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A Short Note On A Method For Applying Fractional Calculus To Solve The Tautochrone Problems

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Abstract
Fractional calculus has been more important in fields such as engineering, physics, economics, and more throughout the past 20 years. It results from the new opportunities fractional calculus opens up for problem modelling. The global nature and linearity of differintegrals are the key ideas. The tautochrone problem, one of the first instances of fractional calculus in action, will be covered in this essay. It shows how useful fractional calculus can be in solving certain kinds of integral equations.

Keywords: *Differintegral, Caputo differintegral, TheMittage –Leffler function,The Tautochrone Problem.*

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I. Introduction

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals, briefly differintegrals). In particular, this discipline involves the notion and methods of solving of differential equations involving fractional derivatives of the unknown function (called fractional differential equations)[2], [5]. The history of fractional calculus started almost at the same time when classical calculus was established. [7]it was first mentioned in leibniz’s letter to l’hospital in 1695, where the idea of semiderivative was suggested. During time fractional calculus was built on formal foundations by many famous mathematicians, e.g. Liouville, gr¨unwald, riemann, euler, lagrange, heaviside, fourier, abel etc.[8] a lot of them proposed original approaches, which can be found chronologically in [10]. The theory of fractional calculus includes even complex orders of differintegrals and left and right differintegrals (analogously to left and right derivatives).[9] the fact, that the differintegral is an operator which includes both integer-order derivatives and integrals as special cases, is the reason why in present fractional calculus becomes very popular and many applications arise. [1],[10],[15]the fractional integral may be used e.g. For better describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model as podlubn´y presents in [1]. Analogously the fractional derivative is sometimes used for describing damping.other applications occur in the following fields: fluid flow, viscoelasticity, control theory of dynamical systems, diffusive transport akin to diffusion, electrical networks, probability and statistics, dynamical processes in self-similar and porous structures, electrochemistry[14] in this paper we consider only the most common definitions named after riemann and liouville, caputo, miller and ross let us only note that we use the name “differintegral” which can mean both derivative and integral of arbitrary order. Due to simplicity we will work only differintegrals of real order[11][22][20].

In section 1: introduction, the paper is organized as follows. In section 2: basic fractional calculus gives the definitions of differintegrals, their most important properties, composition rules, as well as laplace and fourier transforms. At the end we give several differintegrals of simple functions.section 1. Introductionfractional derivatives, so called fractional differential equations (fdes)[23]. We restrict ourselves to linear fdes because there is a more compact theory. In section3: lfdes and their solutions we investigate the main methods of solving for linear fdes and illustrate them on several examples. Finally[10] insection 4: applications of fractional calculus we discuss some concrete problems like the tautochrone problem, advection-dispersion equation, oscillations with fractional damping and fractional models of viscoelasticity.this thesis tries to be self-contained, however if you find a part which is not perfectly clear, all answers are surely included in one of the book listed in the bibliography[15]. Classical calculus focuses on derivatives and integrals of functions, which are inversely related. By putting the derivatives of a function f(t) on the left and integrating on the right, we may create an endless series on both sides [13].

II. Basic concepts of Fractional calculus

Classical calculus focuses on derivatives and integrals of functions, which are inversely related. By putting the derivatives of a function f(t) on the left and integrating on the right, we may create an endless series on both sides.

$$\dots, \frac{d^2 f(t)}{dt^2}, \frac{df(t)}{dt}, f(t), \int_a^t f(\tau)d\tau, \int_a^t \int_a^{\tau_1} f(\tau)d\tau\, d\tau_1, \dots$$

Fractional calculus aims to interpolate this sequence thus this operation unites the classical derivatives and integrals and generalizes them for arbitrary order. The term "differential" is commonly used, however the term "α-derivative" (which can also refer to an integral if α < 0) or "fractional derivative" may also be used[17].

Various techniques to defining the differential integral are referred to by their authors. The Grunwald-Letnikov concept of differintegral begins with classical definitions of derivatives and integrals using infinitesimal division and limits[19]. The downsides of this technique include technical difficulties in computations and proofs, as well as significant function constraints. Fortunately, there are more elegant alternatives available[20].

The Riemann-Liouville Differintegral

The Riemann-Liouville technique relies on the Cauchy formula (2.1) for the nth integral, a simple integration that allows for extension.

$$I_a^n f(t) = \int_a^t \int_a^{\tau_{n-1}} \dots \int_a^{\tau_1} f(\tau)d\tau d\tau_1 \dots d\tau_{n-1} = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau \quad (2.1)$$

Proof. The formula (2.1) can be proven by the help of mathematical induction[13]. The case n = 1 is obviously fulfilled, so we show the case n = 2 which demonstrates the mechanism of the entire proof in a better way

$$\frac{1}{1!}\int_a^t (t-\tau) f(\tau)d\tau = \begin{vmatrix} u = t-\tau & u' = -1 \\ v' = f(t) & v = \int_a^\tau f(r)dr \end{vmatrix} = \left[(t-\tau)\int_a^\tau f(r)dr\right]_{\tau=a}^{\tau=t} \int_a^t \int_a^\tau f(r)dr = I_a^2 f(t)$$

In the higher limit, the polynomial is zero, whereas in the lower limit, we integrate over a set with measure zero. Therefore, the first term is zero.

Assume the formula holds for generic n. Then, we integrate it once more to observe the results.

$$\int_a^t I_a^n f(r)dr = \int_a^t \frac{1}{(n-1)!}\int_a^t (r-\tau)^{n-1} f(\tau)d\tau\, dr \quad \begin{vmatrix} change\ of\ order \\ of\ integration \end{vmatrix}$$

$$= \frac{1}{(n-1)!}\int_a^t f(\tau)\int_\tau^t (r-\tau)^{n-1} dr\, d\tau = \frac{1}{(n-1)!}\int_a^t f(\tau)\left[\frac{(t-\tau)}{n}\right]_\tau^t d\tau$$

$$= \frac{1}{n!}\int_a^t (t-\tau)^n f(\tau)d\tau = I_a^{n+1} f(t)$$

This completes the proof of the Cauchy formula (2.1).

Remark. The only property of the function $f(t)$ we used during the proof was its integrability.

No other restrictions are imposed. Now it is obvious how to get an integral of arbitrary order[12]. We simply generalize the Cauchy formula (2.1) - the integer n is substituted by a positive real number α and the Gamma function is used instead of the factorial. Notice that the integrand is still integrable because $\alpha - 1 > -1$.

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau \quad (2.2)$$

This formula represents the integral of arbitrary order α > 0, but does not permit order $\alpha = 0$ which formally corresponds to the identity operator. This expectation is fulfilled under certain reasonable assumptions at least if we consider the limit for $\alpha \to 0$ (see [1]).

Hence, we extend the above definition by setting:

$$I_a^0 f(t) = f(t) \quad (2.3)$$

The definition of fractional integrals is very straightforward and there are no complications.

A more difficult question is how to define a fractional derivative[3]. There is no formula for the nth derivative analogous to (2.1) so we have to generalize the derivatives through a fractional integral. First we perturb the integer order by a fractional integral according to (2.2) and then apply an appropriate number of classical derivatives. As we will see later (the formula (2.2)), we can always choose the order of perturbation less than 1.

The result of these ideas is the following ($\alpha > 0$):

$$D_a^\alpha f(t) = \frac{d^n}{dt^n}[I_a^{n-\alpha} f(t)] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau. \quad (2.4)$$

where $n = [\alpha] + 1$. This formula includes even the integer order derivatives. If $\alpha = k$ and $k \in N0$ then $n = k + 1$ and we obtain:

$$D_a^k f(t) = \frac{1}{\Gamma(1)} \frac{d^{k+1}}{dt^{k+1}} \int_a^t f(\tau) d\tau (3) = \frac{d^k f(t)}{dt^k}$$

We can see that classical derivatives are something like singularities among differintegrals because the integration disappears and so there is no dependence on the lower bound a anymore. In this sense the classical derivatives are the only differintegrals which do not depend on history, i.e. are local[4].

If we put $D_a^{-\alpha} = I_a^\alpha$ and note that $f^{(0)}(t) = f(t)$, we can write both fractional integral and derivative by one expression and formulate the definition of the Riemann-Liouville differintegral.

Definition 3.1.1 (The Riemann-Liouville differintegral). Let a, T, α be real constants $(a < T), n = max(0, [\alpha] + 1)$ and $f(t)$ an integrable function on $\langle a, T \rangle$. For $n > 0$ additional assume that $f(t)$ is n −times differentiable on $\langle a, T \rangle$ except on a set of measure zero. Then the Riemann-Liouville differintegral is defined for $t \in \langle a, T \rangle$ by the formula:

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau. \quad (2.5)$$

Remark. In this thesis we will denote the differintegrals by various symbols according to the used approach. For the Riemann-Liouville approach the bold face capital letter D is reserved from now on [3].

The Caputo Differintegral

We will denote the Caputo differintegral by the capital letter with upper-left index ${}^{C}D$. The fractional integral is given by the same expression like before, so for $\alpha > 0$ we have

$${}^{c}D_a^{-\alpha} f(t) = D_a^{-\alpha} f(t) \quad (2.6)$$

The difference occurs for fractional derivative. A non-integer-order derivative is again defined by the help of the fractional integral, but now we first differentiate $f(t)$ in the common sense and then go back by fractional integrating up to the required order. This idea leads to the following definition of the Caputo differintegral[5].

Definition 2.1 (The Caputo differintegral). Let a, T, α be real constants $(a < T)$, $n_c = max(0, -[\alpha] + 1)$and $f(t)$ a function which is integrable on $\langle a, T \rangle$ in case $n_c = 0$ and n_c −times differentiable on $\langle a, T \rangle$ except on a set of measure zero in case $nc > 0$. Then the Caputo differintegral is defined for $t \in \langle a, T \rangle$by formula:

$${}^{c}D_a^{-\alpha} f(t) = I_a^{-\alpha} \left[\frac{d^{n_c} f(t)}{dt^{n_c}}\right] \quad (2.7)$$

Remark. For $\boldsymbol{\alpha} > \mathbf{0}, \boldsymbol{\alpha} \notin \mathbb{N}_\mathbf{0}$, formula 3.7 is often weitten in the form:

$${}^{c}D_a^{-\alpha} f(t) = \frac{1}{\Gamma(n_c - \alpha)} \int_a^t (t-\tau)^{n_c-\alpha-1} f^{(n_c)}(\tau) d\tau. \quad (2.8)$$

The reason why $n_c$in the definition of the Caputo derivative is different from n introduced in the Riemann-Liouville case, is correspondence with integer-order derivatives[6].

We cannot use n even in the Caputo definition because we would get wrong results for the kth derivative of a function with zero $(k + 1)th$ derivative. This would be an effect of the paradox that we would need for the kth derivative a $(k + 1)$ −times differentiable function.

On the contrary, we could use n_c with the Riemann-Liouville derivative, but will use n because then we do not need a limit relationship (2.1).

Anyway, the only difference between the values of n and nc is for integers as we can see in figures 2.1 and 2.2. In addition, we know that both cases coincide with classical derivatives at those points, hence there should not be any problems[7].

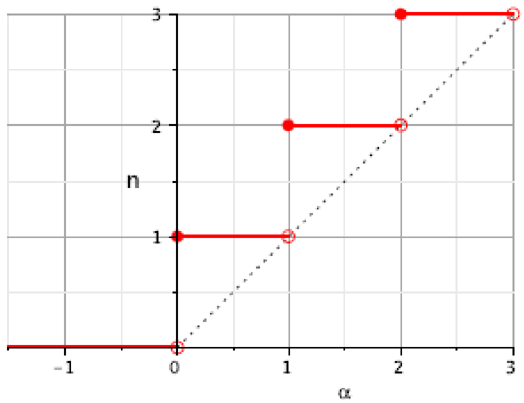

Figure 2.1: Function $n = [\alpha] + 1$ used for the Riemann-Liouville derivative.

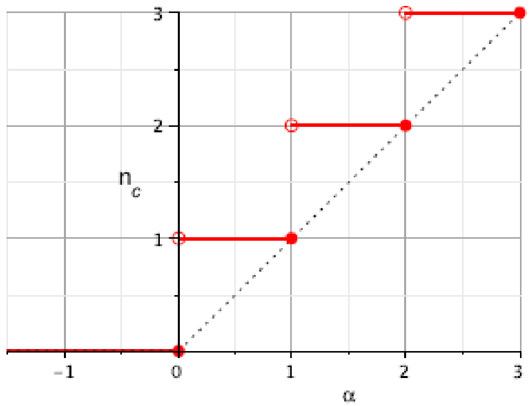

Figure 2.2: Function $n = -[-\alpha]$ used for the Caputo derivative.

Clearly, the Caputo derivative can also be written by the help of fractional integrals of the Riemann-Liouville type:

$${}^{c}D_a^{-\alpha} f(t) = D_a^{-(n_c-\alpha)}\left(\frac{d^{n_c} f(t)}{dt^{n_c}}\right) \qquad (2.9)$$

Here we see that if we consider formula (3.3), the Caputo derivative of order $\alpha = nc$ is equal to the classical ${n_c}^{\text{th}}$ derivative.

The reasons which led to the definition of the Caputo derivative are mainly practical. As we will see in section 3.6, the Riemann-Liouville approach requires the initial conditions for differential equations in terms of non-integer derivatives which are hardly physical interpreted, whereas the Caputo approach uses integer-order initial conditions[8]. Moreover, we sometimes also need fractional derivatives of constants to be zero. The Riemann- Liouville derivative with finite lower bound a does not satisfy this while the Caputo derivative does. More about the correspondence of these approaches can be found

III. The Method of the Transformation to ODE

Some initial-value problems may be solved by the transformation to an ordinary differential equation. In principal there is a condition w.r.t. the order of derivatives - if there is only one derivative in the equation, its order has to be a rational number. If there are more differential terms, all their orders have to be rational numbers and moreover we should manage to get only integer-order terms by combining the used fractional orders. Other problems may occur during the computation because it is difficult to apply a fractional derivative on terms containing an unknown function [16].

We will demonstrate this method on examples with one differential term. For better understanding the method's spirit, let us first introduce the following simple example with the Riemann-Liouville semiderivative[17].

Fractional Ordinary Differential Equations.

The general formula of fractional linear ordinary differential equation of order (m,α is given by

$$\left[D^{qm} + a_{(m-1)} D^{q(m-1)} + \cdots + a_0 D^0\right] x(t) = h(t) \quad x \geq 0 \qquad (3.1)$$

Where $q = \frac{1}{\alpha}$ if $\alpha = 1$ then $q = 1$and equation (3,1) is a simple ODE of order$m$,

Where $a_0, a_1, \dots a_{m-1}$are functions of the independent variable t .

we introduce the definition of important function, Mittage–leffler function

Mittage –Leffler function [Dzherbashyan, 1966].

The Mittage –Leffler function is an important function that finds widespread in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittage –Leffler function plays an analogous role in the solution of noninteger order differential equations. In fact, the exponential function itself a very specific form, one of an infinite series, of this seemingly omnipresent function[18]. The standard definition of the Mittage –Leffler is given by

$$E_{\beta,m}(t) = \sum_{k=0}^{\infty} \frac{t^{\beta k}}{\Gamma(\beta k + a)} \qquad (3.2)$$

Where $\beta \in \not\subset, Re(m), Re(\beta) > 0$.

The function $E\ (t\ , m\ , a\ )$ is used to solve differential equation of fractional order which is defined by:

a) $E_{\beta,m}(t\,,m\,,a\,) = t^m \sum_{k=0}^{\infty} \frac{(at^{\beta})^k}{\Gamma(\beta k+m+1)} = t^m\, E_{\beta,m+1}(at^{\beta}\,)$

b) $E_{\beta,m}(t\,,m-1,a\,) = t^{m-1} \sum_{k=0}^{\infty} \frac{(at^{\beta})^k}{\Gamma(\beta k+m)} = t^{m-1}\, E_{\beta,m}(at^{\beta}\,)$

(3.4)

Example 3.1. Solve the initial-value problem1

$$D_0^{\frac{1}{2}} y(t) = y(t)$$

$$D_0^{-\frac{1}{2}} y(t)\Big|_{t=0} = b$$

We see that if we apply again the semiderivative on the entire equation, we get the ordinary differential equation of the first order. During computation it is necessary to remember the formula for the composition of fractional derivatives (3.2).

$$y'(t)\, D_0^{-\frac{1}{2}} y(t)\Big|_{t=0} = \frac{t^{\frac{-3}{2}}}{\Gamma\left(-\frac{1}{2}\right)} = D_0^{\frac{1}{2}} y(t)$$

Now we use the initial condition and even the original equation and after some calculations we obtain the nonhomogeneous linear ODE with constant coefficients[21].

$$y'(t) - y(t) = -\frac{-b}{2\sqrt{\pi}} t^{\frac{-3}{2}}$$

First we can calculate the general solution of the appropriate homogeneous equation which is yh(t) = C et, where C is a constant. We know from the theory of linear ODEs that the solution of the original nonhomogeneous equation can be found in the form y(t) = C(t) et.

The function C(t) can be determined by substituting back into ODE.

$$C'(t)\, e^t + \lambda 2C(t)\, e^t - C(t)\, e^t = \frac{-b}{2\sqrt{\pi}}$$

$$C(t) = -\frac{-b}{2\sqrt{\pi}} \int_0^t \tau^{\frac{-3}{2}} e^{-\tau} d\tau$$

This integral diverges at the first glance, but we may identify so-called incomplete Gamma function there, hence we may write C(t) in the form (for more details see [10])

$$C(t) = b\left(erf(\sqrt{t})\frac{e^{-t}}{\sqrt{\pi t}}\right)$$

Thus the solution of the ODE is

$$y(t) = C\, e^t + b\, erf(\sqrt{t}) e^t = \left(\frac{b}{\sqrt{\pi t}}\right)$$

The last thing that remains, isto determine the unknown constant C. The only condition we did not use yet, is the original FDE. We are not going to compute the semiderivative of $y(t)$ here but only introduce the result.

$$D_0^{\frac{1}{2}} y(t) = C\left(\frac{1}{\sqrt{\pi t}} + erf(\sqrt{t})e^t\right) + \frac{e^t}{\sqrt{\pi t}}$$

Let us substitute it and find the constant C.

$C\left(\frac{1}{\sqrt{\pi t}} + erf(\sqrt{t})e^t\right) + e^t = Ce^t + berf(\sqrt{t})e^t + \frac{b}{\sqrt{\pi t}}$

It is clear that $C = b$ and then the solution of the initial-value problem is

$$y(t) = b\left(e^t + erf(\sqrt{t})e^t + \frac{1}{\sqrt{\pi t}}\right)$$

It is easy to check that we would get the same result by using the Laplace transform method.

The method of the transformation to ODE is more general and it could be used even for linear FDEs with nonconstant coefficients but the problem is the big technical complication[23].

Applications of Calculus of Fractions

Fractional calculus has been more important in fields such as engineering, physics, economics, and more throughout the past 20 years. It results from the new opportunities fractional calculus opens up for problem modeling. The global nature and linearity of differintegrals are the key ideas. The tautochrone problem, one of the first instances of fractional calculus in action, will be covered in this essay[24]. It shows how useful fractional calculus can be in solving certain kinds of integral equations

The Tautochrone Problem

Abel examined this well-known example for the first time in the early 1800s. Although it is not strictly necessary, it was one of the fundamental issues where the fractional calculus framework was used [10].

The challenge is to locate a curve in the (x, y) −plane such that, under the assumption of a homogeneous gravity field and no friction, the time it takes a particle to go down the curve to its lowest point is independent of where it started[25]. Let's establish a curve's lowest point at the origin and its location in the positive quadrant of the plane, indicating the beginning point (x, y) and any intermediate point (x^*, y^*) between $(0,0)$ and (x, y). The energy conservation law states that we may write

$$m\left(\frac{d\sigma}{dt}\right)^2 = mg(y - y^*)$$

where σ is the length along the curve measured from the origin, m the mass of the particle, g the gravitational acceleration[24]. Considering $\frac{d\sigma}{dt} < 0$ and $\sigma = \sigma\ (y^*(t))$, we rewrite the

$$\sigma'\frac{dy^*}{dx} = -\sqrt{2g(y - y^*)}$$

Which we integrate from $y^* = y$ to $y^* = 0$ and from $t = 0\ to\ t = T$. After some calculations we get the integral equation

$$\int_0^y \sigma'\frac{\sigma'(y^*)}{\sqrt{(y - y^*)}}dy^* = \sqrt{2g}T$$

Here one can easily recognize the Caputo differential and write

$${}^{C}D_0^{\frac{1}{2}}\sigma(y) = \frac{\sqrt{2g}}{\Gamma(\frac{1}{2})}T$$

Let us note that T is the time of descent, so it is a constant. By applying the $\frac{1}{2}$-integral to both sides of the equation and by using the formulas for the composition of the Caputo differintegrals (3.26) and for the fractional integral of the constant (3.34), we get the relation between the length along the curve and the initial position in y direction

$$\sigma(y) = \frac{\Gamma(1)\sqrt{2g}T}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}\ y^{\frac{1}{2}} = \frac{2\sqrt{2g}T}{\pi}\ y^{\frac{1}{2}}$$

The formula describing coordinates of points generating the curve can be written by the help of the relation:

$$\frac{d\sigma}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

Which after the substitution of $\sigma(y)$ gives

$$\frac{dx}{dy} = \sqrt{\frac{2gT^2}{\pi^2 y} - 1}$$

It can be shown that the solution of this equation is so-called tautochrone, i.e. one arch of the cycloid which arises by rolling of the circle along the green line in figure 6.1. The parametric equations of the tautochrone are

$$x = \frac{A}{2}[u + \sin(u)]$$
$$y = \frac{A}{2}[1 + \cos(u)]$$

where $A = \frac{2gT^2}{\pi^2 y}$ In particular for$T = \frac{\pi}{\sqrt{2g},}$ i.e. for $A\ =\ 1$, the tautochrone is drawn in figure 6.1 by the red colour.

We have seen that the knowledge of the rules of fractional calculus is very useful for solving this type of integral equations.

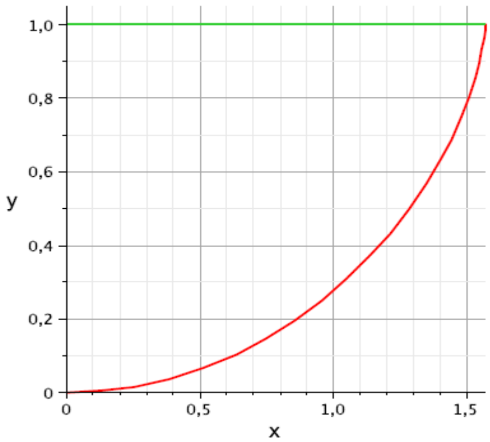

Figure 4.1: the Tautochrona for $A = \frac{2gT^2}{\pi^2} = 1$

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