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Solving The Goldbach's Problem

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Abstract
The paper proposes a method for proving the Goldbach's binary conjecture, based on the properties of representations of even numbers and the rules of formal logic. The paper provesthat an even number, in the representation of which there are no pairs of prime conjugate numbers, is identical to a number that does not exist. The connection between the Goldbach's conjecture and the Legendre hypothesis is considered.

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I. Introduction. General Relations

One of the unresolved problems of the number theory is the proof of the Goldbach's binary conjecture. Experimental studies and calculations confirm its validity for large numbers [1– 3]. In this paper, we consider a method for proving a hypothesis based on the properties of representations of even numbers and the rules of formal logic. As is known, every natural number admits a trivial representation as a sum of units. Combining (grouping) units in different ways, we obtain all the representations. Consider the even numbers and their representations as asum of two odd summands. For an arbitrary even p, the trivial representation has the form $p = 1 + 1 + ... + 1$ (p units). If p is divisible by 4, then its representation as a sum of two odd numbers can be written as a chain of equalities

$$p = [1+(p-1)] = [3+(p-3)] = ... = [(p/2-1)+(p/2+1)], \quad (1)$$

where $p/2$ is the center of representation (even number). The terms in square brackets, we callthe conjugate numbers. The first term is to the left of the center and the second term is to the right of the center. If p is not divisible by 4, its representation has the form

$$p = [1+(p-1)] = [3+(p-3)] = ... = [(p/2-2)+(p/2+2)].= [p/2+ p/2], \quad (2)$$

where $p/2$ is the center (odd number). The number of pairs depends on the parity of center. These regularities are of a general nature. If we move from the number p, which is divisible by 4, to thenext even number $p + 2$, which is not divisible by 4, then the even center of the representation becomes odd ($p/2 \rightarrow p/2 + 1$), and the number of pairs of conjugate numbers increases by 1 ($p/4 \rightarrow p/4 + 1$). If we move from the number $p + 2$ to the next even number $p + 4$, which is divisible by 4, the odd center becomes even ($p/2 + 1 \rightarrow p/2 + 2$), and the number of pairs of conjugate numbers does not change ($p/4 + 1 \rightarrow p/4 + 1$). Acceptable combinations of ends in conjugate numbers and their sequence are completely determined by the end of even number. The prime numbers to the left of the center of the representation of an arbitrary even number willbe called prime representation generators. With increasing of even number p the number of conjugate pairs and the number of primes included in the representation of even number increase. From (1) it can be seen that initial prime numbers 3, 5, 7, 11, 13, 17, 19, 23 etc. are included in all representations for sufficiently large p and their number increases with increasing p.

There is an asymmetry in the distribution of prime numbers to the left and to the right of the center, which depends on two processes. The number of primes to the left of the center rises (not decreases), as they transfer from the right to the left. Therefore the number of primes on the rightof the center can decrease (does not increase), but not much. It is compensated by appearance of new primes and does not vanish when p increases. With increasing p between these two processes is the dynamic equilibrium that determines an asymmetry of the distribution of prime numbers and depends on order of magnitude of number p. Prime numbers regularly appear both on the left and on the right of the center. The appearance of pairs of prime conjugate numbers depends on the total number of primes less than p and on the irregularity of their distribution on the left and on the right of the center. Representations of even numbers form a connected system.It turns out that the only assumption that does not lead to contradiction is the validity of the Goldbach's conjecture for all even numbers.

II. Proof Of Goldbach's Binary Conjecture

We prove the Goldbach's binary conjecture by induction. We are going to prove the assertion that every even integer more than or equal to 4 can be represented as the sum of two primes. The validity of this statement can be verified for any finite even number and, what is more important for us, for any interval of even numbers when it is doubled. Consider the set of even numbers. Suppose that the conjecture is valid for all even numbers $p \leq 2n1$, in particular, itis valid for numbers in the interval $n1 \leq p \leq 2n1$. It is required to prove that the binary hypothesis is valid for $p > 2n1$. We will prove the induction transition not for one value of p, but for an interval of values. We prove that the conjecture is valid for the interval $(2n1 + 2) \leq p \leq 2(2n1 + 2)$.The choice of the interval is determined by the fact that the ratio of the even number p to the center of its representation $p/2$ is equal to the ratio of the end of the considered interval to its beginning. Thus, there is vertical-horizontal symmetry in the representation of even numbers.

Three assumptions are possible: 1) the Goldbach's binary conjecture is not valid for any number from the considered interval; 2) the conjecture is not valid only for some numbers from the interval; 3) the conjecture is valid for all numbers from the interval. These assumptions are mutually exclusive and form a complete group of events. The representations of evennumbers from the interval are interconnected and have common parts with each other and the representations of numbers from the previous interval. When even numbers increase from $2n1 + 2$ to $2(2n1 + 2)$, the numbers to the right of the center of the representation move to the left of the center. At the same time, new numbers appear and the number of pairs in the representation increases, as well as the number of prime and composite numbers on the left and on the right of the center. When the even number is increased from $2n1 + 2$ to $2(2n1 + 2)$, the numbers to the right of center in the representation of the number $2n1 + 2$ appear (pass) to the leftof center in the representation of the number $2(2n1 + 2)$. When even numbers decrease from$2n1 + 2$ to $n1 + 1$, the numbers to the left of the center move to the right of the center and, at the same time, the number of pairs in the representation decreases. When the even number decreases from $2n1 + 2$ to $n1 + 1$, the numbers to the left of the center in the representation of the number$2n1 + 2$ appear (pass) to the right of the center in the representation of the number $n1 + 1$. Therefore, the representations of even numbers form a connected system. It turns out that this system is consistent only if the Goldbach's conjecture is valid for all even numbers from the considered interval. However, we cannot verify this statement directly, since we do not know the quantitative relationship between the number of prime and composite numbers participating inthe representation of an arbitrary even number from a given interval. We prove the validity of thehypothesis by the elimination method, i.e. we prove that the first and the second assumptions cannot be valid. The proposed method of proof does not require taking into account the properties of representation associated with the belonging of even numbers to different classes. For example, if an even number is divisible by 3, then in any pair, if the first number is divisible by 3, then the second number is also divisible by 3. If an even number ends in zero, then in any pair, if the first number is divisible by 5, then the second number is also divisible by 5.

Assume that **the first assumption** is valid, i.e. the Goldbach's binary conjecture is not satisfied for any even number from the considered interval. We show that this leads to acontradiction. The representation of an arbitrary even number p from the considered interval consists of a set of pairs of conjugate numbers of the form $(m, p - m)$. Let m be a prime numberto the left of the center of the representation. When p changes from $2n1 + 2$ to $2(2n1 + 2)$, the number $p - m$ changes from $2n1 + 2 - m$ to $4n1 + 4 - m$. The ratio $((4n1 + 4 - m)/(2n1 + 2 - m)) > 2$, therefore, according to Bertrand's postulate, there is at least one prime number between $2n1 + 2 - m$ and $4n1 + 4 - m$. It follows that there is an even number in the considered intervalthat has a pair of prime conjugate numbers in representation. This result contradicts our assumption. Therefore, **the first assumption** is not satisfied and can be rejected.

Assume that **the second assumption** is valid, i.e. the Goldbach's conjecture does not hold for some even numbers (at least one number) from the considered interval. Take an arbitrary number p from the considered interval. If the conjecture is valid, then the representationof p contains at least one pair of primes. If the conjecture is not valid, then the representation of $p$does not contain pairs consisting of two primes. In this case, one of two conditions must be satisfied. The first condition: there are no prime numbers to the right of the center in the representation of p. This condition should be rejected since it contradicts Bertrand's postulate. The second condition: all prime numbers to the right of center form pairs only with composite numbers to the left of center, or, in an equivalent formulation, all prime numbers to the left of center form pairs only with composite numbers to the right of center. We prove that this is impossible. Consider a representation of an arbitrary even number $2n1 + 2l$ from the considered interval, where $l = 1, 2, 3$, etc. up to $n1 + 2$. For $l = 0$, we obtain the number $2n1$ for which the conjecture is valid by assumption. The representation of the number $2n1 + 2l$ consists of the set ofpairs $(m, 2n1 + 2l - m)$, where $m = 1, 3, 5$, etc. up to mt ($mt = n1 + l$ if $n1 + l$ is odd, or $mt = n1 + l - 1$ if $n1 + l$ is even). This representation has common parts with representations of all numbers from the considered interval, as well as with representations of numbers $2n1$ and less than $2n1$, namely, $2n1 - 2$, etc. (depending on l). It is convenient to compare the representation of the numbers $2n1 + 2l$ and $2n1$ by considering the pairs ($2l + m$, $2n1 - m$). Changing the value of l, we obtain all even numbers from the considered interval, and changing the value of m, we obtain theright parts of the representation of an even number $2n1 + 2l$ for some l, which are common with the representation of the number $2n1$. The first number in these pairs is to the left of the center of the representation of the corresponding even number, and the second is to the right of the center of the representation. The number $2l + m$ should not be greater than $2n1 - m$, otherwise they should be swapped.

In the representation of even numbers from the interval under consideration, we take a line in which the first number in the pair (m, $2n1 + 2l - m$) is prime, i.e. $m = 3, 5, 7, 11$, etc. When l changes from 1 to $n1 + 2$, the number $2n1 + 2l - m$ changes from $2n1 + 2 - m$ to $4n1 + 4 - m$, i.e. more than 2 times. Therefore, at least one number $2n1 + 2l - m$ must be prime for every prime m. Otherwise, Bertrand's postulate is violated. Therefore, the hypothesis will be valid for at least one even number from the considered interval. If the first number in the pair (m,$2n1 + 2l - m$) is composite or 1, then when l changes from 1 to $n1 + 2$, at least one number$2n1 + 2l - m$ must be prime for every composite m. For different m, the numbers $2n1 + 2l - m$will be prime for different even numbers from the interval under consideration. The prime numbers $2n1 + 2l - m$ cannot be concentrated in the representation of only one or part of the numbers of the interval under consideration, since, according to Bertrand's postulate, there is at least one prime number on the right of the center in the representation of any even number. In addition, if primes were localized in the representation of only one even number, we would have several pairs of twin-primes to the right of the center of the representation of an even number, which contradicts the facts. The distance between pairs of twin-primes for sufficiently large evennumbers $2n$ exceeds n and increases with the even number.

Compare the representations of the numbers $2n1 + 2l$ for different values of l with the representation of the number $2n1$, which corresponds to the value $l = 0$. For a given l, the numbers $2l + m$, depending on the value of m, can be prime or composite, as well as the numbers$2n1 - m$. In the representation of the numbers $2n1 + 2l$ and $2n1$, the numbers $2l + m$ and $2n1 - m$ will form pairs that consist of prime and composite numbers. When l changes, the result will change, and a different number of pairs will consist of prime and composite numbers. For example, if $2n1 - m$ in the representation of the number $2n1$ is a prime number, then the hypothesis will be valid for all even numbers in representation of which $2l + m$ is a prime number. If the number $2n1 - m$ is composite, then the pairs ($2l + m$, $2n1 - m$) can be excluded from the representation. In this case, the hypothesis will hold for all even numbers in representation of which the pairs (m, $2n1 + 2l - m$) are prime when l and m change. The result also depends on our assumption for which even numbers the hypothesis is not valid. For example, if $2l + m$ is a prime number, then $2n1 - m$ cannot be a prime number in the representation of even numbers for which we assume that the hypothesis is not valid. If $2l + m$ is a composite number, then $2n1 - m$ can be both a prime and a composite number, depending on our assumption for which l the hypothesis fails. For different values of l, the number of prime pairs in the representation of different even numbers will be different.

The set of even numbers from the considered interval is divided into groups (subsets). The representation of even numbers in the same group has the same number of pairs of prime numbers. Since the representation of any even number is unique, the same number cannot simultaneously belong to different groups. Thus, we have a partition of the set of even numbers from the interval under consideration, formed by a finite number of groups. Representations of even numbers in the same group are invariant under transformations that keep the number of pairs consisting of prime numbers. Such transformations include the elimination or addition of pairs in which one or both numbers are composite. A representation consisting only of pairs of prime numbers we call irreducible. The number of pairs of primes in the representation of even numbers of one group we call the index of the group. In particular, the index of a group consisting of even numbers in representation of which there are no pairs of primes is equal to zero. From an arbitrary number of a group, any other number from the same group can be obtained using a permissible transformation. Such a transformation is the addition or subtraction of some even number $2l$ so that the number of pairs of primes participating in the representation does not change. The value $2l = 0$ corresponds to identical transformation. If the index of a group is not equal to 0, then there is an order relation on even numbers of the same group, determined by the values of the number $2l$. If the index of a group is equal to 0, then comparison of even numbers is impossible, since their irreducible representations do not contain pairs of conjugate numbers. Only the identity transformation is permissible, and even numbers from this group are indistinguishable (identical). We use the properties of irreducible representations from the group with index 0 to prove the Goldbach's conjecture.

Suppose that for some even number $2n1 + 2l0$ the Goldbach's conjecture is not valid. Such a number belongs to the group of index 0. This group also includes some even number $2n0$, which is divisible by all prime numbers located to the left of center of its representation. Number $2n0$ does not necessarily lie within the considered interval. If we exclude from the representation of the number $2n1 + 2l0$ all pairs in which the number to the right of the center is prime, as well as the pair in which the first number is 1 (such a transformation is permissible), then the representation of the number $2n1 + 2l0$ will be equivalent to the representation of the number $2n0$. The irreducible representations of the numbers $2n1 + 2l0$ and $2n0$ coincide (identical). We prove that such a number $2n0$ does not exist. From this it will follow that the number $2n1 + 2l0$ does not exist too, i.e., the order of the group of index 0 is zero. Suppose the contrary, i.e. that there is a number $2n0$ that is divisible by all prime numbers to the left of the center of its representation (prime representation generators). Then in the representation of the number $2n0$, the number $2n0 – 1$ is prime, since, otherwise, $2n0$ cannot be divided into all prime representation generators. In the future, we exclude the pair $(1, 2n0 – 1)$ from consideration, and then all numbers to the right of the center in the representation of the number $2n0$ will be composite. Write the product ofall primes to the left of the center of the representation of the number $2n0$ as $\Pi \equiv g1 \cdot g2 \cdot \ldots \cdot gi \cdot \ldots \cdot gk$. We have the following chain of inequalities $\Pi \equiv g1 \cdot g2 \cdot \ldots \cdot gi \cdot \ldots \cdot gk > gj \cdot gl \cdot gk > ((2n0)^{1/2})^3 > 2n0$. Here, we designate gi is the largest generator of the number $2n0$, $gi \le [(2n0)^{1/2}]$; gk is the largest prime number to the left of the center of the representation of the number $2n0$. Therefore, we assume that there are at least two primesgj and gl between gi and gk. We solve the inequality $((2n0)^{1/2})^3 > 2n0$. From this, we obtain $2n0 > 1$, which contradicts our assumption. Thus, we have proved Lemma 1. **Lemma 1**: "Any even number that has in the representation at least two primes between the largest generator and the largest prime to the left of the center of the representation is less than the product of the primes to the left of the center of the representation." When proving Lemma 1, we use strict condition (requirement), which allows us to prove it in general form. We carry out the theoreticalestimate of the lower bound of even numbers for which the condition of Lemma 1 is satisfied. Denote, as above, gi is the largest generator of the even number $2n$; $gi \le [(2n)^{1/2}]$. According to Bertrand's postulate, in each of the intervals $(2n)^{1/2} \ldots 2(2n)^{1/2}$, $2(2n)^{1/2} \ldots 4(2n)^{1/2}$, $4(2n)^{1/2} \ldots 8(2n)^{1/2}$ is a prime number. A prime number gk from the interval $4(2n)^{1/2} \ldots 8(2n)^{1/2}$ participatesin the representation of the even number $2n$ if $8(2n)^{1/2} \le n$. From here, we get the estimate$2n \ge 256$. The condition of Lemma 1 is satisfied for all even numbers $2n \ge 256$, since there are two prime numbers between gi and gk. For smaller even numbers, the validity of Lemma 1 is verified directly. The condition we used to prove Lemma 1 is satisfied for even numbers $2n > 20$.In fact, this condition is not necessary and the statement of Lemma 1 is valid for all even numbers $2n > 8$, which can be verified directly. Lemma 1 implies Lemma 2. **Lemma 2**: "Any even number $2n > 6$ is not divisible by all primes to the left of the center of its representation."

The condition used in the proof of Lemmas 1 and 2 can be weakened. Let 3, 5, 7, etc. up to gk, the prime numbers involved in the representation of an even number and located to the left of the center. Designate the product of these numbers as B; $B = 3 \cdot 5 \cdot 7 \cdot \ldots \cdot gk$. The smallest even number that is divisible by B is $A = 2B$. The center of the representation of the number A is the number B. If $gk > 3$, then $B/gk > 2$. Then, according to Bertrand's postulate, between B and gk there is a prime number $gk +1$ by which an even number A cannot be divided. We also have $Bgk +1 > A$. We add $gk +1$ to the other primes. We form the product $B1 = Bgk +1$. The smallest even number that is divisible by $B1$ is $A1 = 2B1$. The center of the representation of the number $A1$ is the number $B1$. If $gk +1 > 3$, then $B1/gk +1 > 2$. According to Bertrand's postulate, between $B1$ and $gk +1$ there is a prime number $gk +2$, by which the even number $A1$ cannot be divided. We also have $B1gk +2 > A1$. This process can be continued with the same result. This implies the validity of Lemmas 1 and 2. The smallest even number for which Lemmas 1 and 2 hold is 10. In this case,$B = 3 \cdot 5$; $A = 30$; $gk = 5$. By direct check, we verify that for the number 8 Lemma 2 is satisfied, but Lemma 1 is not valid. If we consider even numbers more than $2A$, for example, $4A$, $6A$, $8A$, etc., then the product of prime numbers located to the left of the center of the representation of these even numbers will be more than these numbers, since new prime numbers will appear to the left of the center. Indeed. The subsequent term of the series of even numbers $2A$, $4A$, $6A$, $8A$, etc., increases compared to the previous term by $(2t + 2)/2t$ times, therefore, no more than 2 times, where $t = 1, 2, 3$, etc., and the product of prime representation generators for neighboring numbers increases by more than 7 times, since $gk > 3$ in representation of A.

Thus, there is no even number $2n$ more than 6 that is divisible by all primes to the left of the center of its representation.

The number $2n1 + 2l0$ belongs to the group of index 0 as well as the number $2n0$, which is divisible by all prime representation generators, and one number can be obtained from another using the permissible transformation. We exclude from the representation of the number $2n1 + 2l0$ all pairs in which the number to the right of the center is prime, as well as the pair in which the first number is 1 (such a transformation is permissible), then the representation of the number $2n1 + 2l0$ will be equivalent to the representation of the number $2n0$. The irreducible representations of these numbers coincide (identical), since they have no one pair of conjugate numbers. In this case, only the identity transformation is permissible. Since an even number is completely determined by its representation, the numbers $2n1 + 2l0$ and $2n0$ are indistinguishable (identical). But according to Lemma 2, the number $2n0$ does not exist. Consequently, we can conclude, that such an even number $2n1 + 2l0$ for which the second assumption is valid does not exist. Therefore, **the second assumption** is incorrect and can be rejected. Thus, we have proven the induction transition. **The third assumption** is valid and the conjectureholds for all numbers in the considered interval. This implies the validity of Goldbach's conjecture for all even numbers.

III. Connection Of The Binary Hypothesis With The Legendre's Conjecture

Legendre's conjecture consists of the assertion that for any natural n in the interval between n^2 and $(n + 1)^2$ there is always a prime number. It is easy to see that when n changes, these intervals cover the positive part of the numerical axis, and the values of adjacent intervals relate to each other as odd numbers; so, we have $[(n + 1)^2 – n^2]/ [n^2 – (n – 1)^2] = (2n + 1)/(2n –1)$. Let us prove that the validity of this conjecture follows from the validity of Goldbach's binaryconjecture. The proof is carried out by induction. It is easy to verify that for $n = 1$ the Legendre conjecture is valid, since the interval (1, 4) contains prime numbers 2 and 3. Suppose that for $n = p$ the hypothesis is valid, i.e. the interval $(p^2, (p + 1)^2)$ contains a prime number. We put $n =p + 1$; we prove that the interval $((p + 1)^2, (p + 2)^2)$ contains a prime number. Suppose the contrary. For definiteness, let p be an odd number, then $(p + 1)$ is an even number, and $(p + 2)$ is an odd number. Consider the interval $(p^2, (p + 2)^2)$. We extend the interval by decreasing the left boundary by 2: $(p^2 − 2, (p + 2)^2)$. The center of this new interval is $(p + 1)^2 = (p^2 − 2 + (p + 2)^2)/2$.However, at the same time $(p + 1)^2$ is the center of the representation when the number $2(p + 1)^2$ is represented as the sum of two odd numbers. We get that there is a prime number on the left of this center, since for $n = p$ the Legendre conjecture is valid. *A fortiori* it is valid for the extended interval, but there are no prime numbers on the right of the center, that is, in the interval $((p + 1)^2, (p + 2)^2)$, since we assumed that the Legendre conjecture is not satisfied for $n = p + 1$. Then for the number $2(p + 1)^2$ there is no representation as a sum of two prime conjugate numbers, which contradicts Goldbach's binary hypothesis, the validity of which is established. Hence, there is a prime number in the interval $((p + 1)^2, (p + 2)^2)$. The case when p is even, $p + 1$ is odd and $p + 2$ is an even number is considered similarly. The interval is given in the form $(p^2 −1, (p + 2)^2 − 1)$. Since the numbers p^2 and $(p + 2)^2$ are both even numbers, shifting the boundariesof the interval to the left by 1 does not affect the final result. The center of this interval is again$(p + 1)^2$, and at the same time it is the center of the representation when the number $2(p + 1)^2$ is represented as a sum of two odd numbers. Further arguments are the same as in the first case for odd p and $p + 2$.

IV. Conclusion

The proof of the Goldbach's binary conjecture is obtained by induction based on the analysis of the representation of even numbers in an arbitrary interval. Numbers for which the conjecture is not valid form a group with index 0. In the irreducible representation of numbers from the group with index 0, there are no pairs of conjugate numbers, so numbers from thisgroup are indistinguishable and they are identical to a number that does not exist (Lemma 2). The validity of the Goldbach's binary conjecture implies the validity of the Legendre hypothesis.

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