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The Solution Of Positive Integer Forced Sine Gordon Equation By The Modified Homotopy Perturbation Method

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Abstract
In this paper we try to find the approximation solution of Forced Sine Gordon equation with Positive Integer forcing term by the Modified Homotopy Perturbation Method, Firstly we formulae the model of our equation by using the Modified Homotopy Perturbation method so as to construct our Homotopy equations which can be solved by a normal integration. The summation solutions of these integration equations give us the final solution of our eqution. Then by use Mathematica programmed we found our graphs solutions with positive and negative soliton solutions.
Keywords: *Forced Sine Gordon equation, modified homotopy Perturbation, Positive force*

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I. Introduction

In this Section we talk about the Modified Homotopy Perturbation Method. In general the modified homotopy perturbation method proposed for solving the nonlinear differential equation is based upon a small parameter and the homotopy method one (He, J., 1999 a). After substituting the assumed approximation solution into the homotopy and solving corresponding equations, then the approximation solution will comes**.** **Lu, J. (2009)** , he proposed the modified homotopy perturbation method for solving sine Gordon equation without forcing terms. . He introduced a small parameter and Taylor series expansion to modify the homotopy perturbation method. After that he gets a new analytical approach for solving the initial value problem for the following sine Gordon equation,

$$u_{tt} - u_{xx} + \sin u = 0 \qquad (1)$$

Subject to the initial conditions $u(x,t_0) = g_1(x) \qquad u_t(x,t_0) = g_2(x)$

The main feature of his method has ability to obtain the analytical or approximate solution to the sine Gordon equation without linearization, discretization process so as to avoid the difficulties of calculation which is involved or appears in the polynomials in the old method such as Adomain decomposition or the old homotopy method one.

II. The Solution of Sine Gordon Equation without Forcing Terms

In the case of sine Gordon equation, it is clear that we have to solve equation involve **sine u**, this makes it very complicated to get the solution. To avoid this difficulty, we consider another approach for dealing with this equation, on the basis of the homotopy perturbation method. We introduce a variable parameter $p \in [0,1]$ in the sine Gordon equation. Then equation (1) becomes,

$$u_{tt} - pu_{xx} + \sin pu = 0 \qquad (2)$$

Subject to the initial conditions $u(x,t_0) = g_1(x) \qquad u_t(x,t_0) = g_2(x)$

It is easy to see that when **p = 0**, equation (2) corresponds to a linear equation, while when **p = 1**, it is just the original nonlinear one. The parameter **p** is introduced naturally and not affected by artificial factors. Furthermore, it can be considered as a small parameter.

By applying the modified homotopy perturbation technique, we assume that the solution of equation (2) ,can be expressed in terms of **p** as follows,

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + ... \qquad (3)$$

Substituting p = 1, the results is the approximate solution of equation (1).

$$u = \lim_{p \to 1} u \quad \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + ... \qquad (4)$$

To obtain the approximate solution of equation (2), we consider the Taylor series expansion of **sin u** in the following form,

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - ....(-1)^n \frac{u^{2n-1}}{(2n-1)!} + ... \qquad (5)$$

Substituting equation (3) and equation (5) into equation (2), and comparing the coefficients of identical degrees of p. Then we get the following equations.

$$p^0 \quad : u_{0tt} = 0, \quad u_0(x,t_0) = g_1(x), u_{0t}(x,t_0) = g_2(x)$$

$$p^1 \quad : u_{1tt} - u_{0xx} + u_0 = 0, \quad u_1(x,t_0) = 0, u_{1t}(x,t_0) = 0$$

$$p^2 \quad : u_{2tt} - u_{1xx} + u_1 = 0, \quad u_2(x,t_0) = 0, u_{2t}(x,t_0) = 0$$

$$p^3 \quad : u_{3tt} - u_{2xx} + u_2 - \frac{u_0^3}{3!} = 0, \quad u_3(x,t_0) = 0, u_{3t}(x,t_0) = 0\,,$$

and so on. Then we solve the above equations by simple integration, we found the values of u_0, u_1, u_2, u_3, u_4 and so on. After that we can get the approximate solution of equation (1) as follows

$$u = u_0 + u_1 + u_2 + u_3 + ...u_n \qquad (6)$$

Then we can obtain the **n^th^** order approximate solutions .It is clear that and very easy to calculate more components to improve our final solutions

III. Approximate Solution of Our Model

In this Section we need to solve our model that means the Positive Forced Sine Gordon with the following initial conditions, by modified homotopy perturbation method,

$$u_{tt} - u_{xx} + \sin u = 1 \qquad (7)$$

Subject to the initial conditions, $u(x,0) = 0 \qquad u_t(x,0) = 4\sec hx$.

We know that in this case the forcing terms in equation (7) , which is equal 1 is called constant nonlocal forcing terms. Then by modified homotopy perturbation method, we construct the following homotopy and then equation (7) becomes,

$$u_{tt} - pu_{xx} + \sin pu - p = 0 \qquad (8)$$

Subject to the initial conditions, $u(x,0) = 0 \quad and \quad u_t(x,0) = 4\sec hx$

and where **p** is small parameter belong to [0, 1], and by applying the modified homotopy perturbation technique, we assume that the solution of equation (8) can be expressed in terms of **p** as follows,

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + ... \qquad (9)$$

Substituting p = 1 in the above equation, the results is the approximate solution of equation (8).

$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + u_3 + ... \qquad (10)$$

To obtain the approximate solution of equation (7), we consider the Taylor series expansion of **sin u** in the following form,

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - ....(-1)^n \frac{u^{2n-1}}{(2n-1)!} + ... \qquad (11)$$

Substituting equation (9) and equation (11) into equation (8),
and then by comparing the coefficients of identical degrees of p,
We will get the following equations,

$$p^0 \quad : u_{0tt} = 0, \quad u_0(x,0) = 0, u_{0t}(x,0) = 4\sec h\, x$$

$$p^1 \quad : u_{1tt} - u_{0xx} + u_0 - 1 = 0, \quad u_1(x,0) = 0, u_{1t}(x,0) = 0$$

$$p^2 \quad : u_{2tt} - u_{1xx} + u_1 = 0, \quad u_2(x,0) = 0, u_{2t}(x,0) = 0$$

$$p^3 \quad : u_{3tt} - u_{2xx} + u_2 - \frac{u_0^3}{3!} = 0, \quad u_3(x,0) = 0, u_{3t}(x,0) = 0$$

and so on

Normally to find an approximation solution using homotopy method we take, $u_0 = g_1(x) + t\, g_2(x)$ as initial solutions. Then by solving the above equations by simple integration,

we get the following results, as example we found the procedure of u_0.

$$u_0 = g_1(x) + t\, g_2(x) = 0 + t4\sec hx = 4t\sec hx$$

$$u_0' = -4t\sec hx\tanh x$$

$$u_0'' = -4t\left(\sec hx.\sec h^2 x + \tanh .-\sec hx\tanh x\right) = -4t\sec h^3 x + 4t\sec hx\tanh^2 x$$

we know that $\tanh^2 x = 1 - \sec h^2 x$ *sub. in the above equation we get*

$$u_0'' = -4t\sec h^3 x + 4t\sec hx\left(1 - \sec h^2 x\right) - 4t\sec h^3 x + 4t\sec hx - 4t\sec h^3 x$$

$$u_0'' = -8t\sec h^3 x + 4t\sec hx$$

then we needs to find $u_1 = \int_0^t\int_0^t \left(u_0'' - u_0' - 1\right) dt\, dt$ *then*

$$\int_0^t\int_0^t\left(-8t\sec h^3 x + 4t\sec hx - 4t\sec hx\right) dt\, dt = \int_0^t\int_0^t\left(-8t\sec h^3 x - 1\right) dt\, dt = -\frac{8t^3}{6}\sec h^3 x = -\frac{4}{3}t^3\sec h^3 x - \frac{t^2}{2} \quad then$$

$u_1 = -\frac{4t^3}{3}\sec h^3 x - \frac{t^2}{2}$, *then by the same way we get*, $u_2 = \frac{4t^5}{15}\left(2 - \cosh x\right)\left(\sec h^5 x\right)$ *and*

$$u_3 = \frac{8t^5}{15}\sec h^3 x - \frac{t^7}{315}\left(96 - 80\cosh 2x\right) + 4\cosh 4x.$$

In the same way, more components can be calculated. Then the approximation solution of our equation (7) for 3rd order can be

$$u(x,t) = 4t\sec hx - \frac{4t^3}{3}\sec h^3 x - \frac{t^2}{2} + \frac{4t^5}{15}\left(2 - \cosh x\right)\left(\sec h^5 x\right) + \frac{8t^5}{15}\sec h^3 x - \frac{t^7}{315}\left(96 - 80\cosh 2x\right) + 4\cosh 4x.$$

All the graphs of our solutions were found by using Mathematica programs. All graphs showed the solutions in time equal 0.5 with differences values of x so as to compare between them.

Figure 1 gives the solution with t values equal 0.5 and x between -50 and 50.

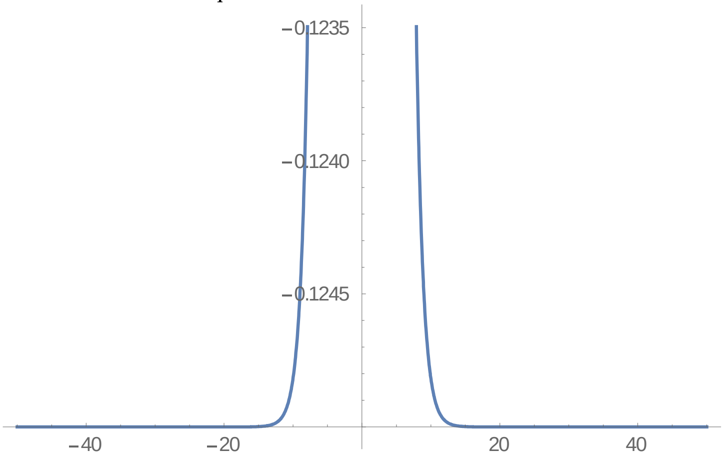

Figure 1

Figure 2 gives the solution of t values equal 0.5 and x between -5 and 5.

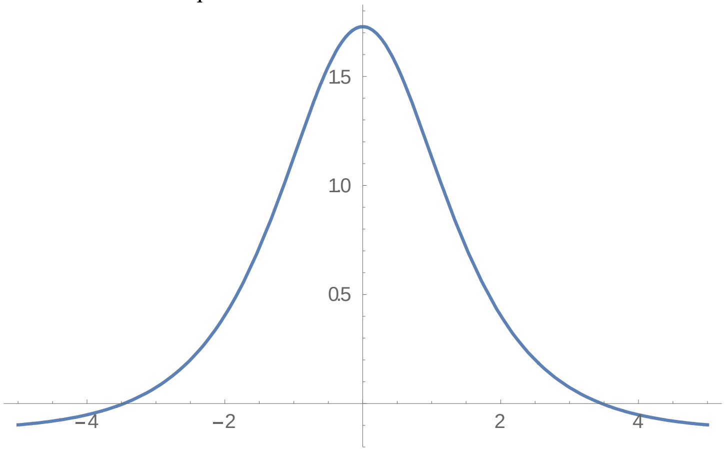

Figure 2

Figure 3 gives the solution of t equal 0.5 and x between -10 and 10.

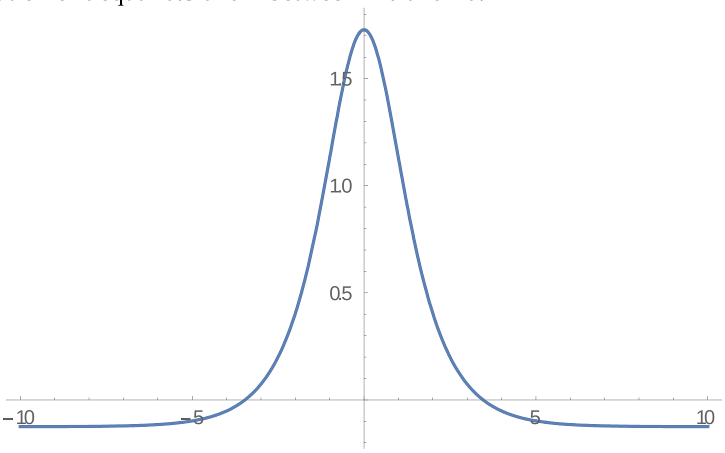

Figure 3

Figure 4 gives the solution of t values equal 0.5 and x between -8 and 2.

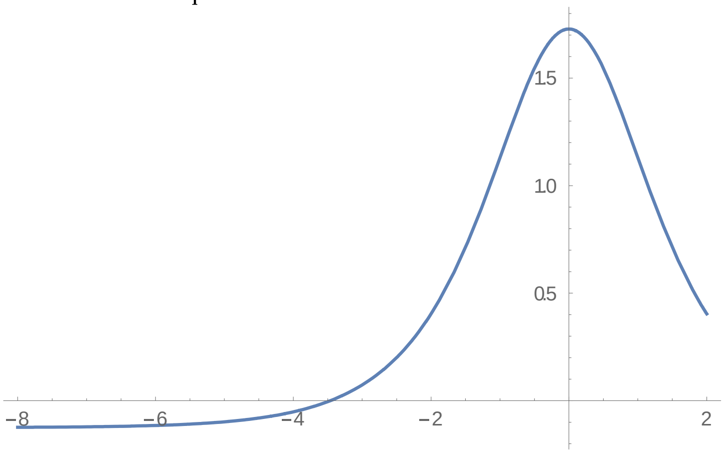

Figure 4

All graph solutions of Figures No 2, 3 and 4 gives us full shapes of soliton solutions in the positive and negative areas. Figure 1 gives us soliton shape but not complete one.

IV. Conclusions

In this paper we studied the approximation solution of forced Sine Gordon equation with positive Integer forcing term by the modified homotopy perturbation method. We found that all solutions give us soliton solutions in positive and negative one. The future work will be for negative Integer Forced one