold N of a Riemannian Banach manifold M are established.

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I. Introduction

Let M be a Banach manifold of class $C^r (r \geq 2, \infty)$ modeled on a Banach space E and $N \subset M$ is a submanifold of M of the same class C^r modeled on a Banach space $F \subset E$ [1]. Also, let $\bar{i} : \bar{x} \in N \longrightarrow \bar{i}(\bar{x}) = \bar{x} \in M$ be the inclusion map. If $c = (U, \phi, E)$ is a chart on M at a point $\bar{x} \in M$ and $d = (V, \psi, F)$ is a chart on N at $\bar{x} \in N \subset M$. Furthermore, if $\psi(\bar{x}) = x$ and $Z = \phi(\bar{x})$ are the models of $\bar{x}$ with respect to the charts d and c respectively. Also, if i is the local representation of the mapping $\bar{i}$ with respect to the charts c and d, then we have :

$$i : x \in \psi(V) \subset F \longrightarrow i(x) = Z \in \phi(U) \subset E. \qquad (1.1)$$

Equation (1.1) is called the local equation of the submanifold N with respect to the charts c and d.

Let $(M, \bar{g}^1)$ be a Riemannian manifold and $N \subset M$ be its Riemannian submanifold with a metric $\bar{g}^2$. This means that $\bar{g}^2$ is induced on N by $\bar{g}^1$ according to the rule:

$$\forall\, \bar{x} \in N, \forall\, \bar{X}_1, \bar{X}_2 \in T_{\bar{x}} N$$

$$\bar{g}^2_{\bar{x}}(\bar{X}_1, \bar{X}_2) = \bar{g}^1_{\bar{i}(\bar{x})=\bar{x}}\left(T_{\bar{x}}\, \bar{i}(\bar{X}_1), T_{\bar{x}}\, \bar{i}(\bar{X}_2)\right), \qquad (1.2)$$

where $T_{\bar{x}}\, \bar{i} : T_{\bar{x}} N \longrightarrow T_{\bar{x}} M$ is the tangent mapping to the mapping $\bar{i}$ at the point $\bar{x}$. Furthermore $T_{\bar{x}} N$ and $T_{\bar{x}} M$ are the spaces of all tangent vectors of N and M respectively. We assume that $\bar{g}^1$ and $\bar{g}^2$ are strong non-singular [3]. Now, if $X_1, X_2 \in F$ are the models of the vectors $\bar{X}_1, \bar{X}_2 \in T_{\bar{x}} N$ with respect to the chart d, then the models of these vectors in the chart c take the form:

$$Y_1 = Di_x(X_1) \qquad , \qquad Y_2 = Di_x(X_2),$$

where Di_x is the Frechet derivative of the mapping i [1]. In this case the local representation of (1.2) has the form:

$$g^2_x(X_1, X_2) = g^1_z\big(Di_x(X_1), Di_x(X_2)\big), \qquad (1.3)$$

where g^1 and g^2 are the models of $\bar{g}^1$ and $\bar{g}^2$ with respect to the charts c and d respectively.

Since M and N are Riemannian manifolds, then there exists unique torsion-free connections $\bar{\Gamma}^1$ and $\bar{\Gamma}^2$ [3], such that $\bar{\nabla}^1\, \bar{g}^1 \equiv 0$ and $\bar{\nabla}^2\, \bar{g}^2 \equiv 0$, on M and N respectively, where $\bar{\nabla}^1$ and $\bar{\nabla}^2$ are the operators of the covariant differentiation on M and N respectively [3]. Also, we assume Γ^1 and Γ^2 are the models of the connections $\bar{\Gamma}^1$ and $\bar{\Gamma}^2$ with respect to the charts c and d respectively.

Assuming that at every point $\bar{x} \in N$, the tangent space $T_{\bar{x}} N$ to the submanifold $N \subset M$ has a topological orthogonal complement $(T_{\bar{x}} N)^{\perp}$ where,

$$(T_{\bar{x}} N)^{\perp} = \{\bar{Y} \in T_{\bar{x}} M :\ \bar{g}^1_{\bar{x}}(\bar{Y}, \bar{X}) = 0\ \ \forall\, \bar{X} \in T_{\bar{x}} N\}$$

such that $T_{\bar{x}} N \oplus (T_{\bar{x}} N)^{\perp} = T_{\bar{x}} M$, also the Banach spaces $T_{\bar{x}} N \times (T_{\bar{x}} N)^{\perp}$ and $T_{\bar{x}} M$ are isomorphic (Here $\oplus$ is the operation of the direct sum of mutually orthogonal subspaces $T_{\bar{x}} N$ and $(T_x N)^{\perp}$) [3].

From the definition of the submanifold [1], there exist charts $c = (U, \phi, E)$ on M at the point $\bar{x} \in M$ and $d = (V = U \cap N, \psi = \phi|_N, F \subset E)$ on N at the point $\bar{x} \in N$ such that $\phi(V) \subset F$.

Now, assume that the chart c is fixed at $\bar{x} \in M$ and define a mapping $w_{c,\bar{x}}$: $T_{\bar{x}} M \longrightarrow E$ as follows:

Let $\bar{h} \in T_{\bar{x}} M$. From all equivalence pairs which define the vector $\bar{h}$, we take the pair (c, h) whose first component is our fixed chart c, then the second component h can be taken as the image of $w_{c,x}$ at $\bar{h}$.

Then, we define:

$$w_{c,\bar{x}}(T_{\bar{x}} N)^{\perp} = F_x^{\perp} \subset E,$$

to be the orthogonal topological complement of $F \subset E$ with respect to $g^1_{i(x)}$. This means that $\forall\, \bar{x} \in N, \forall\, \bar{X} \in T_{\bar{x}}\, N, X = w_{d,\bar{x}}(\bar{X}) \in F$ and for all $S \in F^{\perp}_{x=\psi(\bar{x})}$, we have

$g^1_{i(x)}\big(S, D\, i_x(X)\big) = 0,$ (1.4)

where $w_{c,\bar{x}} : T_{\bar{x}}\, M \longrightarrow E$ is an isomorphism between the normed spaces with respect to the chart c [7]. Similarly $w_{d,\bar{x}} : T_{\bar{x}}\, N \longrightarrow F$ is an isomorphism with respect to the chart d.

Now, we shall prove that all these orthogonal topological complements $F^{\perp}_x$ to the Banach space F are isomorphic to a unique Banach space G.

Proof: We assume that the Banach subspaces $G_1, G_2 \subset E$ are orthogonal complement to F, This means that $F \oplus G_1 = E = F \oplus G_2$ and we will prove that G_1 and G_2 are isomorphic.

Denoting $f_i : F \times G_i \longrightarrow F \oplus G_i = E, i = \overline{1,2}$ are isomorphisms between the Banach spaces. We define the mappings:

$$h = proj_2 \circ f_2^{-1} :\ G_1 \longrightarrow G_2$$

by the rule: $h : X_1 = Y + X_2 \in G_1 \subset E = F \oplus G_2 \xrightarrow{f_2^{-1}} (Y, X_2) \xrightarrow{proj_2} X_2 \in G_2$, and $h^{-1} = proj_2 \circ f_1^{-1} :\ G_2 \longrightarrow G_1$ by the rule $h^{-1} : X_2 = -Y + X_1 \in G_2 \subset E = F + G_1 \xrightarrow{f_1^{-1}} (-Y, X) \in F \times G_1 \xrightarrow{proj_2} X_1 \in G_1$.

It is clear that h is an isomorphism of the Banach spaces G_1 and G_2. Hence all the topological orthogonal complements of F are isomorphic. Thus, they are isomorphic to a unique Banach space G.

Now, we shall prove the following theorem:

Theorem (1.1): For all $\bar{x} \in N \subset M$, there exists an isomorphism of the Banach spaces $\bar{n}_{\bar{x}_0} : G \longrightarrow (T_x\, N)^{\perp} \subset T_{\bar{x}}\, N$ satisfies the following property:

$\forall\, \bar{x}_0 \in N$, there exists a chart $d = (V, \psi, F)$ at the point $\bar{x}_0$ on and $c = (U, \phi, E)$ at the point $\bar{i}(\bar{x}_0) = \bar{x}_0$ on M such that the mapping:

$n : x = \psi(\bar{x}) \in \psi(V) \subset F \longrightarrow n_x = w_{c,\bar{x}} \circ \bar{n}_{\bar{x}} \in L(G; E)$ is differentiable of class c^{r-1}.

Proof: Let $\bar{x}_0 \in N$ be a fixed point and $c = (U, \phi, E)$, $d = (V = U \cap N, \psi = \phi|_V, F \subset E)$ are charts at $\bar{x}_0$ on M and N respectively.

Now, for all $\bar{x} \in V \subset N$, we have $G_x = w_{c,\bar{x}}((T_{\bar{x}}\, N)^{\perp})$ orthogonal complement to F with respect to $g^1_{i(x)}$. We take $G = G_{x_0} = w_{c,\bar{x}_0}\left(\left(T_{\bar{x}_0}\, N\right)^{\perp}\right)$.

Now, for all $\bar{x} \in V \subset N$, we define a linear continuous operator $n_x \in L(E; E)$ and its inverse $\tilde{n}_x = n^{-1}{}_x \in L(E; E)$ as a solution of the equations:

$g^1_x(n_x(Y_1), Y_2) = g^1_{x_0}(Y_1, Y_2),$ (1.5)

$g^1_{x_0}(\tilde{n}_x(Y_1), Y_2) = g^1_x(Y_1, Y_2).$ (1.6)

In this case, it is clear that

$n_x(G) = G_x,$ (1.7)

$\tilde{n}_x(G_x) = G.$ (1.8)

Furthermore, if we denote:

$$g^{1*}_x :\ L_2(E; R) \longrightarrow L(E; E)$$

as an isomorphism of Banach spaces, and taking into account that $g^1{}_x$ is strong non-singular [3], then from (1.5), (1.6) we have that:

$n_x = g^{1*}_x(g^1_{x_0}),$ (1.9)

$\tilde{n}_x = g^{1*}_{x_0}(g^1_x).$ (1.10)

Therefore taking into account that g^1 is differentiable , we deduce that:

$$\tilde{n} : x \longrightarrow \tilde{n}_x \in Lis(E; E) \subset L(E; E)$$

is differentiable of class C^{r-1}(Here, $Lis(E; E)$ is open subset of $L(E; E)$ and it is the set of all automorphisms on the Banach space E).

Now, from the fact that

$$f \in Lis(E; E) \subset L(E; E) \longrightarrow f^{-1} \in Lis(E; E)$$

is differentiable [4], we deduce that $n : x \longrightarrow n_x = (\tilde{n}_x)^{-1} \in Lis(E; E) \subset L(E; E)$ is also differentiable of class C^{r-1}.

Now, for all $\bar{x} \in V \subset N$, we define:

$\bar{n}_{\bar{x}} = w^{-1}_{c,\bar{x}} \circ n_x|_G : G \longrightarrow (T_{\bar{x}}\, N)^{\perp} \subset T_{\bar{x}}\, M,$ (1.11)

where $x = \psi(\bar{x})$.

Therefore, differentiability of the mapping $\bar{n} :\ \bar{x} \longrightarrow Lis(G; (T_{\bar{x}}\, N)^{\perp})$ of class C^{r-1} exists at least locally.

Remark 1.1: Let $c' = (U', \phi', E)$ and $d' = (V' = U' \cap N, \psi' = \phi'|_{V'}, F_1 \subset E)$ are charts on M and N at the point $\bar{x} \in V' \subset N$ respectively.

Hence, if $\bar{A}: \bar{x} \in V' \longrightarrow \bar{A}_{\bar{x}} \in (T_{\bar{x}} N)^{\perp}$ is a differentiable vector field of class C^{r-1} on $V' \subset N$, then we define the mapping:

$$\tilde{A}: \bar{x} \in V' \longrightarrow \tilde{A}_{\bar{x}} = \bar{n}_{\bar{x}}^{-1}(\bar{A}_{\bar{x}}) \in G$$

which is also differentiable of class C^{r-1}.

Proof: Using (1.11) with respect to the chart d' on $V' \subset N$, we get:

$\tilde{A}_{\bar{x}}: \left((n_{x|_{G_x}})^{-1} \circ w_{c',\bar{x}}\right)(\bar{A}_{\bar{x}}) = \left((n_x^{-1})_{|_{G_x}} \circ w_{c',\bar{x}}\right)(\bar{A}_{\bar{x}}) = \left(n_x^{-1} \circ w_{c',\bar{x}}\right)(\bar{A}_{\bar{x}})$, this means that the mapping:

$\tilde{A}: \bar{x} \longrightarrow \tilde{A}_{\bar{x}}$ can be represented as composition of the mappings:

$\bar{x} \xrightarrow{\bar{A} \times id_{V'}} (\bar{A}_{\bar{x}}, \bar{x}) \xrightarrow{\psi \times id_{V'}} \left(w_{c',\bar{x}}(\bar{A}_{\bar{x}}, \bar{x})\right) \xrightarrow{\alpha_1 = id_E \times \tilde{n}} \left(w_{c',\bar{x}}(\bar{A}_{\bar{x}}), \tilde{n}_x \overset{\text{def}}{=} n_x^{-1}\right) \overset{\alpha_2}{\to} \tilde{A}_{\bar{x}}$, where $x = \phi'(\bar{x})$ such that:

(1) $\bar{A}: \bar{x} \longrightarrow \bar{A}_{\bar{x}}$ is differentiable of class C^{r-1} by condition,

(2) $\psi: y \in (T\, V')^{\perp} \subset (T\, N)^{\perp} \subset T\, E \longrightarrow w_{c',\pi(y)=Z \in V'}(y) \in E$, is of class C^{r-1}, since the mapping ψ; locally, can be written as:

$$\hat{\psi}: \left(\phi'(Z), w_{c',Z}(y)\right) \xrightarrow{proj_2} w_{c',Z}(y),$$

this means:

$\hat{\psi} = proj_2: \phi'(V') \times E \longrightarrow E$ is of class C^{∞},

(3) The mapping $\alpha_1: (X, \bar{x}) \in E \times V' \longrightarrow (X, \tilde{n}_{\bar{x}}) \in E \times L(E; E)$ is of class C^{r-1},

(4) The mapping $\alpha_2: (X, B) \in E \times L(E; E) \longrightarrow B(X) \in E$ is of class C^{∞}.

Therefore, we have that the mapping $\alpha_2 \circ \alpha_2 \circ (\psi \times id_{V'}) \circ (\bar{A} \times id_{V'}): \bar{x} \longrightarrow (\bar{A}_{\bar{x}}, \bar{X}) \longrightarrow (\psi(\bar{A}_{\bar{x}}), \bar{X}) = \left(w_{c',\bar{x}}(\bar{A}_{\bar{x}}), \bar{X}\right) \longrightarrow \left(w_{c',\bar{x}}(\bar{A}_{\bar{x}}), \tilde{n}_x\right) \longrightarrow \tilde{n}_x\left(w_{c',\bar{x}}(\bar{A}_{\bar{x}})\right) = \tilde{A}_{\bar{x}}$ is differentiable of class C^{r-1} (Here $T\, V', T\, N, T\, E$ are tangent spaces of the manifolds V', N and E respectively [5], furthermore the mapping $\bar{x} \longrightarrow x = \phi'(\bar{x}) \overset{\tilde{n}}{\to} \tilde{n}_x$ is differentiable of class C^{r-1} by condition).

Also, since $\forall\, Z \in G, n_x(Z) \in F_x^{\perp}$, then similarly (1.4) we get

$g_{i(x)}^1\left(n_x(Z), D\, i_x(X)\right) = 0, \forall\, \bar{X} \in T_{\bar{x}}\, N.$ (1.12)

Now, mixed covariant differentiation of equality (1.3) with respect to the mixed covariant differentiation $\nabla^{1,2}$ taking into account that $\bar{g}^1 \in T_{0+2}^{0+0}(N), \bar{g}^2{}_{|_N} \in T_{2+0}^{0+0}(N)$ and $T\,\bar{i} \in T_{0+1}^{1+0}(N)$ [6], we get:

$g_{i(x)}^1\left(\nabla^{1,2} D\, i_x(X_1; X_3), D\, i_x(X_2)\right) + g_{i(x)}^1\left(D\, i_x(X_3), \nabla^{1,2} D\, i_x(X_1; X_2)\right) + g_{i(x)}^1\left(\nabla^{1,2} D\, i_x(X_2; X_1), D\, i_x(X_3)\right) + g_{i(x)}^1\left(D\, i_x(X_1), \nabla^{1,2} D\, i_x(X_2; X_3)\right) - g_{i(x)}^1\left(\nabla^{1,2} D\, i_x(X_3; X_1), D\, i_x(X_2)\right) - g_{i(x)}^1\left(D\, i_x(X_1), \nabla^{1,2} D\, i_x(X_3; X_2)\right) = 0.$ (1.13)

But, for a mixed tensor $S \in T_{0+1}^{1+0}(N)$, we have [6].

$$\nabla^{1,2} S(\underline{X}, \underline{Y}) = \nabla^{1,2} S(X; Y) - \nabla^{1,2} S(Y; X) = \Gamma^1\left(S(\underline{Y}), D\, i_x(\underline{X})\right) - S\left(\Gamma^2(\underline{Y}, \underline{X})\right) + \Gamma^2\left(S(\underline{Y}), D\, i_x(\underline{X})\right).$$

Also, we take $S(Y) = D\, i_x(Y)$, therefore $\nabla^{1,2} D\, i_x(\underline{X}, \underline{Y}) = 0$ and from (1.13) we get:

$2\, g_{i(x)}^1\left(D\, i_x(X_3), \nabla^{1,2} D\, i_x(X_1; X_2)\right) = 0.$ (1.14)

Now, from (1.14) we obtain:

$$\nabla^{1,2} D\, i_x(X_1; X_2) \in F_x^{\perp}.$$

But, since $n_x: G \longrightarrow F_x^{\perp}$ is an isomorphism, then there exists α vector $A_x(X_1, X_2) \in G$ such that:

$\nabla^{1,2} D\, i_x(X_1; X_2) = n_x\left(A_x(X_1, X_2)\right).$ (1.15)

Lemma 1.1: $\forall\, x = \psi(\bar{x}) \in \psi(V) \subset F, A_x \in L_2(F; G)$, this means: A_x is bilinear continuous mapping.

Proof: From theorem (1.1), we have $n_x \in L(G; E)$, furthermore $\forall\, x = \psi(\bar{x}) \in \psi(V) \subset F, n_x(G)$ is a closed vector subspace of E.

Then $n_x: G \longrightarrow n_x(G)$ is a linear isomorphism of the two Banach spaces. Therefore by Banach theorem of inverse mapping [7], we have that the mapping $n_x^{-1}: n_x(G) \longrightarrow G$ is, also linear and continuous.

This means $n_x^{-1} \in Lis(n_x(G); G)$.

Now, from (1.13) we get:

$A_x(X_1, X_2) = n_x^{-1}\left(\nabla^{1,2} D\, i_x(X_1; X_2)\right)$, where $\nabla^{1,2} D\, i_x \in L_2(F; E)$ [2].

Thus, we obtain:

$$A_x \in L_2(F; G).$$

Also, we consider the first derivative $D\, n_x(X; Z)$ at the point $x \in \psi(V) \subset F$, where $x \in F$ and $Z \in G$. Then we can get:

$D\, n_x(X; Z) = D\, i_x\left(H_x(X, Z)\right) + n_x\left(S_x(X, Z)\right).$ (1.16)

Now, we give the following Lemma:

Lemma 1.2:

1- $H_x(X, Z) \in L(F, G; F)$, this means H_x is bilinear and continuous;

2- $S_x \in L(F,G;G)$ and this, also means that S_x is bilinear and continuous.

Proof:

1- Scalar multiplication (1.16) by $D\,i_x(Y)$ with respect to $g^1_{i(x)}$ where $Y \in F$, taking into account (1.3) and (1.12) we get:

$$g^1_{i(x)}\big(D\,i_x(Y), D\,n_x(X,Z)\big) = g^2_x\big(Y, H_x(X,Z)\big),$$

or denoting the left hand side of the last equality as following:

$\beta_x(Y,X,Z) = g^2_x\big(Y, H_x(X,Z)\big),$ (1.17)

where $\beta : x \in \psi(V) \subset F \longrightarrow \beta_x \in L(F,F,G;R)$.

Thus equality (1.17), can rewrites in the from:

$H_x(X,Z) = (g^{2*}_x)^{-1}\big(\beta_x(.\,,X,Z)\big) = (g^{2*}_x)^{-1}\left(\tilde{\beta}_x(X,Z)\right),$ (1.18)

where $\tilde{\beta}_x : (X,Z) \in F \times G \longrightarrow \tilde{\beta}_x(X,Z) = \beta_x(.\,,X,Z) \in L(F;R) = F^*$ and $g^{2*}_x : F \longrightarrow F^*$ is an isomorphism between the Banach spaces F and its dual F^*, taking into account that $g^2_x \in L_2(F;R)$ is strong non-singular [3]. Hence, from (1.18) we get:

$H_x = (g^{2*}_x)^{-1} \circ \tilde{\beta}_x,$ (1.19)

where $\tilde{\beta}_x \in L(F,G;F^*)$ and we deduce that: $H_x \in L(F,G;F)$;

2- From (1.14) we have:

$\gamma_x = n_x \circ S_x,$ (1.20)

where $\gamma_x \overset{\text{def}}{=} D\,n_x - D\,i_x \circ H_x$. Hence we obtain: $\gamma_x \in L(F,G;E)$, furthermore from theorem (1.1) we get: $n_x^{-1} \in L(n_x(G);G)$. Finally, it is clear that: $S_x = n_x^{-1} \circ \gamma_x \in L(F,G;G)$.

Lemma 1.3:

1- The mapping:

$H : x \in F \longrightarrow H_x \in L(F,G;F)$, is differentiable of class C^{r-1}.

2- The mapping:

$S : x \in F \longrightarrow S_x \in L(F,G;G)$, is differentiable of class C^{r-2}.

Proof:

1- At first we prove that the mapping $g^{2*} : x \in F \longrightarrow g^{2*}_x \in L(F;F^*)$ is differentiable and its inverse $(g^{2*})^{-1} : x \in F \longrightarrow (g^{2*}_x)^{-1} \in L(F^*;F)$ is also differentiable of class C^{r-1}. For this aim, we have that the Banach spaces $L(F;F^*)$ and $L_2(F;R)$ are isomorphic [2]. Then, $g^{2*}_x = K(g^2_x)$, where $K : L_2(F;R) \longrightarrow L(F;F^*)$ is an isomorphism of the Banach spaces. But the mapping $g^1 : x \longrightarrow g^1_x \in L_2(F;R)$ is differentiable of class C^{r-1} by condition, and hence the mapping: $g^{2*} : x \in F \longrightarrow g^{2*}_x \in L(F;F^*)$ is differentiable of class C^{r-1}.

Now, $(g^{2*})^{-1} : x \in F \longrightarrow g^{2*}_x \longrightarrow (g^{2*}_x)^{-1}$ is differentiable, since the mapping $u \in L(F;F^*) \longrightarrow u^{-1} \in L(F^*;F)$ is differentiable [4].

Furthermore, from (1.19) we get $H_x = (g^{2*}_x)^{-1} \circ \tilde{\gamma}_x$, such that $\tilde{\gamma}_x$ is differentiable of class C^{r-2} (see Lemma (1.1)). Therefore, it is clear that the mapping $x \longrightarrow H_x$ is differentiable of class C^{r-2};

2- From (1.19) we have:

$S_x = n_x^{-1} \circ \beta_x$, furthermore the mapping β_x is differentiable of class C^{r-2}. Also, from remark (1.1) it follows that the mapping n_x^{-1} is differentiable of class C^{r-1}, $\forall\, x' \in \psi'(\bar{x}) \in \psi'(V') \subset F_1 \subset E$.

Hence, we deduce that the mapping S_x is differentiable of class C^{r-2}.

Equations (1.15) and (1.16) are called the first and the second derivative equations of the Riemannian submanifold N of the Banach Riemannian manifold M.

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