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Graphs With Representation Number 3

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ABSTRACT. Let $G = (V(G), E(G))$ be a graph with n vertices $a_1, a_2, \ldots, a_n$. The graph G is word representable if and only if there is a word w in which letters correspond to vertices of G and $(a_i, a_j) \in E(G)$ for each $a_i \neq a_j, (1 \leq i < j \leq n)$ if and only if a_i and a_j alternate in w. A graph G is word representable then it is l-word representable for some l (a word containing l copies of each letter). The minimum l for which a graph G is l-word-representable is called the representation number of G. In this paper, we study the word representation of Book graph, Stacked Book graph along with its representation number.

Keywords: word-representable graph, book graph, representation number.

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1. INTRODUCTION

The theory of word representable graph was introduced in 2004 and many interesting results has been discussed in recent years by S. Kitev [6]. Since graph representation corresponds to any word w exists, but not the case that every graph is word-representable [6]. For example, the wheel graph $W_{2n+1}, n \geq 2$ is non-word representable [6]. using semi-transitive orientation, one can easily determine whether a graph is word representable or not.

Definition 1.1. *Let w be a word of length n. The letters x and y alternate in w if we obtain either a word $xyxy$... (of even or odd length) or a word $yxyx$... (of even or odd length) after deleting all other letters in w except the copies of x and y. A Graph $G(V(G), E(G))$ represents a word w is defined as follows, the vertex set $V(G)$ are the letters of w and an edge $xy \in E(G)$ iff the letters x and y are alternate in w.*

Example 1.1. *The graph of the word $w = 12312434$ is given below in figure 1.*

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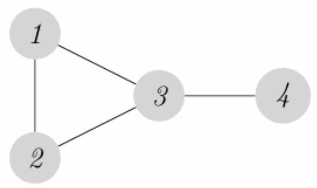

FIGURE 1. Graph of $w = 12312434$

Example 1.2. *The word-reprsentation of tree given in the following figure 2 is as follows:*
step 1: 1212
Step 2: 123132
step 3: 12341432
step 4: 1234515432
step 5: 123451564632
step 6: 12345156747632
Therefore, $w = 12345156747632$.

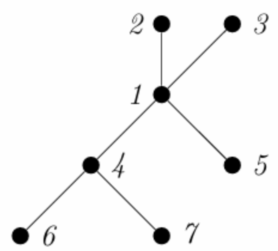

FIGURE 2. Tree

Definition 1.2. *A word is l-uniform if each letter in w occurs l times.*

Example 1.3. *12312434 is a 2-uniform word, while 2341 is a 1-uniform word.*

Definition 1.3. *A graph is l-word-representable or k-representable if there exists a l-uniform word representing it.*

Definition 1.4. *Graph's representation number is the least l such that the graph is l-representable and it denoted by $R(G)$. A complete graph k_n is represented by the word $123\ldots n$. Therefore, $R(k_n) = 1$*

The following table gives us the word representation and its representation number of some class of graphs.

S No	Graph	Word Representation	Representation number
1	Path Graph	1213243545. . . n(n-1)n	2
2	Cycle Graph	1n213243545. . . n(n-1)	2
3	Empty Graph	1234. . . (n-1)nn(n-1). . . 4321	2
4	star Graph	1234. . . (n-1)n1n(n-1). . . 432	2

Definition 1.5. *Let $G = (V, E)$ is semi-transitive if it admits an acyclic orientation such that for any directed path $v_1 \to v_2 \to \ldots \to v_k$ with $v_i \in V$ for all $i, 1 \leq i \leq k$, either there is no edge $v_1 \to v_k$, or the edge $v_1 \to v_k$ is present and there are edges $v_i \to v_j$ for all $1 \leq i \leq j \leq k$.(In other words, the (acyclic) subgraph induced by the vertices $v_1, v_2, \ldots, v_k$ is transitive (with unique starting vertex v_1 and unique ending vertex v_k). Such an orientation is called semi-transitive orientation.*

Theorem 1.1. *A graph G is word-representable if and only if it admits semi-transitive orientation.*

Marc Glen et.al proved that, the crown graph $H_(n,n), n \geq 5$ is $\lceil n/2 \rceil$-representable which is bipartite having high representation number [3]. Section 2, contains a word representation of book graph B_m, a word construction expressing stacked book graph $B_{n,m}$ and its representation number.

2. Graph with Representation number 3

Book graphs play a significant role in graph theory and this is one of the complicated graphs to represent. Since book graphs are bipartite and bipartite graphs admits semi-transitive orientation. Therefore, book graphs are word-representable.

In this section, completely solve the representation and representation number of book graphs and stacked book graphs.

The m-book graph is defined as the Cartesian product $S_{m+1} \times P_2$, Where S_{m+1} is a star graph with m vertex positions and P_2 is a path graph with two vertex positions. The (m, n) - stacked book graph is a generalization of the book graph to n stacked pages.

Theorem 2.1. *Every Book graph B_m is 3 word-representable.*

Proof. The book graph $B_m, m \geq 2$ can be represented by the word w_m to be constructed below. we begin with the 3-uniform word $w_2 = a_2a_1b_2a_2b_1a_1b_2b_1a_2b_2a_1b_1$ which represents the cycle $a_2a_1b_1b_2$. Note that w_2 contains the factor $a_{21}a_{11}$ and $b_{22}b_{12}$. where a_i denotes the i-th occurrence of a letter a in the word. Add the path $a_1a_3b_3b_1$ to the cycle in w_2 using the third case in the proof of theorem (5.2.2 in [4]), namely the substitutions, $a_1^1 \rightarrow a_3b_3a_1a_3$ and $b_1^2 \rightarrow b_3b_1a_3b_3$ then, $w_3 = a_2a_3b_3a_1a_3b_2a_2b_1a_1b_2b_3b_1a_3b_3a_2b_2a_1b_1$. Generally, use the substitution $a_1^1 \rightarrow a_mb_ma_1a_m$ and $b_1^2 \rightarrow b_mb_1a_mb_m$ in w_{m-1}, we will obtain the word representation of book graph Bm. Hence every Book graph B_m is 3 word- representable.

m	Representation
2	$a_2a_3b_3a_1a_3b_2a_2b_1a_1b_2b_3b_1a_3b_3a_2b_2a_1b_1$
3	$a_2a_3b_3a_4b_4a_1a_4a_3b_2a_2b_1a_1b_2b_3b_4b_1a_4b_4a_3b_3a_2b_2a_1b_1$
4	$a_2a_3b_3a_4b_4a_5b_5a_1a_5a_4a_3b_2a_2b_1a_1b_2b_3b_4b_5b_1a_5b_5a_4b_4a_3b_3a_2b_2a_1b_1$

□

Corollary 2.1. *We have $R(B_m) \leq 3$, that is each book graph's representation number is at most 3. In the following two theorems we prove that each book graph's representation number is exactly 3.*

Theorem 2.2. *The representation number of book graph B_3 is 3.*

Proof. By theorem 2.1, it remains to show that B_3 is not 2-word representable. In our proof we refer the graph in figure 2. Suppose that B_3 is 2-word representable by a word w. By the definition of B_3, there are exactly 3 neighbors of a_1 and b_1. Let us try to find the word of B_3 using 2 copies of each vertex. Due to the symmetry and proposition (3.2.7 in [4]) we may assume that the word starts with a_1. Let the word w starts with $w = a_1a_1 = a_1b_1a_1b_1$. The following illustration provides that no word exists for B_3 using 2 copies of each vertex from w.

- $a_1b_1a_1b_1 \to a_2a_1a_2b_1a_1b_1 \to b_1a_2a_1a_2b_1a_1 \to b_2b_1a_2b_2a_1a_2b_1a_1 \to b_2b_1a_2b_2a_3a_1a_3a_2b_1a_1$. There is no factor a_3b_1 to insert b_3.
- $a_1b_1a_1b_1 \to a_2a_1a_2b_1a_1b_1 \to b_1a_2a_1a_2b_1a_1 \to b_2b_1a_2b_2a_1a_2b_1a_1 \to b_2b_1a_2b_2a_1a_2b_1a_3a_1a_3 \to b_2b_1a_2b_2a_1a_2b_3b_1a_3b_3a_1a_3 \to b_2b_1a_2b_2a_1a_2b_3b_1a_3b_3a_4a_1a_4a_3$. There is no factor a_4b_1 to insert b_4.
- $a_1b_1a_1b_1 \to a_2a_1a_2b_1a_1b_1 \to b_1a_2a_1a_2b_1a_1 \to b_2b_1a_2b_2a_1a_2b_1a_1 \to b_2b_1a_2b_2a_1a_2b_1a_3a_1a_3 \to b_2b_1a_2b_2a_1a_2b_3b_1a_3b_3a_1a_3 \to b_2b_1a_2b_2a_4a_1a_4a_2b_3b_1a_3b_3a_1a_3$. There is no factor a_4b_1 to insert b_4.
- $a_1b_1a_1b_1 \to a_2a_1a_2b_1a_1b_1 \to b_1a_2a_1a_2b_1a_1 \to b_1a_2a_1b_2a_2b_1b_2a_1 \to b_1a_2a_3a_1a_3b_2a_2b_1b_2a_1$. There is no factor a_3b_1 to insert b_3.
- $a_1b_1a_1b_1 \to a_2a_1a_2b_1a_1b_1 \to b_1a_2a_1a_2b_1a_1 \to b_1a_2a_1b_2a_2b_1b_2a_1 \to a_3b_1a_2a_1b_2a_2b_1b_2a_3a_1 \to a_3b_1a_2a_1b_2a_2b_1b_2a_3a_4a_1a_4$. There is no factor a_4b_1 to insert b_4.
- $a_1b_1a_1b_1 \to a_2a_1a_2b_1a_1b_1 \to b_1a_2a_1a_2b_1a_1 \to b_1a_2a_1b_2a_2b_1b_2a_1 \to a_3b_1a_2a_1b_2a_2b_1b_2a_3a_1 \to a_3b_1a_2a_4a_1a_4b_2a_2b_1b_2a_3a_1$. There is no factor a_4b_1 to insert b_4.
- $a_1b_1a_1b_1 \to a_1b_1a_2a_1a_2b_1 \to a_1b_2b_1a_2b_2a_1a_2b_1 \to a_3a_1a_3b_2b_1a_2b_2a_1a_2b_1 \to b_1a_3a_1a_3b_2b_1a_2b_2a_1a_2 \to b_3b_1a_3b_3a_1a_3b_2b_1a_2b_2a_1a_2 \to b_3b_1a_3b_3a_4a_1a_4a_2$. There is no factor a_4b_1 to insert b_4.
- $a_1b_1a_1b_1 \to a_1b_1a_2a_1a_2b_1 \to a_1b_2b_1a_2b_2a_1a_2b_1 \to a_3a_1a_3b_2b_1a_2b_2a_1a_2b_1 \to b_1a_3a_1a_3b_2b_1a_2b_2a_1a_2 \to b_3b_1a_3b_3a_1a_3b_2b_1a_2b_2a_1a_2 \to b_3b_1a_3b_3a_1a_3b_2b_1a_2b_2a_4a_1a_4a_2$. There is no factor a_4b_1 to insert b_4.
- $a_1b_1a_1b_1 \to a_1b_1a_2a_1a_2b_1 \to a_1b_2b_1a_2b_2a_1a_2b_1 \to a_1b_2b_1a_2b_2a_3a_1a_3a_2b_1$. There is no factor a_3b_1 to insert b_3.
- $a_1b_1a_1b_1 \to a_1b_1a_2a_1a_2b_1 \to a_1b_1a_2a_1b_2a_2b_1b_2 \to a_1b_2b_1a_2b_2a_3a_1a_3a_2b_1$. There is no factor a_3b_1 to insert b_3.
- $a_1b_1a_1b_1 \to a_1b_1a_2a_1a_2b_1 \to a_1b_1a_2a_1b_2a_2b_1b_2 \to a_3a_1a_3b_2b_1a_2b_2a_1a_2b_1 \to b_1a_3a_1a_3b_2b_1a_2b_2a_1a_2 \to b_1a_3a_1a_3b_2b_1a_2b_2a_1a_2 \to b_1a_3a_1a_3b_2b_1a_2b_2a_4a_1a_4a_2$. There is no factor a_4b_1 to insert b_4.

Also, if w represents G then the reverse of the word w also represents the same graph G. This implies we can conclude that there is no word exist using 2 copies of each vertex (using table 3 and 4) if w starts with $b_1(w = b_1b_1 = b_1a_1b_1a_1)$Therefore, the book graph B_3 is not 2-word representable and by theorem 2.1, $R(B_3) = 3$. □

Theorem 2.3. *For* $m \geq 3, R(B_m) = 3$

Proof. By theorem 2.1, it remains to show that for $m \geq 3, B_m$ is not 2-word representable. By theorem 2.2, the representation number of book graph B_3 is 3. Since every B_m contains the subgraph B_3. Therefore, B_m cannot be represented by 2-word representation. Hence $R(B_m) = 3$. □

Theorem 2.4. *Every stacked book graph* $B_{m,n}$ *is 3-word-representable.*

Proof. The stacked book graph $B_{m,n}$ with 3 stacks is presented figure 3,4,5.Let $w_{m,n}$ be the word representing $B_{m,n}$. if $n = 2$, then $B_{m,2} = B_m$. By theorem 2.3, $B_{m,2}$ is 3-word-representable. Let us derive the word $w_{2,3}$ from $w_{2,2}$.

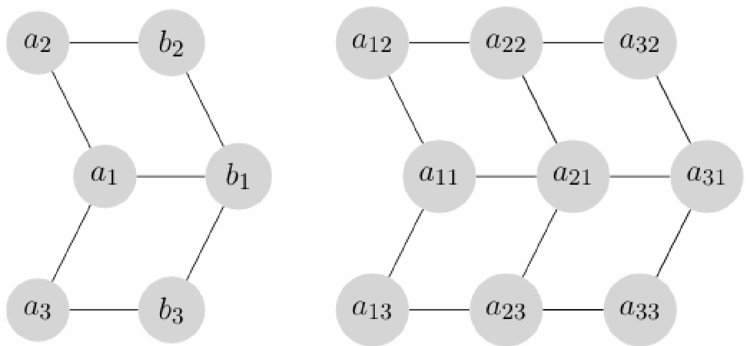

FIGURE 3. Book Graph $B_2, B_{2,3}$

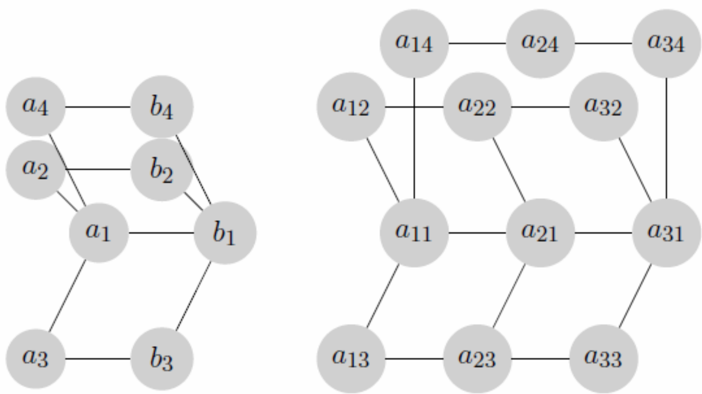

FIGURE 4. Book Graph $B_3, B_{3,3}$

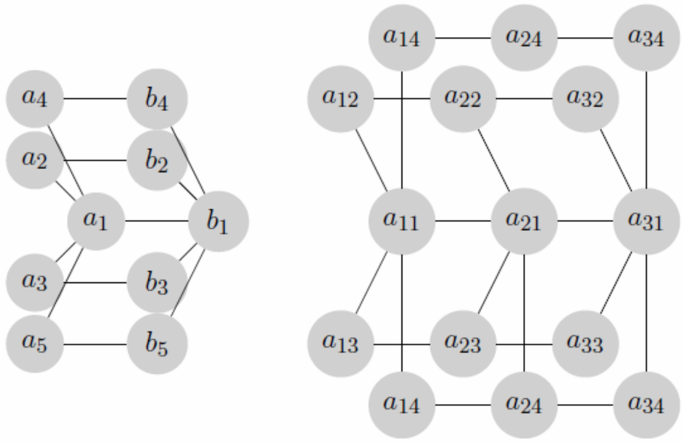

FIGURE 5. Book Graph $B_4, B_{4,3}$

Since $w_{2,2} = a_{12}a_{13}a_{23}a_{11}a_{13}a_{22}a_{12}a_{21}a_{11}a_{22}a_{23}a_{21}a_{13}a_{23}a_{12}a_{22}a_{11}a_{21}$. Now add the paths $a_{21}a_{31}a_{32}a_{22}$ and $a_{21}a_{31}a_{33}a_{23}$ to $w_{2,2}$. In $w_{2,2}$, we have the factor $a_{22}a_{23}a_{21}$
$a_{13}a_{23}a_{12}a_{22}a_{11}a_{21}$. substitute $a^2_{2i} \rightarrow a^1_{3i}a^2_{2i}, i = 1,2,3, a^3_{2i} \rightarrow a^2_{3i}a^3_{2i}, i = 2,3$ and $a^3_{21} \rightarrow a^2_{31}a^3_{21}a^3_{32}a^3_{33}a^3_{31}$.

Therefore, $w_{2,3} = a_{12}a_{13}a_{23}a_{11}a_{13}a_{22}a_{12}a_{21}a_{11}a_{32}a_{22}a_{33}a_{23}a_{31}a_{21}a_{13}\,a_{33}a_{23}a_{12}a_{32}a_{22}$ $a_{11}a_{31}a_{21}a_{32}a_{33}a_{31}$ and $B_{2,3}$ is 3-word representable.

Assume $n = 4$, add the paths $a_{31}a_{41}a_{42}a_{32}$ and $a_{31}a_{41}a_{43}a_{33}$ to $w_{2,3}$. In $w_{2,3}$, we have the factor $a_{32}a_{22}a_{33}a_{23}a_{31}a_{21}a_{13}a_{33}a_{23}a_{12}a_{32}a_{22}a_{11}a_{31}a_{21}a_{32}a_{33}a_{31}$. Now substitute $a^1_{3i}a^2_{2i} \rightarrow a^1_{3i}a^2_{2i}a^1_{4i}, i = 1,2,3, a^2_{3i}a^3_{2i} \rightarrow a^2_{3i}a^2_{4i}a^3_{2i}, i = 2,3$ and $a^3_{31} \rightarrow a^2_{41}a^3_{31}a^3_{42}a^3_{43}a^3_{41}$.

Therefore, $w_{2,4} = a_{12}a_{13}a_{23}a_{11}a_{13}a_{22}a_{12}a_{21}a_{11}a_{32}a_{22}a_{42}a_{33}a_{23}a_{43}a_{31}a_{21}a_{41}a_{13}a_{33}a_{43}a_{23}$ $a_{12}a_{32}a_{42}a_{22}a_{11}a_{31}a_{21}a_{32}a_{33}a_{41}a_{31}a_{42}a_{43}a_{41}$. More generally, the 3-word-representation $w_{2,n}$ of $B_{2,n}$ from $w_{2,n-1}$ of $B_{2,n-1}$ is derived by using the following substitutions.

when $j =$ even substitute,

i) $a^1_{3i}a^2_{2i}a^1_{5i}a^1_{4i}a^1_{7i}a^1_{6i}...a^1_{(n-1)i}a^1_{(n-2)i} \rightarrow a^1_{3i}a^2_{2i}a^1_{5i}a^1_{4i}a^1_{7i}a^1_{6i}...a^1_{(n-1)i}a^1_{(n-2)i}a^1_{ni}$ for $i = 1,2,3$.

ii)$a^2_{(n-1)i}a^2_{(n-3)i}a^2_{(n-2)i}a^2_{(n-5)i}a^2_{(n-4)i}...a^2_{7i}a^2_{8i}a^2_{5i}a^2_{6i}a^2_{3i}a^2_{4i}a^3_{2i} \rightarrow a^2_{(n-1)i}a^2_{ni}a^2_{(n-3)i}a^2_{(n-2)i}a^2_{(n-5)i}$ $...a^2_{7i}a^2_{8i}a^2_{5i}a^2_{6i}a^2_{3i}a^2_{4i}a^3_{2i}$ for $i = 2,3$.

iii) $a^3_{(n-1)1} \rightarrow a^2_{n1}a^3_{(n-1)1}a^3_{n2}a^3_{n3}a^3_{n1}$.

when $j =$ odd, substitute

i) $a^1_{3i}a^2_{2i}a^1_{5i}a^1_{4i}a^1_{7i}a^1_{6i}...a^1_{(n-3)i}a^1_{(n-1)i} \rightarrow a^1_{3i}a^2_{2i}a^1_{5i}a^1_{4i}a^1_{7i}a^1_{6i}...a^1_{(n-3)i}a^1_{ni}a^1_{(n-1)i}$ for $i = 1, 2, 3$.

ii) $a^2_{(n-2)i}a^2_{(n-1)i}a^2_{(n-4)i}a^2_{(n-3)i}a^2_{(n-6)i}...a^2_{7i}a^2_{8i}a^2_{5i}a^2_{6i}a^2_{3i}a^2_{4i}a^3_{2i} \rightarrow a^2_{ni}a^2_{(n-2)i}a^2_{(n-1)i}a^2_{(n-4)i}a^2_{(n-3)i}$ $...a^2_{7i}a^2_{8i}a^2_{5i}a^2_{6i}a^2_{3i}a^2_{4i}a^3_{2i}$ for $i = 2, 3$.

iii) $a^3_{(n-1)1} \rightarrow a^2_{n1}a^3_{(n-1)1}a^3_{n2}a^3_{n3}a^3_{n1}$.

Thus $B_{2,n}$ is 3-word-representable.

Similarly from B_m, we can derive the representation for $B_{m,n}, m \geq 3$ using the following substitutions

when $j =$ even substitute,

i) $a^1_{3i}a^2_{2i}a^1_{5i}a^1_{4i}a^1_{7i}a^1_{6i}...a^1_{(n-1)i}a^1_{(n-2)i} \rightarrow a^1_{3i}a^2_{2i}a^1_{5i}a^1_{4i}a^1_{7i}a^1_{6i}...a^1_{(n-1)i}a^1_{(n-2)i}a^1_{ni}$ for $i = 1, 2, \ldots, m + 1$.

ii)$a^2_{(n-1)i}a^2_{(n-3)i}a^2_{(n-2)i}a^2_{(n-5)i}a^2_{(n-4)i}...a^2_{7i}a^2_{8i}a^2_{5i}a^2_{6i}a^2_{3i}a^2_{4i}a^3_{2i} \rightarrow a^2_{(n-1)i}a^2_{ni}a^2_{(n-3)i}a^2_{(n-2)i}a^2_{(n-5)i}$ $...a^2_{7i}a^2_{8i}a^2_{5i}a^2_{6i}a^2_{3i}a^2_{4i}a^3_{2i}$ for $i = 2, 3, \ldots, m + 1$.

iii) $a^3_{(n-1)1} \rightarrow a^2_{n1}a^3_{(n-1)1}a^3_{n2}a^3_{n3}a^3_{n1}$.

when $j =$ odd, substitute

i) $a^1_{3i}a^2_{2i}a^1_{5i}a^1_{4i}a^1_{7i}a^1_{6i} \ldots a^1_{(n-3)i}a^1_{n-1,i} \rightarrow a^1_{3i}a^2_{2i}a^1_{5i}a^1_{4i}a^1_{7i}a^1_{6i}...a^1_{(n-3)i}a^1_{ni}a^1_{(n-1)i}$ for $i = 1, 2, \ldots, m + 1$.

ii)$a^2_{(n-2)i}a^2_{(n-1)i}a^2_{(n-4)i}a^2_{(n-3)i}a^2_{(n-6)i}...a^2_{7i}a^2_{8i}a^2_{5i}a^2_{6i}a^2_{3i}a^2_{4i}a^3_{2i} \rightarrow a^2_{ni}a^2_{(n-2)i}a^2_{(n-1)i}a^2_{(n-4)i}a^2_{(n-3)i}\cdots$ $a^2_{7i}a^2_{8i}a^2_{5i}a^2_{6i}a^2_{3i}a^2_{4i}a^3_{2i}$ for $i = 2, 3, \ldots, m + 1$.

iii) $a^3_{(n-1)1} \rightarrow a^2_{n1}a^3_{(n-1)1}a^3_{n2}a^3_{n3}a^3_{n1}$.

Hence every stacked book graph $B_{m,n}$ is 3-word-representable.

□

Theorem 2.5. *Every stacked book graph* $R(B_{m,n}) = 3, n \geq 2$.

Proof. By Theorem 2.3, every book graph $R(B_m) = 3, m \geq 3$. since every stacked book graph $B_{m,n}$ is a book graph and by Theorem 2.4, $B_{m,n}$ is 3-word-representable for $n \geq 3$. Hence $R(B_{m,n}) = 3$. □

3. Conclusion

In this paper We discussed the Word Representation number of Book graph and stacked book graph. Book graphs are one of the hardest graphs to be represented by words among the class of bipartite graphs. which means that they may be require longest words for its representation. Here we derived the word representation of book graph, stacked book graph.

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