IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 21, Issue 1, Ser. 1 (Jan. – Feb. 2025), PP 04-29
www.iosrjournals.org

Laplace Transform And Laplace Decomposition Method For Of Order (A,1) Of Fractional Differential Difference Equations With Interval Conditions Linear And Nonlinear

Kamble Rajratna M.A, Kulkarni Pramod Rb*
Shri Vitthal Rukmini Arts, Commerce And Science College, Sawana, Maharashtra State445205, India.
Science College, Nanded, Maharashtra State, India.

Abstract:
This paper uses the Laplace transform and laplace decomposition method to solve fractional differential difference equations with interval conditions. To do this, we use the Caputo fractional derivative definition. we get laplace transform form the given fractional differential difference equation to Laplace form and taking an inverse Laplace transform, we get the solution of the fractional-differential difference equation. For the Laplace transform of a difference-type function. The equation becomes to a differential-difference equation if α=1. our proved result is valid for α=1

Keywords: *Differential-difference equation, caputo derivative, Laplace transform method,laplace decomposition method.*

Date of Submission: 16-01-2025 Date of Acceptance: 26-01-2025

introduction:

Differential-Difference Equations is given in [1]

Let u, v be a function is piecewise continuous and of exponential order, i.e laplace transform exists. Next, the equation for fractional differential-difference

$u_c{}^{\alpha}(t) - u(t-w) =$

$v(t)$ . . .(1)) of fractional differential order α, n-1<α≤n, n∈N, and difference order 1 with initial conditions of the form $u^{(k)}(0) = u_k,\ k = 0, 1, 2, \dots, n-1,\ u(t) = 0\ for\ t < 0, w > 0$has a unique solution,

$u(t) = \int_0^t \sum_{k=0}^{[\eta]} \frac{(\eta - wk)^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]} v(t-\eta)\ d\eta +$

$\int_0^t \sum_{k=0}^{[\eta]} \frac{(\eta - wk)^{\alpha k - 1}}{\Gamma(\alpha k)} \sum_{k=0}^{n-1} \frac{u_k (t-\eta)^k}{k!}\ d\eta$

w is any positive real number of fractional differential order $\alpha, n-1<\alpha\leq n, n\in N$ and difference order 1.

where $u^{(k)}(0)$ denotes the ordinary derivative of integer order k at t=0 of the function u, and $u_c{}^{\alpha}(t)$ denotes the Caputo fractional derivative of order α is given in [2]

With the initial conditions $x(t-1)=\psi(t), 0\leq t<1, x(0)=x_0$, Sugiyama S.[3] has illustrated the differential-difference equation

$x'(t)=f[t,x(t),x(t-1)], t\in[0,t_0]$ and given the existence of a continuous solution x(t) valid for $0\leq t\leq \min(t_0, K/M)$, supposing that $f(t,x,y)$ is bounded by M>0 and continuous in the region $|x-x_0|\leq K, |y-y_0|\leq K$. otherwise, the uniqueness of the solution x(t) , not ensured by the continuation of the function f is given in [3].

Review of literature:

Basic of difference Equation given in [4]

Basic of laplace transform is given in [5]

Linear differential difference

$$\frac{du(t)}{dt}+cu(t-w)=v(t), t>w, \ldots 1)$$

$$u(t)=a+bt, 0\leq t\leq w \ldots 2)$$

$$c\neq 0, a, b \text{ are real Constant}$$

, u(t),v(t) is a function of exponential order, and w is a positive constant i.e difference parameter.is given solution is given in [6]

Non linear differential difference Equation

$$\frac{du}{dt}=v(u(t-w), t>w$$

$$u(t)=a+bt, 0\leq t\leq w$$

Solution is given in[6]

Basic of differential equation is given in [7,8,9]

Fourier series and boundary value problem is given in [10]

Basic of fractional differential equation and fractional calculus is given in

[10,11,12,13,14,15,16,17]

Differential-Difference Equation:

Differential-Difference Equation: differential Difference Equation contains both the derivatives of an unknown function and also some of its derivatives at arguments that differ by a predetermined number of values. The order of the highest derivative in a differential-difference equation is its differential order, and the number of distinct arguments that appear in the differential-difference equation minus one is the difference order.

An ordinary differential-difference equation of differential order n and difference order m in its most general form. $F\left[t, u(t), u(t-w_1), \ldots, u(t-w_m), u'(t), u'(t-w_1), \ldots, u'(t-w_m), \ldots, u^{(n)}(t), u^{(n)}(t-w_1), \ldots, u^{(n)}(t-w_m)\right] = f(t)$

, the constants $w_1, w_2, \ldots, w_m$ are spans or retardations, and $t, w_i \in R, 1 \le i \le m$.

Or this equation is :

$\sum_{i=0}^{m} \sum_{j=0}^{m} a_{ij}(t) u^{(j)}(t-w_i) = f(t)$The (m+1)(n+1) functions aij(t) are defined on some interval [a,b] on the real t-axis,

An fractional differential-difference equation of differential order $n\alpha$ and difference order m in its most general form.

$F\left[t, u(t), u(t-w_1), \ldots, u(t-w_m), u^{\alpha}(t), u^{\alpha}(t-w_1), \ldots, u^{\alpha}(t-w_m), \ldots, u^{(\alpha n)}(t), u^{(\alpha n)}(t-w_1), \ldots, u^{(\alpha n)}(t-w_m)\right] = v(t)$

Here $t, w_i \in R, 1 \le i \le m, n \le \alpha < n+1$, the constants $w_1, w_2, \ldots, w_m$ are called as spans or retardations.

Or This equation is

$$\sum_{i=0}^{m} \sum_{j=0}^{m} a_{ij}(t) u^{(j)}(t-w_i) = v(t)$$

where the $(m+1)(n+1)$ functions $a_{ij}(t)$ are defined on some interval $[a,b]$of real t-axis.

2 Main Results

Linear fractional differential difference solution

$$\frac{d^{\alpha} u(t)}{dt} + cu(t-w) = v(t), t > w, \ldots 1)$$

$$u(t) = a + bt^{\alpha}, 0 \leq t \leq w \dots 2)$$

$$c \neq 0, a, b \text{ are real Constant}$$

, u(t),v(t) is a function of exponential order, and w is a positive constant i.e difference parameter.

Laplace Transform method

Method 1:

Multipy both sides by e^{-st} , $s > 1$ to and integrate between w to ∞.

And

$$\int_0^w \frac{d^{\alpha}u(t)}{dt} e^{-st} dt = \int_0^w b\alpha e^{-st} dt = b\alpha[\frac{e^{-ws}}{-s} + \frac{1}{s}]$$

$$\int_0^{\infty} \frac{d^{\alpha}u(t)}{dt} e^{-st} dt = L\left[\frac{d^{\alpha}u(t)}{dt}\right] + \frac{b\alpha}{s}[1 - e^{-ws}]$$

$$\int_w^{\infty} \frac{d^{\alpha}u(t)}{dt} e^{-st} dt + c\int_w^{\infty} u(t-w)e^{-st}\, dt = \int_w^{\infty} v(t)e^{-st} dt$$

Applying laplace transform to 1,

$$\int_0^{\infty} \frac{d^{\alpha}u(t)}{dt} e^{-st} dt - \frac{b\alpha}{s}[1 - e^{-ws}] + ce^{-ws}\int_0^{\infty} u(t)e^{-st}\, dt$$

$$= e^{-ws}\int_0^{\infty} v(t+w)e^{-st} dt$$

$$\int_0^{\infty} \frac{d^{\alpha}u(t)}{dt} e^{-st} dt + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws}\int_0^{\infty} u(t)e^{-st}\, dt$$

$$= e^{-ws}\int_0^{\infty} v(t+w)e^{-st} dt$$

$$L\left[\frac{d^{\alpha}u(t)}{dt}\right] + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws}L[u(t)] = e^{-ws}L[v(t+w)]$$

caputo derivative of fractional order α laplace transform

$$s^{\alpha}L[u(t)] - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, u^{(k)}(0)]$$

$$s^{\alpha}L[u(t)]-\sum_{k=0}^{n-1}s^{\alpha-k-1}u^{(k)}(0)+\frac{b\alpha}{s}[e^{-ws}-1]+ce^{-ws}L[u(t)]$$
$$=e^{-ws}L[v(t+w)]$$
$$s^{\alpha}L[u(t)]-s^{\alpha-1}a+\frac{b\alpha}{s}[e^{-ws}-1]+ce^{-ws}L[u(t)]=e^{-ws}L[v(t+w)]$$

$$\{s^{\alpha}+ce^{-ws})\}L[u(t)]-s^{\alpha-1}a+\tfrac{b\alpha}{s}[e^{-ws}-1]=e^{-ws}L[v(t+w)]$$
$$(1+\frac{ce^{-ws}}{s^{\alpha}})\}L[u(t)]-s^{-1}a+\frac{b\alpha}{s^{\alpha+1}}[e^{-ws}-1]=\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]$$
$$(1+\frac{ce^{-ws}}{s^{\alpha}})\}L[u(t)]=\frac{a}{s}-\frac{b\alpha}{s^{\alpha+1}}[e^{-ws}-1]+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]$$
$$L[u(t)]=\{\frac{a}{s}+-\frac{b\alpha}{s^{\alpha+1}}[e^{-ws}-1]+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\}(1+\frac{ce^{-ws}}{s^{\alpha}})\}^{-1}$$
$$L[u(t)]=\left\{\frac{a}{s}-\frac{b\alpha}{s^{\alpha+1}}[e^{-ws}-1]+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\right\}\{(1-\frac{ce^{-ws}}{s^{\alpha}}+\frac{c^2e^{-2ws}}{s^{2\alpha}}$$
$$-+\cdots+(-1)^n\frac{c^ne^{-nws}}{s^{n\alpha}}+\cdots\}$$
$$L[u(t)]=\left\{\frac{a}{s}-\frac{b\alpha e^{-ws}}{s^{\alpha+1}}+\frac{b\alpha}{s^{\alpha+1}}+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\right\}-\frac{ce^{-ws}}{s^{\alpha}}\{\frac{a}{s}-\frac{b\alpha e^{-ws}}{s^{\alpha+1}}$$
$$+\frac{b\alpha}{s^{\alpha+1}}+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\}$$
$$+\frac{c^2e^{-2ws}}{s^{2\alpha}}\left\{\frac{a}{s}-\frac{b\alpha e^{-ws}}{s^{\alpha+1}}+\frac{b\alpha}{s^{\alpha+1}}+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\right\}+\cdots$$
$$+(-1)^n\frac{c^ne^{-nws}}{s^{n\alpha}}\left\{\frac{a}{s}-\frac{b\alpha e^{-ws}}{s^{\alpha+1}}+\frac{b\alpha}{s^{\alpha+1}}+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\right\}$$

u(t) is inversei.e Laplace inverse

Laplace inverse of

$$\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]$$

Let $L^{-1}[V(t+w)]=v(t)$

Laplace inverse $L^{-1}[\frac{e^{-ws}}{s^{\alpha}}\}]=\frac{(t-w)^{\alpha-1}}{(\alpha-1)!}=p(t)$

And using convolution theorem

$$\int_0^t p(\eta)v(t-\eta)\,d\eta$$

$$\int_0^t \frac{(\eta-w)^{\alpha-1}}{(\alpha-1)!}v(t-\eta)d\eta$$

We have to find inverse Laplace transform of

$(-1)^n \frac{c^n e^{-nws}}{s^{n\alpha}}\left\{\frac{a}{s}-\frac{b\alpha e^{-ws}}{s^{\alpha+1}}+\frac{b\alpha}{s^{\alpha+1}}+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\right\}$

Since the inverse Laplace transform of $(-1)^n \frac{c^n e^{-nws}}{s^{n\alpha}} =$

$(-1)^n \frac{(t-nw)^{n\alpha-1}}{(n\alpha-1)!}\ for\ t > nw$

And the inverse laplace of

$\{\left\{\frac{a}{s}-\frac{b\alpha e^{-ws}}{s^{\alpha+1}}+\frac{b\alpha}{s^{\alpha+1}}+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\right\}\} = a+\frac{b\alpha t^{\alpha}}{\alpha!}-\frac{b\alpha(t-w)^{\alpha}}{\alpha!}+$
$\int_0^t \frac{(\eta-w)^{\alpha-1}}{(\alpha-1)!}v(t-\eta)d\eta$

Therefore by convolution theorem

Laplace inverse of $(-1)^n \frac{c^n e^{-nws}}{s^{n\alpha}}\left\{\frac{a}{s}-\frac{b\alpha e^{-ws}}{s^{\alpha+1}}+\frac{b\alpha}{s^{\alpha+1}}+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\right\}$

$$\int_0^{\mu}\frac{(-1)^n(t-nw-\mu)^{n\alpha-1}}{(n\alpha-1)!}\{a+\frac{b\alpha u^{\alpha}}{\alpha!}-\frac{b\alpha(u-w)^{\alpha}}{\alpha!}+\int_0^t\frac{(\eta-w)^{\alpha-1}}{(\alpha-1)!}v(u-\eta)d\eta\}d\mu$$

Therefore

$u(t) = a+\frac{b\alpha t^{\alpha}}{\alpha!}-\frac{b\alpha(t-w)^{\alpha}}{\alpha!}+\int_0^t \frac{(\eta-w)^{\alpha-1}}{(\alpha-1)!}v(t-\eta)d\eta+\ldots..+$

$$\int_0^{\mu}\frac{(-1)^n(t-nw-\mu)^{n\alpha-1}}{(n\alpha-1)!}\{a+\frac{b\alpha u^{\alpha}}{\alpha!}-\frac{b\alpha(u-w)^{\alpha}}{\alpha!}+\int_0^t\frac{(\eta-w)^{\alpha-1}}{(\alpha-1)!}v(u-\eta)d\eta\}d\mu+\cdots\ldots$$

Method 2:

Multipy both sides by e^{-st} , $s > 1$ to and integrate between w to ∞ to obtain

Note that $\int_0^w \frac{d^\alpha u(t)}{dt} e^{-st} dt = \int_0^w b e^{-st} dt = b[\frac{e^{-ws}}{-s} + \frac{1}{s}]$

$$\int_0^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt = L\left[\frac{d^\alpha u(t)}{dt}\right] + \frac{b}{s}[1 - e^{-ws}]$$

$$\int_w^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt + c\int_w^\infty u(t-w) e^{-st}\, dt = \int_w^\infty v(t) e^{-st} dt$$

Applying laplace transform ,

$$\int_0^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt - \frac{b\alpha}{s}[1 - e^{-ws}] + ce^{-ws}\int_0^\infty u(t) e^{-st}\, dt$$

$$= e^{-ws}\int_0^\infty v(t+w) e^{-st} dt$$

$$\int_0^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws}\int_0^\infty u(t) e^{-st}\, dt$$

$$= e^{-ws}\int_0^\infty v(t+w) e^{-st} dt$$

$$L\left[\frac{d^\alpha u(t)}{dt}\right] + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws} L[u(t)] = e^{-ws} L[v(t+w)]$$

Caputo fractional derivative Laplace transform formula

$$s^\alpha L[u(t)] - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, u^{(k)}(0)]$$

$$s^\alpha L[u(t)] - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, u^{(k)}(0) + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws} L[u(t)]$$

$$= e^{-ws} L[v(t+w)]$$

$$s^\alpha L[u(t)] - s^{\alpha-1} a + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws} L[u(t)] = e^{-ws} L[v(t+w)]$$

$$\{s^\alpha + ce^{-ws})\} L[u(t)] - s^{\alpha-1} a + \frac{b\alpha}{s}[e^{-ws} - 1] = e^{-ws} L[v(t+w)]$$

Dividing

$$(1 + \frac{ce^{-ws}}{s^\alpha})\} L[u(t)] - s^{-1} a + \frac{b\alpha}{s^{\alpha+1}}[e^{-ws} - 1] = \frac{e^{-ws}}{s^\alpha} L[v(t+w)]$$

$$(1+\frac{ce^{-ws}}{s^{\alpha}})\}L[u(t)]=\frac{a}{s}-\frac{b\alpha}{s^{\alpha+1}}[e^{-ws}-1]+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]$$

$$L[u(t)]=\{\frac{a}{s}-\frac{b\alpha}{s^{\alpha+1}}[e^{-ws}-1]+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)]\}(1+\frac{ce^{-ws}}{s^{\alpha}})\}^{-1}$$

$$L[u(t)]=\{\frac{a}{s}-\frac{b\alpha}{s^{\alpha+1}}[e^{-ws}-1]+\frac{e^{-ws}}{s^{\alpha}}L[v(t+w)](1-\frac{ce^{-ws}}{s^{\alpha}}+\frac{c^{2}e^{-2ws}}{s^{2\alpha}}$$

$$-+\cdots+(-1)^{n}\frac{c^{n}e^{-nws}}{s^{n\alpha}}+\cdots\}$$

$$\left[\frac{a}{s}+\frac{b\alpha}{s^{\alpha+1}}\right]+e^{-ws}[-\frac{b\alpha}{s^{\alpha+1}}-\frac{ace^{-ws}}{s^{\alpha+1}}+\frac{1}{s^{\alpha}}L[v(t+w)]+$$

$$\frac{-c}{s^{\alpha}}\left[-\frac{ace^{-ws}}{s^{\alpha+1}}-\frac{cb\alpha}{s^{2\alpha+1}}+\frac{1}{s^{\alpha}}L[v(t+w)]\right]e^{-2ws}+\cdots+\frac{(-1)^{n}}{s^{(n-1)\alpha}}[-\frac{ace^{-ws}}{s^{\alpha+1}}-\frac{cb\alpha}{s^{2\alpha+1}}+$$

$$\frac{1}{s^{\alpha}}L[v(t+w)]e^{-nws}$$

$$\sum_{n=0}^{\infty}L\big(U_{n}(t)\big).e^{-nws}$$

The the inverse Laplace transform
We get

$$u(t)=\sum_{n=0}^{\infty}u_{n}(t-nw)e(t-nw)$$

Where $e(t-nw)$ is function defined as

$$e(t-nw)=\begin{Bmatrix}0,t<nw\\1,t>nw\end{Bmatrix}$$

Hence the Exact solution for each interval is given by

$u(t)=\sum_{n=0}^{\infty}u_{n}(t-nw),\ Nw\leq t\leq(N+1)w$.

4.4Laplace decomposition method:

$$\frac{d^{\alpha}u(t)}{dt}+cu(t-w)=v(t),t>w,\ldots1)$$

$$u(t)=a+bt^{\alpha},0\leq t\leq w\ldots2)$$

$$c\neq0,a,b\ are\ real\ Constant$$

u(t) is a given function of Exponential order and w is a positive Constant and called difference parameter.

Multipy both sides by e^{-st} , $s > 1$ to and integrate between w to ∞ to obtain

Note that $\int_0^w \frac{d^\alpha u(t)}{dt} e^{-st} dt = \int_0^w b e^{-st} dt = b[\frac{e^{-ws}}{-s} + \frac{1}{s}]$

$$\int_0^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt = L\left[\frac{d^\alpha u(t)}{dt}\right] + \frac{b}{s}[1 - e^{-ws}]$$

$$\int_w^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt + c\int_w^\infty u(t-w) e^{-st}\, dt = \int_w^\infty v(t) e^{-st} dt$$

Applying laplace transform and changing variable t-w to t we get,

$$\int_0^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt - \frac{b\alpha}{s}[1 - e^{-ws}] + ce^{-ws}\int_0^\infty u(t) e^{-st}\, dt$$

$$= e^{-ws}\int_0^\infty v(t+w) e^{-st} dt$$

$$\int_0^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws}\int_0^\infty u(t) e^{-st}\, dt$$

$$= e^{-ws}\int_0^\infty v(t+w) e^{-st} dt$$

$$L\left[\frac{d^\alpha u(t)}{dt}\right] + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws} L[u(t)] = e^{-ws} L[v(t+w)]$$

Caputo fractional derivative Laplace transform formula

$$s^\alpha L[u(t)] - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, u^{(k)}(0)]$$

$$s^\alpha L[u(t)] - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, u^{(k)}(0) + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws} L[u(t)]$$

$$= e^{-ws} L[v(t+w)]$$

$$s^\alpha L[u(t)] - s^{\alpha-1} a + \frac{b\alpha}{s}[e^{-ws} - 1] + ce^{-ws} L[u(t)] = e^{-ws} L[v(t+w)]$$

$$L[u(t)] - s^{-1} a + \frac{b\alpha}{s^{\alpha+1}}[e^{-ws} - 1] + \frac{1}{s^\alpha} ce^{-ws} L[u(t)] = \frac{1}{s^\alpha} e^{-ws} L[v(t+w)]$$

$$L[u(t)] = s^{-1} a - \frac{b\alpha}{s^{\alpha+1}}[e^{-ws} - 1] - \frac{1}{s^\alpha} ce^{-ws} L[u(t)] + \frac{1}{s^\alpha} e^{-ws} L[v(t+w)]$$

Now

$Lu(t) = \sum_{n=0}^{\infty} e^{-nws} L\{u_n(t)\}$….(1)

Now the main theme of laplace decomposition is to set iteration as follows

$\sum_{n=0}^{\infty} e^{-nws} L\{u_n(t)\} = s^{-1}a - \frac{b\alpha}{s^{\alpha+1}}[e^{-ws} - 1] - \frac{1}{s^{\alpha}} c e^{-ws} L[u_0(t)] - \frac{1}{s^{\alpha}} c \sum_{n=2}^{\infty} e^{-nws} L\{u_{n-1}(t)\} + \frac{1}{s^{\alpha}} e^{-ws} L[v(t+w)]$…(2)

For n=0,1,2,..

Equate the coefficient of e^{-nws} on both sides

$$L\{u_0(t)\} = \frac{a}{s} + \frac{b\alpha}{s^{\alpha+1}}$$

$$L\{u_1(t)\} = -\frac{b\alpha}{s^{\alpha+1}} - \frac{1}{s^{\alpha}} c L[u_0(t)] + \frac{1}{s^{\alpha}} L[v(t+w)]$$

$$L\{u_n(t)\} = -\frac{1}{s^{\alpha}} c \sum_{n=2}^{\infty} L\{u_{n-1}(t)\}, n \geq 2$$

By applying inverse laplace transform to equation (1)

We get

$$u(t) = \sum_{n=0}^{\infty} u_n(t - nw) e(t - nw)$$

Where $e(t - nw)$ is unit step function

$$e(t - nw) = \begin{Bmatrix} 0, t < nw \\ 1, t > nw \end{Bmatrix}$$

Hence the Exact solution for each interval is given by

$u(t) = \sum_{n=0}^{\infty} u_n(t - nw),\ Nw \leq t \leq (N+1)w$

Applications:

Lapace transform method

Example 2.1:

$$\frac{d^{\alpha}u(t)}{dt} - u(t - w) = 1, t > w, \ldots 1)$$

$$u(t) = 1 \quad ,0 \leq t \leq w \ldots 2)$$

Multipy both sides by e^{-st} , $s > 1$ to and integrate between w to ∞ to obtain

$$\int_0^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt - e^{-ws} \int_0^\infty u(t) e^{-st}\, dt = e^{-ws} \int_0^\infty e^{-st} dt$$

$$\int_0^\infty \frac{d^\alpha u(t)}{dt} e^{-st} dt - \int_0^\infty u(t) e^{-st}\, dt = \frac{e^{-ws}}{s}$$

$$L\left[\frac{d^\alpha u(t)}{dt}\right] - e^{-ws} L[u(t)] = [\frac{e^{-ws}}{s}]$$

caputo fractional derivative laplace transform

$$s^\alpha L[u(t)] - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, u^{(k)}(0)]$$

Since u(t)=1

$$s^\alpha L[u(t)] - s^{\alpha-1} - e^{-ws} L[u(t)] = \frac{e^{-ws}}{s}$$

$$(s^\alpha - e^{-ws}) L[u(t)] - s^{\alpha-1} = \frac{e^{-ws}}{s}$$

$$(1 - \frac{e^{-ws}}{s^\alpha}) L[u(t)] - s^{-1} = \frac{e^{-ws}}{s^{\alpha+1}}$$

$$\left(1 - \frac{e^{-ws}}{s^\alpha}\right) L[u(t)] = \frac{1}{s} + \frac{e^{-ws}}{s^{\alpha+1}}$$

$$L[u(t)] = [\frac{1}{s} + \frac{e^{-ws}}{s^{\alpha+1}}] \left(1 - \frac{e^{-ws}}{s^\alpha}\right)^{-1}$$

$$L[u(t)] = \left[\frac{1}{s} + \frac{e^{-ws}}{s^{\alpha+1}}\right] \{1 + \frac{e^{-ws}}{s^\alpha} + \frac{e^{-2ws}}{s^{2\alpha}} + \frac{e^{-3ws}}{s^{3\alpha}} + \cdots + \frac{e^{-nws}}{s^{n\alpha}} + \cdots$$

$$L[u(t)] = \frac{1}{s} + \frac{e^{-ws}}{s^{\alpha+1}} + \frac{e^{-ws}}{s^{\alpha+1}} + \frac{e^{-2ws}}{s^{2\alpha+1}} + \frac{e^{-2ws}}{s^{2\alpha+1}} + \frac{e^{-3ws}}{s^{3\alpha+1}} + \frac{e^{-3ws}}{s^{3\alpha+1}} - \frac{e^{-4ws}}{s^{4\alpha+1}} \dots + \frac{e^{-nws}}{s^{n\alpha}} + \cdots$$

$$L[u(t)] = \frac{1}{s} + 2\sum_{n=1}^{\infty} \frac{e^{-nws}}{s^{n\alpha+1}}$$

Applying the inverse Laplace Transform

$$u(t) = L^{-1}\left[\frac{1}{s}\right] + 2L^{-1}\left[\sum_{n=1}^{\infty} \frac{e^{-nws}}{s^{n\alpha+1}}\right]$$

$$u(t) = 1 + 2\left[\sum_{n=1}^{\infty} \frac{(t-nw)^{n\alpha}}{n\alpha!} e(t-nw)\right]$$

Whrere

$$e(t-nw) = \begin{Bmatrix} 0, t < nw \\ 1, t > nw \end{Bmatrix}$$

Is the Exact Solution

Let us note that

$$u(t) = 1, 0 \le t \le w$$

$$u(t) = 1 + 2(t-w)^{\alpha}, w \le t \le 2w$$

$$u(t) = 1 + 2(t-w)^{\alpha} + \frac{(t-2w)^{2\alpha}}{2\alpha!} \quad , 2\,w \le t \le 3w$$

$$u(t) = 1 + 2(t-w)^{\alpha} + \frac{(t-2w)^{2\alpha}}{2\alpha!} + \frac{(t-3w)^{3\alpha}}{3\alpha!} \,, 3w \le t \le 4w.$$

.

.

$$u(t) = 1 + 2(t-w)^{\alpha} + \sum_{n=2}^{n-1} \frac{(t-(n-1)w)^{(n-1)\alpha}}{(n-1)\alpha!} \quad , (N-1)w \le t \le Nw$$

$$u(t) = 1 + 2(t-w)^{\alpha} + \sum_{n=2}^{n-1} \frac{(t-(nw))^{n\alpha}}{(n\alpha)!} \quad , Nw \le t \le (N+1)w$$

and

$$u^{\alpha}(t) = 0, 0 \le t \le w$$

$$u^{\alpha}(t) = 2\alpha, \qquad w \le t \le 2w$$

$$u^{\alpha}(t) = 2\alpha + \frac{\alpha(t-2w)^{\alpha}}{\alpha!}, 2w \le t \le 3w$$

$$u^{\alpha}(t) = 2\alpha + \frac{\alpha(t-2w)^{\alpha}}{\alpha!} + , 3w \le t \le 4w.$$

$$.u^{\alpha}(t) = \sum_{n=1}^{N-1} \frac{2(n-1)\alpha(t-(n-1)w)^{(n-2)\alpha}}{(n-1)\alpha!}, (N-1)w \le t \le Nw$$

$$u^{\alpha}(t) = 2\alpha + \sum_{n=1}^{N} \frac{(t-nw)^{(n-1)\alpha}}{(n-1)\alpha!}, Nw \le t \le (N+1)w$$

We modify the interval to get continuity at t=w for $u^{\alpha}(t)$.

The Example is becomes

$$\frac{d^{\alpha}u(t)}{dt} - u(t-w) = 1, t > w, \dots 1$$
$$u(t) = 1 + 2t^{\alpha}, 0 \leq t \leq w$$

integrate between w and ∞ and multiply by $e^{-st}, s > 1$
The Caputo fractional derivative's Laplace transform is

$$s^{\alpha}L[u(t)] - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, u^{(k)}(0)]$$

$$\int_0^{\infty} \frac{d^{\alpha}u(t)}{dt} e^{-st} dt + \frac{2\alpha}{s}[e^{-ws} - 1] - e^{-ws}\int_0^{\infty} u(t)e^{-st} = \frac{e^{-ws}}{s}$$

$$L\left[\frac{d^{\alpha}u(t)}{dt}\right] + \frac{2\alpha}{s}[e^{-ws} - 1] - 1e^{-ws}L[u(t)] = \frac{e^{-ws}}{s}$$

$$s^{\alpha}L[u(t)] - s^{\alpha-1} + \frac{2\alpha}{s}[e^{-ws} - 1] - 1e^{-ws}L[u(t)] = \frac{e^{-ws}}{s}$$

$$(s^{\alpha} - e^{-ws})L[u(t)] - s^{\alpha-1} + \frac{2\alpha}{s}[e^{-ws} - 1] = \frac{e^{-ws}}{s}$$

$$(1 - \frac{e^{-ws}}{s^{\alpha}})L[u(t)] - s^{-1} + \frac{2\alpha}{s^{\alpha+1}}[e^{-ws} - 1] = \frac{e^{-ws}}{s^{\alpha+1}}$$

$$\left(1 - \frac{e^{-ws}}{s^{\alpha}}\right)L[u(t)] = s^{-1} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{2\alpha}{s^{\alpha+1}}\right] + \frac{e^{-ws}}{s^{\alpha+1}}$$

$$\left(1 - \frac{e^{-ws}}{s^{\alpha}}\right)L[u(t)] = \frac{1}{s} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{2\alpha}{s^{\alpha+1}}\right] + \frac{e^{-ws}}{s^{\alpha+1}}$$

$$L[u(t)] = \{\frac{1}{s} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{2\alpha}{s^{\alpha+1}}\right] + \frac{e^{-ws}}{s^{\alpha+1}}\}\left(1 - \frac{e^{-ws}}{s^{\alpha}}\right)^{-1}$$

$$L[u(t)] = \left\{\frac{1}{s} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{2\alpha}{s^{\alpha+1}}\right] + \frac{e^{-ws}}{s^{\alpha+1}}\right\}\{1 + \frac{e^{-ws}}{s^{\alpha}} + \frac{e^{-2ws}}{s^{2\alpha}} + \cdots + \frac{e^{-nws}}{s^{n\alpha}} + \cdots\}$$

$$L[u(t)] = \left\{\frac{1}{s} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{2\alpha}{s^{\alpha+1}}\right] + \frac{e^{-ws}}{s^{\alpha+1}}\right\} + \frac{e^{-ws}}{s^{\alpha+1}} - \frac{2\alpha e^{-2ws}}{s^{2\alpha+1}} + \left[\frac{2\alpha e^{-ws}}{s^{2\alpha+1}}\right]$$
$$+ \frac{e^{-2ws}}{s^{2\alpha+1}} + \frac{e^{-2ws}}{s^{2\alpha+1}} - \frac{2\alpha e^{-2ws}}{s^{3\alpha+1}} + \frac{2\alpha e^{-2ws}}{s^{3\alpha+1}} + \frac{e^{-3ws}}{s^{3\alpha+1}} + \cdots +$$

$$L[u(t)] = \left\{\frac{1}{s} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{2\alpha}{s^{\alpha+1}}\right] + \frac{e^{-ws}}{s^{\alpha+1}}\right\} + \frac{e^{-ws}}{s^{\alpha+1}} + \frac{e^{-2ws}}{s^{2\alpha+1}} + \frac{e^{-2ws}}{s^{2\alpha+1}} + \frac{e^{-3w}}{s^{3\alpha+}}$$
$$+ \cdots +$$

$$L[u(t)] = \left\{\frac{1}{s} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{2\alpha}{s^{\alpha+1}}\right] + \frac{e^{-ws}}{s^{\alpha+1}}\right\} + \frac{e^{-ws}}{s^{\alpha+1}} + \frac{e^{-2ws}}{s^{2\alpha+1}} + \frac{e^{-2ws}}{s^{2\alpha+1}} + \frac{e^{-3w}}{s^{3\alpha+}}$$
$$+ \cdots +$$

$$L[u(t)] = \frac{1}{s} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{2\alpha}{s^{\alpha+1}}\right] + 2\sum_{n=1}^{\infty} \frac{e^{-nws}}{s^{n\alpha+1}}$$

Taking inverse laplace transform

$$u(t) = 1 + 2\alpha\frac{t^\alpha}{(\alpha)!} - \frac{2\alpha(t-w)^\alpha}{\alpha!} + 2\sum_{n=1}^{\infty} \frac{(t-nw)^{n\alpha}}{(n\alpha)!} e(t-nw)$$

U(t)=

$$= \left\{1 + 2\alpha\frac{t^\alpha}{(\alpha)!} - \frac{2\alpha(t-w)^\alpha}{\alpha!} + \frac{(t-w)^\alpha}{\alpha!}\right\} + \frac{(t-w)^\alpha}{\alpha!} + \frac{2\alpha(t-2w)^{2\alpha}}{2\alpha!}$$
$$- \left[\frac{2\alpha(t-w)^{2\alpha}}{2\alpha!} 1\right] + \frac{(t-2w)^{2\alpha}}{2\alpha!} + \cdots +$$

$$= 1 + 2\alpha\frac{t^\alpha}{(\alpha)!} - \frac{2\alpha(t-w)^\alpha}{\alpha!} + \frac{(t-w)^\alpha}{\alpha!} + \frac{(t-w)^\alpha}{\alpha!} + \frac{(t-2w)^{2\alpha}}{2\alpha!}$$
$$+ \frac{(t-2w)^{2\alpha}}{2\alpha!} \dots +$$

Example 2.2

$$\frac{d^\alpha u(t)}{dt} - u(t-w) = 1, t > w, \dots 1$$

$$u(t) = 1 + t^\alpha, 0 \le t \le w$$

$$s^\alpha L[u(t)] - \sum_{k=0}^{n-1} s^{\alpha-k-1} u^{(k)}(0)]$$

$$\int_0^\infty \frac{d^\alpha u(t)}{dt} e^{-st}dt + \frac{\alpha}{s}[e^{-ws} - 1] - e^{-ws}\int_0^\infty u(t)e^{-st} = \frac{e^{-ws}}{s}$$

$$L\left[\frac{d^\alpha u(t)}{dt}\right] + \frac{\alpha}{s}[e^{-ws} - 1] - 1e^{-ws}L[u(t)] = \frac{e^{-ws}}{s}$$

$$s^\alpha L[u(t)] - s^{\alpha-1} + \frac{\alpha}{s}[e^{-ws} - 1] - 1e^{-ws}L[u(t)] = \frac{e^{-ws}}{s}$$

$$(s^\alpha - e^{-ws})L[u(t)] - s^{\alpha-1} + \frac{\alpha}{s}[e^{-ws} - 1] = \frac{e^{-ws}}{s}$$

$$(1 - \frac{e^{-ws}}{s^\alpha})L[u(t)] - s^{-1} + \frac{\alpha}{s^{\alpha+1}}[e^{-ws} - 1] = \frac{e^{-ws}}{s^{\alpha+1}}$$

$$\left(1 - \frac{e^{-ws}}{s^\alpha}\right)L[u(t)] = s^{-1} - \frac{\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{\alpha}{s^{\alpha+1}}1\right] + \frac{e^{-ws}}{s^{\alpha+1}}$$

$$\left(1 - \frac{e^{-ws}}{s^\alpha}\right)L[u(t)] = \frac{1}{s} - \frac{\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{\alpha}{s^{\alpha+1}}1\right] + \frac{e^{-ws}}{s^{\alpha+1}}$$

$$L[u(t)] = \{\frac{1}{s} - \frac{\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{\alpha}{s^{\alpha+1}}1\right] + \frac{e^{-ws}}{s^{\alpha+1}}\}\left(1 - \frac{e^{-ws}}{s^\alpha}\right)^{-1}$$

$$L[u(t)] = \left\{\frac{1}{s} - \frac{\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{\alpha}{s^{\alpha+1}}1\right] + \frac{e^{-ws}}{s^{\alpha+1}}\right\}\{1 + \frac{e^{-ws}}{s^\alpha} + \frac{e^{-2ws}}{s^{2\alpha}} + \cdots + \frac{e^{-nws}}{s^{n\alpha}} + \cdots\}$$

$$L[u(t)] = \frac{1}{s} - \frac{\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{\alpha}{s^{\alpha+1}}1\right] + \frac{e^{-ws}}{s^{\alpha+1}} + \frac{e^{-ws}}{s^{\alpha+1}} - \frac{\alpha e^{-2ws}}{s^{2\alpha+1}} + \frac{\alpha e^{-2ws}}{s^{2\alpha+1}} + \frac{e^{-2ws}}{s^{2\alpha+1}} + \cdots +..$$

$$L[u(t)] = \frac{1}{s} - \frac{\alpha e^{-ws}}{s^{\alpha+1}} + \left[\frac{\alpha}{s^{\alpha+1}}\right] + 2\sum_{n=1}^{\infty}\frac{e^{-nws}}{s^{n\alpha+1}}$$

Taking the inverse Laplace transform

$$u(t) = 1 + \frac{\alpha t^\alpha}{\alpha!} - \frac{\alpha(t-w)^\alpha}{\alpha!}e(t-w) + 2\sum_{n=1}^{\infty}\frac{(t-nw)^{n\alpha}}{(n\alpha)!}e(t-nw)$$

Example 2.3:

$$u'(t) - u(t-w) = 1, t > w$$

With initial interval condition

$$u(t) = 1 + 2t$$

Here $\alpha = 1, a = 1, b = 2, c = 1, k = o$

The solution is

$$\int_w^\infty \frac{du(t)}{dt} e^{-st} dt - \int_w^\infty u(t-w) e^{-st}\, dt = \int_w^\infty e^{-st} dt$$

$$\int_0^\infty \frac{du(t)}{dt} e^{-st} dt - \frac{2}{s}[1 - e^{-ws}] + e^{-ws} \int_0^\infty u(t) e^{-st}\, dt = \frac{e^{-ws}}{s}$$

Applying laplace transform ,

$$L\frac{du(t)}{dt} + \frac{2}{s}[e^{-ws} - 1] + e^{-ws} L[u(t)] = \frac{e^{-ws}}{s}$$

$$sL(u(t) - 1 + \frac{2}{s}[e^{-ws} - 1] + e^{-ws} L[u(t)] = \frac{e^{-ws}}{s}$$

$$(s - e^{-ws}) L(u(t) - 1 + \frac{2}{s}[e^{-ws} - 1] = \frac{e^{-ws}}{s}$$

$$(1 - \frac{e^{-ws}}{s}) L(u(t) - \frac{1}{s} - \frac{2}{s^2} + \frac{2e^{-ws}}{s^2} = \frac{e^{-ws}}{s^2}$$

$$(1 - \frac{e^{-ws}}{s}) L(u(t) = \frac{1}{s} + \frac{2}{s^2} - \frac{2e^{-ws}}{s^2} + \frac{e^{-ws}}{s^2}$$

$$L(u(t) = \{\frac{1}{s} + \frac{2}{s^2} - \frac{2e^{-ws}}{s^2} + \frac{e^{-ws}}{s^2}\}(1 - \frac{e^{-ws}}{s})^{-1}$$

$$L(u(t) = \{\frac{1}{s} + \frac{2}{s^2} - \frac{e^{-ws}}{s^2}\}\{1 + \frac{e^{-ws}}{s} + \frac{e^{-2ws}}{s^2} + \cdots + \frac{e^{-nws}}{s^n}$$

$$L(u(t) = \{\frac{1}{s} + \frac{2}{s^2} + 2\sum_{n=1}^{\infty} \frac{e^{-nws}}{s^{n+2}}$$

Taking inverse laplace transform

$u(t) = 1 + 2t + 2\sum_{n=1}^{N} \frac{(t-nw)^{n+1}}{(n+1)!} . e(t - nw), \quad Nw \le t \le (N+1)w, N =$

1,2,3

Laplace Decomposition Method:

Example

$$\frac{d^\alpha u}{dt} - u(t - w) = 1, \qquad t > w$$

$$u(t) = 1 + 2t^{\alpha}, \qquad t > w$$

$$\int_0^{\infty} \frac{d^{\alpha}u(t)}{dt} e^{-st} dt + \frac{2\alpha}{s}[e^{-ws} - 1] - e^{-ws} \int_0^{\infty} u(t) e^{-st} = \frac{e^{-ws}}{s}$$

$$s^{\alpha} L\big(u(t)\big) - 1s^{\alpha-1} + \frac{2\alpha e^{-ws}}{s} - \frac{2\alpha}{s} + e^{-ws} L(u(t) = \frac{e^{-ws}}{s}$$

$$L\big(u(t)\big) = \frac{1}{s} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \frac{2\alpha}{s^{\alpha+1}} + \frac{e^{-ws}}{s^{\alpha}} L\big(u_n(t)\big) + \frac{e^{-ws}}{s^{\alpha+1}}$$

$$\sum_{n=0}^{\infty} e^{-nws} L\{u_n(t)\}$$

$$= \frac{1}{s} + \frac{2\alpha}{s^{\alpha+1}} - \frac{2\alpha e^{-ws}}{s^{\alpha+1}} + \frac{e^{-ws}}{s^{\alpha+1}} + \frac{e^{-ws}}{s^{\alpha}} L\big(u_0(t)\big) + \sum_{n=2}^{\infty} \frac{e^{-nws} L(u_{n-1})}{s^{\alpha}}$$

$$\sum_{n=0}^{\infty} e^{-nws} L\{u_n(t)\}$$

$$= \frac{1}{s} + \frac{2\alpha}{s^{\alpha+1}} + \frac{(-2\alpha + 1)e^{-ws}}{s^{\alpha+1}} + \frac{e^{-ws}}{s^{\alpha}} L\big(u_0(t)\big) + \sum_{n=2}^{\infty} \frac{e^{-nws} L(u_{n-1})}{s^{\alpha}}$$

Taking

$$L(u(t)) = \sum_{n=0}^{\infty} e^{-nws} L\{u_n(t)\}$$

n=0

$$\frac{1}{s} + \frac{2\alpha}{s^{\alpha+1}}$$

n=1

$$\frac{(-2\alpha+1)}{s^{\alpha+1}} + \frac{1}{s^{\alpha}}\{\frac{1}{s} + \frac{2\alpha}{s^{\alpha+1}}\}$$

$$\frac{(-2\alpha + 1)}{s^{\alpha+1}} + \frac{1}{s^{\alpha+1}} + \frac{2\alpha}{s^{2\alpha+1}}$$

Ingeneral for $n \geq 2$

$$\frac{(-2\alpha+1)}{s^{\alpha+1}}+\frac{1}{s^{\alpha+1}}+\frac{1}{s^{\alpha}}\sum_{n=2}^{\infty}\frac{2\alpha}{s^{n+2}}$$

Therefore

$$L\left(u_n(t)\right)=$$
$$\frac{1}{s}+\frac{2\alpha}{s^{\alpha+1}}+\frac{(-2\alpha+1)}{s^{\alpha+1}}+\frac{1}{s^{\alpha+1}}+2\sum_{n=1}^{\infty}\frac{\alpha e^{-nws}}{s^{\alpha n+2}}$$

Therefore

$$u(t)=1+\frac{2\alpha t^{\alpha}}{\alpha!}+\frac{(-2\alpha+1)t^{\alpha}}{(\alpha)!}+\frac{t^{\alpha}}{\alpha!}+2\alpha\sum_{n=1}^{\infty}\frac{(t-nw)^{\alpha n+1}}{(\alpha n+1)!}$$

Put $\alpha=1$

$$u(t)=1+2t+2\sum_{n=1}^{\infty}\frac{(t-nw)^{n+1}}{(n+1)!}, Nw\leq t\leq (N+1)t$$

Non linear Fractional differential difference Equation

$$\frac{d^{\alpha}u}{dt}=v(u(t-w),t>w$$

$$u(t)=a+bt^{\alpha},0\leq t\leq w$$

Multiply by e^{-st} and integrate between w to ∞

$$\int_{w}^{\infty}\frac{d^{\alpha}u}{dt}=\int_{w}^{\infty}v(u(t-w))$$

Changing the variable

$$\int_{0}^{\infty}\frac{d^{\alpha}u}{dt}-\frac{b\alpha[1-e^{-ws}]}{s}=e^{-ws}\int_{0}^{\infty}v(u(t))$$

$$\int_{0}^{\infty}\frac{d^{\alpha}u}{dt}+\frac{b\alpha e^{-ws}}{s}-\frac{b\alpha}{s}=e^{-ws}\int_{0}^{\infty}v(u(t))$$

$$s^{\alpha}L[u(t)]-\sum_{k=0}^{n-1}s^{\alpha-k-1}u^{(k)}(0)]$$

$$s^{\alpha}L[u(t)] - s^{\alpha-1}a + \frac{b\alpha e^{-ws}}{s^{\alpha+1}} - \frac{b\alpha}{s^{\alpha+1}} = \frac{e^{-ws}L[v(u(t))]}{s^{\alpha}}$$

$$L[u(t)] = s^{-1}a - \frac{b\alpha e^{-ws}}{s^{\alpha+1}} + \frac{b\alpha}{s^{\alpha+1}} + \frac{e^{-ws}L[v(u(t))]}{s^{\alpha}}$$

The laplace decomposition for $L(u(t)) = \sum_{n=0}^{\infty} e^{-nws}L\{u_n(t)\}$

The decomposition of $L[v(u(t))]$ is given by $L[v(u(t))] = L\{v(u_0(t)) + e^{-ws}L\{\frac{\left([\frac{d}{du}v(u(t))\middle| u = u_0\right)}{1!}u_1(t) + e^{-2ws}L\left\{\frac{\left([\frac{d}{du}v(u(t))\middle| u = u_0\right)}{1!}u_2(t) + \frac{\left([\frac{d^2}{du^2}v(u(t))\middle| u = u_0\right)u_1{}^2(t)}{2!}\right\} + e^{-3ws}L\left\{\frac{\left([\frac{d}{du}v(u(t))\middle| u = u_0\right)}{1!}u_3(t) + \frac{\left([\frac{d^2}{du^2}v(u(t))\middle| u = u_0\right)2u_1(t)u_2(t)}{2!}\right\} + \frac{\left([\frac{d^3}{du^3}v(u(t))\middle| u = u_0\right)u_1{}^3(t)}{3!} + ..$

$$L(u(t)) = \sum_{n=0}^{\infty} e^{-nws}L\{A_n(t)\}$$

Which is the Laplace decomposition

$$A_0(t) = v(u_0(t))$$

$$A_1(t) = \{\frac{\left([\frac{d}{du}v(u(t))\middle| u = u_0\right)}{1!}u_1(t))$$

And $n \geq 2, A_n(t)$

$$A_n(t) = \frac{\left([\frac{d}{du}v(u(t))\middle| u = u_0\right)}{1!}u_n(t)) + [\sum_{k=2}^{n}\frac{\left([\frac{d^k}{du^k}v(u(t))\middle| u = u_0\right)}{3!}][\sum_{i_1+i_2+\cdots+i_k=n} u_{i_1}(t)u_{i_2}(t)\dots u_{i_n}(t)]$$

Now the main idea of laplace decomposion is to set

$$L\big(u(t)\big) = \sum_{n=0}^{\infty} e^{-nws} L\{u_n(t)\}$$

$$= s^{-1}a + \frac{b\alpha}{s^{\alpha+1}} + \left(-\frac{b\alpha}{s^{\alpha+1}} + \frac{L[A_0(t)]}{s^{\alpha}}\right)e^{-ws}$$

$$+ \frac{1}{s^{\alpha}} L \sum_{n=2}^{\infty} e^{-nws} L\{A_{n-1}(t)\}$$

$$L\{u_0(t)\} = \frac{a}{s} + \frac{b\alpha}{s^{\alpha+1}}$$

$$L\{u_1(t)\} = -\frac{b\alpha}{s^{\alpha+1}} + \frac{L[A_0(t)]}{s^{\alpha}}$$

$$L\{u_n(t)\} = \frac{L[A_{n-1}(t)]}{s^{\alpha}}$$

Therefore

$$u(t) = \sum_{n=0}^{\infty} u_n(t - nw) e(t - nw)$$

Where $e(t - nw)$ is unit step function

$$e(t - nw) = \begin{Bmatrix} 0, t < nw \\ 1, t > nw \end{Bmatrix}$$

Hence the Exact solution for each interval is given by

$$u(t) = \sum_{n=0}^{\infty} u_n(t - nw), \qquad Nw \leq t \leq (N+1)w$$

$$N = 0,1,2, \dots$$

Applications

Example 1

$$\frac{d^{\alpha}u}{dt} = u(t-w)(1 - u(t-w) \quad , t > w$$

With initial condition

$$u(t) = \frac{1}{10}, \qquad 0 \leq t \leq w$$

Ans

Taking the laplace transform

The given Equation is

$$\frac{d^\alpha u}{dt} = u(t-w) - u^2(t-w) \quad ,t > w$$

$$L\left(\frac{d^\alpha u}{dt}\right) = e^{-ws}L\big(u(t)\big) - e^{-ws}L(u^2(t))$$

$$s^\alpha L\big(u(t)\big) - s^{\alpha-1}\frac{1}{10} = e^{-ws}L\big(u(t)\big) - e^{-ws}L(u^2(t))$$

$$L\big(u(t)\big) - s^{-1}\frac{1}{10} = \frac{1}{s^\alpha}e^{-ws}L\big(u(t)\big) - \frac{1}{s^\alpha}e^{-ws}L(u^2(t))$$

$$L(u(t) = \frac{1}{s}\frac{1}{10} + \frac{e^{-ws}L\big(u(t)\big)}{s^\alpha} - \frac{e^{-ws}L(u^2(t))}{s^\alpha}$$

The laplace decomposition

$$L\big(u(t)\big) = \sum_{n=0}^{\infty} e^{-nws}L\{u_n(t)\}$$

Therefore

$$u(t) = \sum_{n=0}^{\infty} u_n(t-nw)e(t-nw)$$

Hence

$$u(t) = \sum_{n=0}^{N} u_n(t-nw), \quad Nw \le t \le (N+1)w, \quad N = 0,1,2$$

Now Laplace decomposition of

$$L[u^2(t)] = L[u_0{}^2] + e^{-s}L[2u_0u_1] + e^{-2s}L[2u_0u_2 + u_1{}^2]$$
$$+ e^{-3s}L[2u_0u_3 + 2u_1u_2] + \cdots$$
$$+ e^{-ns}L[u_0u_n + u_1u_{n-1} + \cdots + u_nu_1] + \cdots$$
$$= \sum_{n=0}^{\infty} e^{-ns}L\{A_n(t)\}$$

$A_i{}'$ is Adomian polynomial

$$A_0(t) = u_0{}^2(t)$$

$$A_n(t) = u_0^{\,2}(t)$$

$$\sum_{n=0}^{\infty} e^{-nws} L\{u_n(t)\}$$

$$= \frac{1}{s}\frac{1}{10} + \frac{1}{s^{\alpha}}\sum_{n=1}^{\infty} e^{-nws} L\{u_{n-1}(t)\} - \frac{1}{s^{\alpha}}\sum_{n=1}^{\infty} e^{-nws} L\{A_{n-1}(t)\}$$

Forn=0,1,2

n=0

$$L\{u_0(t)\} = \frac{1}{s}\frac{1}{10} \Rightarrow u_0(t) = \frac{1}{10}$$

n=1

$$L\{u_1(t)\} = \frac{1}{s^{\alpha}} L\{u_0(t)\} - \frac{1}{s^{\alpha}} L\{A_0(t)\}$$

$$L\{u_1(t)\} = \frac{1}{s^{\alpha}}\frac{1}{s}\frac{1}{10} - \frac{1}{s^{\alpha}} L\{u_0^{\,2}\}$$

$$L\{u_1(t)\} = \frac{1}{s^{\alpha}}\frac{1}{s}\frac{1}{10} - \frac{1}{s^{\alpha}}\frac{1}{100}\frac{1}{s}$$

$$L\{u_1(t)\} = \frac{9}{100 s^{\alpha+1}} \Rightarrow u_1(t) = \frac{9}{100}\frac{t^{\alpha}}{\alpha!}$$

n=3

$$L\{u_2(t)\} = \frac{1}{s^{\alpha}} L\{u_1(t)\} - \frac{1}{s^{\alpha}} L\{A_1(t)\}$$

$$L\{u_2(t)\} = \frac{1}{s^{\alpha}}\frac{9}{100 s^{\alpha+1}} - \frac{1}{s^{\alpha}} L\{2\frac{1}{10}\frac{9}{100}\frac{t^{\alpha}}{\alpha!}\}$$

$$L\{u_2(t)\} = \frac{1}{s^{\alpha}}\frac{9}{100 s^{\alpha+1}} - \frac{1}{s^{\alpha}}\frac{1}{s^{\alpha+1}}\{2\frac{1}{10}\frac{9}{100}\}$$

$$L\{u_2(t)\} = \frac{1}{s^{2\alpha+1}}\frac{72}{1000}\} \Rightarrow u_2(t) = \frac{72}{1000}\frac{t^{2\alpha}}{(2\alpha)!}$$

Therefore for $2w \leq t \leq 3w$

$$u(t) = \sum_{n=0}^{2} u_n(t - nw)$$

$$u(t) = \frac{1}{10} + \frac{9}{100}\frac{(t-w)^{\alpha}}{\alpha!} + \frac{72}{1000}\frac{(t-2w)^{2\alpha}}{(2\alpha)!}$$

For $\alpha = 1$

$$u(t) = \frac{1}{10} + \frac{9}{100}\frac{(t-w)}{1!} + \frac{72}{1000}\frac{(t-2w)^{2}}{(2)!}$$

So our proved Results is also valid for $\alpha = 1$

Letting $w \rightarrow 0$

$$u(t) = \frac{1}{10} + \frac{9}{100}\frac{(t)}{1!} + \frac{72}{1000}\frac{(t)^{2}}{(2)!}$$

Example 2)

$$\frac{d^{\alpha}u}{dt} = \sin\left(u(t-w)\right. \quad , t > w$$

With initial condition

$$u(t) = \frac{\pi}{2}, \quad 0 \leq t \leq w$$

Ans

Taking the laplace transform

The given Equation is

$$\frac{d^{\alpha}u}{dt} = \sin\left(u(t-w)\right) \quad , t > w$$

$$L\left(\frac{d^{\alpha}u}{dt}\right) = e^{-ws}L\big(sinu(t)\big)$$

$$s^{\alpha}L\big(u(t)\big) - s^{\alpha-1}\frac{\pi}{2} = e^{-ws}L\big(sinu(t)\big)$$

$$L\big(u(t)\big) - s^{-1}\frac{\pi}{2} = \frac{1}{s^{\alpha}}e^{-ws}L\big(sinu(t)\big)$$

$$L(u(t) = \frac{1}{s}\frac{\pi}{2} + \frac{e^{-ws}L\big(sinu(t)\big)}{s^{\alpha}}$$

The laplace decomposition

$$L\big(u(t)\big) = \sum_{n=0}^{\infty} e^{-nws} L\{u_n(t)\}$$

And $L[sinu(t)]$ at $u = u_0$ is as follows

$$L\big[\sin\big(u(t)\big)\big]$$
$$= L\big[\sin\big(u_0(t)\big)\big] + e^{-ws} L\big[u_1(t)\cos\big(u_0(t)\big)\big]$$
$$+ e^{-2ws} L\left\{u_2(t)\cos\big(u_0(t)\big) - \frac{1}{2}u_1{}^2(t)\sin\big(u_0(t)\big)\right\}$$
$$+ e^{-3ws} L\Big\{u_3(t)\cos\big(u_0(t)\big) - u_1(t)u_2(t)\sin\big(u_0(t)\big)$$
$$- \frac{1}{6}u_1{}^3(t) cos u_0(t)\Big\} + \cdots += \sum_{n=0}^{\infty} e^{-nws} L\{B_n(t)\}$$

Where Adomian polynomial $B_i{}'s$ given below

$$B_0(t) = \sin\big(u_0(t)\big)$$
$$B_1(t) = u_1(t)\cos\big(u_0(t)\big)$$
$$B_2(t) = \left\{u_2(t)\cos\big(u_0(t)\big) - \frac{1}{2}u_1{}^2(t)\sin\big(u_0(t)\big)\right\}$$
$$B_3(t) = \{u_3(t)\cos\big(u_0(t)\big) - u_1(t)u_2(t)\sin\big(u_0(t)\big) - \frac{1}{6}u_1{}^3(t) cos u_0(t)\}$$

And

$$\sum_{n=0}^{\infty} e^{-nws} L\{u_n(t)\} = \frac{1}{s}\frac{\pi}{2} + \frac{e^{-ws}}{s^{\alpha}} \sum_{n=0}^{\infty} e^{-nws} L\{B_n(t)\}$$

Comparing the coefficient on both sides

n=0 $L\{u_0(t)\} = \frac{1}{s}\frac{\pi}{2} \Rightarrow u_0(t) = \frac{\pi}{2}$

n=-1 $L\{u_1(t)\} = \frac{1}{s^{\alpha}} L\{B_0(t)\} = \frac{1}{s^{\alpha}} L\big[\sin\big(u_0(t)\big)\big] = \frac{1}{s^{\alpha}} L\left[\sin\left(\frac{\pi}{2}\right)\right] = \frac{1}{s^{\alpha}} L[1] =$

$L\left[\frac{1}{s^{\alpha+1}}\right] \Rightarrow u_1(t) = \frac{t^{\alpha}}{\alpha!}$

$$n=2, L\{u_2(t)\}=\frac{1}{s^\alpha}L\{B_1(t)\}=\frac{1}{s^\alpha}L\left[u_1(t)\cos\left(u_0(t)\right)\right]=\frac{1}{s^\alpha}L\left[\frac{t^\alpha}{\alpha!}\cos\frac{\pi}{2}\right]$$

$$=\frac{1}{s^\alpha}L[0]=L[0]\Rightarrow u_2(t)=0$$

$$n=3\ , L\{u_3(t)\}=\frac{1}{s^\alpha}L\{B_2(t)\}$$

$$=\frac{1}{s^\alpha}L\left[u_2(t)\cos\left(u_0(t)\right)-\frac{1}{2}{u_1}^2(t)\sin\left(u_0(t)\right)\right]$$

$$=\frac{1}{s^\alpha}L\left[0-\frac{1}{2}{u_1}^2(t)\right]=\frac{1}{s^\alpha}L\left[-\frac{1}{2}\frac{t^{2\alpha}}{(\alpha!)^2}\right]=\frac{1}{s^\alpha}L\left[\frac{1}{\alpha!}\frac{t^{2\alpha}}{2\alpha!}\right]$$

$$=L\left[\frac{1}{\alpha!}\frac{1}{s^\alpha}\frac{1}{s^{2\alpha+1}}\right]\Rightarrow u_3(t)=-\frac{1}{\alpha!}\frac{t^{3\alpha}}{(3\alpha)!}$$

$$n=4\ , L\{u_4(t)\}=\frac{1}{s^\alpha}L\{B_3(t)\}$$

$$=\frac{1}{s^\alpha}L\left[u_3(t)\cos\left(u_0(t)\right)-u_1(t)u_2(t)\sin\left(u_0(t)\right)\right.$$

$$\left.-\frac{1}{6}{u_1}^3(t)cos u_0(t)\right]=\frac{1}{s^\alpha}L[0]\Rightarrow u_4(t)=0$$

Therefore

$$u(t)=\sum_{n=0}^{\infty}u_n(t-nw)e(t-nw)$$

Hence

$$u(t)=\frac{\pi}{2}+\frac{(t-w)^\alpha e(t-w)}{\alpha!}-\frac{1}{\alpha!}\frac{(t-3w)^{3\alpha}e(t-3w)}{(3\alpha)!}+\cdots+$$

For $\alpha=1$

$$u(t)=\frac{\pi}{2}+\frac{(t-w)e(t-w)}{1!}-\frac{1}{1!}\frac{(t-3w)^3e(t-3w)}{(3)!}+\cdots+$$

So our proved Results is also valid for $\alpha=1$

Letting $w\to 0$

$$u(t)=\frac{\pi}{2}+\frac{t}{1!}-\frac{1}{1!}\frac{(t)^3}{(3)!}+\cdots+$$

3 Conclusions : so our proved method for fractional differential difference equation with given condition holds good for ordinary differential difference equation.

1) Bellman, R. and Cooke, K. L., *'Differential-Difference Equations'*, R-374-PR, A Report Prepared for United State Air Force Project Rand, 1963

2) Kamble at.al.'Solution of fractional oeder Differential-difference Equation by using Laplace transform method'journal of herbin Engineering University,volume 45 No.3 March 2024, ISSN1006-7043

3) Sugiyama, S., '*On The Existence and Uniqueness Theorems of Difference-Differential Equations'*, Waseda University, Tokyo, Japan

4) WALTER G. KELLEY, ALLAN C. PETERSON, 'Difference Equations An Introduction with Applications' Second edition, ACADEMIC PRESS

5) MURRAY R. SPIEGE,SCHAUM'S OUTLINE OF THEORY AND PROBLEMSOF LAPLACETRANSFORMS, McGRAW-HILL New York San Francisco Washingtun. D.C. Auckland Rogoid Caracas Lisbon London Madrid Alexien City ,Milan Aluntrcal New Delhi San Juan Singapore Sydney Ibkva "lomnro

6). "Laplace decomposition method for solving certain differential-difference equations both of order 1", Advanced Studies in Contemporary Mathematics, 23(3) (2013), 501-508 (with R. Rangarajan).

7) E. A. Coddington, An Introduction to Ordinary Differential Equation, Prentice-Hall of India Pvt. Ltd., New Delhi.

8)G. F. Simmons, Differential Equations with Applications and Historical Notes, (2nd edition) Mc Graw Hill Book Co.

9) G.Birkhoff and G.C.Rota, Ordinary Differential Equations, John Wiley and Sons.

10) R.V. Churchill and J. Brown.: "Fourier Series and Boundary Value Problems" (7th edition)(Publisher: McGraw-Hill Book Company)

[11] I.Podlubny,Fractional Differential Equations, Academic Press,New York, 1993.

12) K. Oldham, J. Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitrary Order, Academic Press, USA, 1974.

13) A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, in: Math. Studies., North-Holland, New York, 2006. 14)

14) Podlubny, I., *'Fractional Differential Equations'*, Academic Press, New York, USA 1999.

15) .Miller, K. S. and Ross, B., *'An Introduction to the Fractional Calculus and Fractional Differential Equations'*, Wiley-Interscience Publications, USA, 1993.

[16] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[17] Y. Zhou, Basic Theory of Fractional Differential Equations, Xiangtan University, China, 2014.