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On Fuzzy Neutrosophic Pre σ-Baire Spaces

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Abstract: *In this paper, the concepts of fuzzy neutrosophic pre σ-nowhere dense set, fuzzy neutrosophic pre σ-first category set and fuzzy neutrosophic pre σ-second category set in fuzzy neutrosophic topological spaces are introduced and studied. By means of fuzzy neutrosophic pre σ-nowhere dense sets, the concept of fuzzy neutrosophic pre σ-Baire space is defined and several characterizations of fuzzy neutrosophic pre σ-Baire spaces are studied. Several examples are given to illustrate the concepts introduced in this paper.*

Key Words*: Fuzzy neutrosophic pre-open(closed) set, fuzzy neutrosophic pre Fσ-set, fuzzy neutrosophic pre G_δ − -set, fuzzy neutrosophic pre dense, fuzzy neutrosophic pre residual, fuzzy neutrosophic nowhere dense set, fuzzy neutrosophic pre σ-first and second category sets, fuzzy neutrosophic pre σ-Baire space.*

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I. Introduction

The fuzzy idea was invaded all branches of science as far back as the presentation of fuzzy sets by L. A. Zadeh [21]. The important concept of fuzzy topological space was offered by C.L. Chang [3]. The idea of fuzzy σ-Baire Spaces was introduced by G. Thangaraj and E. Poongothai [13]. The concept of neutrosophic sets was defined with membership, non-membership and indeterminacy degrees. In 2017, Veereswari [20] introduced fuzzy neutrosophic topological spaces. The idea of fuzzy neutrosophic Baire spaces was introduced by E. Poongothai and E. Padmavathi [10]. In this paper, the concepts of fuzzy neutrosophic pre σ-nowhere dense set, fuzzy neutrosophic pre σ-first category set and fuzzy neutrosophic pre σ-second category set in fuzzy neutrosophic topological spaces are introduced and studied. By means of fuzzy neutrosophic pre σ-nowhere dense sets, the concept of fuzzy neutrosophic pre σ-Baire space is defined and several characterizations of fuzzy neutrosophic pre σ-Baire spaces are studied. Several examples are given to illustrate the concepts introduced in this paper.

II. Preliminaries

Definition 2.1 [2] A fuzzy neutrosophic set A on the universe of discourse X is defined as $A = (x, T_A(x), I_A(x), F_A(x)), \ x \in X$ where $T, I, F : X \to [0, 1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.2 [2] A fuzzy neutrosophic set A is a subset of a fuzzy neutrosophic set B(i.e.,) $A \subseteq B$ for all x if $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$, $F_A(x) \geq F_B(x)$.

Definition 2.3 [2] Let X be a non-empty set, and
$A = (x, T_A(x), I_A(x), F_A(x))$, $B = (x, T_B(x), I_B(x), F_B(x))$ be two fuzzy neutrosophic sets. Then

$$A \cup B = (x, max(T_A(x), T_B(x)), max(I_A(x), I_B(x)), min(F_A(x), F_B(x)))$$
$$A \cap B = (x, min(T_A(x), T_B(x)), min(I_A(x), I_B(x)), max(F_A(x), F_B(x)))$$

Definition 2.4 [2] The difference between two fuzzy neutrosophic sets A and B is defined as $A \backslash B(x) = (x, min(T_A(x), F_B(x)), min(I_A(x), 1 - I_B(x)), max(F_A(x), T_B(x)))$.

Definition 2.5 [2] A fuzzy neutrosophic set A over the universe X is said to be null or empty fuzzy neutrosophic set if $T_A(x) = 0,\ I_A(x) = 0,\ F_A(x) = 1$ for all $x \in X$. It is denoted by 0_N.

Definition 2.6 [2] A fuzzy neutrosophic set A over the universe X is said to be absolute (universe) fuzzy neutrosophic set if $T_A(x) = 1,\ I_A(x) = 1,\ F_A(x) = 0$ for all $x \in X$. It is denoted by 1_N.

Definition 2.7 [2] The complement of a fuzzy neutrosophic set A is denoted by A^C and is defined as $A^C = (x, T_{A^C}(x), I_{A^C}(x), F_{A^C}(x))$ where $T_{A^C}(x) = F_A(x)$, $I_{A^C}(x) = 1 - I_A(x), F_{A^C}(x) = T_A(x)$.The complement of fuzzy neutrosophic set A can also be defined as $A^C = 1_N - A$.

Definition 2.8 [1] A fuzzy neutrosophic topology on a non-empty set X is a τ of fuzzy neutrosophic sets in X satisfying the following axioms.

(i) $0_N,\ 1_N \in \tau$
(ii) $A_1 \cap A_2 \in \tau$ for any $A_1,\ A_2 \in \tau$
(iii) $\cup A_i \in \tau$ for any arbitrary family $\{A_i : i \in J\} \in \tau$

In this case the pair (X, τ) is called fuzzy neutrosophic topological space and any fuzzy neutrosophic set in τ is known as fuzzy neutrosophic open set in X.

Definition 2.9 [1] The complement A^C of a fuzzy neutrosophic set A in a fuzzy neutrosophic topological space (X, τ) is called fuzzy neutrosophic closed set in X.

Definition 2.10 [1] Let (X, τ) be a fuzzy neutrosophic topological space and A = (x, $T_A(x)$, $I_A(x)$, $F_A(x)$) be a fuzzy neutrosophic set in X . Then the closure and interior of A are defined by

$$\mathrm{int}(A) = \cup\{G : G \text{ is a fuzzy neutrosophic open set in } X \text{ and } G \subseteq A\}$$

$$\mathrm{cl}(A) = \cap\{G : G \text{ is a fuzzy neutrosophic closed set in } X \text{ and } A \subseteq G\}$$

Definition 2.11 [1] Let (X, τ) be a fuzzy neutrosophic topological space over X . Then the following properties hold, (i) $\mathrm{cl}(A^C) = (\mathrm{int}\ A)^C$, (ii) $\mathrm{int}(A^C) = (\mathrm{cl}\ A)^C$.

Definition 2.12 [10] A fy. neutrosophic set A_N in a fy. neutrosophic top. space (P, τ_N) is called a fy. neutrosophic F_σ-set if $A_N = \vee_{i=1}^{\infty}(A_{N_i})$, where $\overline{A_{N_i}} \in \tau_N$ ~~for~~ i∈I.

Definition 2.13 [10] A fy. neutrosophic set A_N in a fy. neutrosophic top. space (P, τ_N) is called a fy. neutrosophic G_δ-set in (P, τ_N) if $A_N = \wedge_{i=1}^{\infty} A_{N_i}$, where $A_{N_i} \in \tau_N$ for $i \in I$.

Definition 2.14 [10] A fy. neutrosophic set A_N in a fy. neutrosophic top. space (P, τ_N) is called a fy. neutrosophic dense if there exist no fnCS B_N in (P, τ_N) s.t $A_N \subset B_N \subset 1_X$. That is $\mathrm{fn}(A_N)^- = 1_N$.

Definition 2.15 [10] A fy. neutrosophic set A_N in a fy. neutrosophic top. space (P, τ_N) is called a fy. neutrosophic nowh. dense set if there exist no non zero fnOS B_N in (P, τ_N) s.t $B_N \subset fn(A_N)^-$. That is, $fn(((A_N)^-)^+) = 0_N$.

Definition 2.16 [10] Let (P, τ_N) be a fy. neutrosophic top. space. A fy. neutrosophic set A_N in (P, τ_N) is called fy. neutrosophic one category set if $A_N = \vee_{i=1}^{\infty} A_{N_i}$, where A_{N_i}'s are fy. neutrosophic nowh. dense sets in (P, τ_N). Any other fy. neutrosophic set in (P, τ_N) is said to be of fy. neutrosophic two category.

Definition 2.17 [10] A fy.neutrosophic top.space (P, τ_N) is called fy. neutrosophic one category space if the fy. neutrosophic set 1_X is a fy. Neutrosophic one category set in (P, τ_N). That is $1_X = \vee_{i=1}^{\infty} A_{N_i}$, where A_{N_i}'s are fy. Neutrosophic nowh. dense sets in (P, τ_N). Otherwise (P, τ_N) will be called a fy. neutrosophic two category space.

Definition 2.18 [10] Let A_N be a fy. neutrosophic one category set in (P, τ_N). Then $\overline{A_N}$ is called fy. neutrosophic re. set in (P, τ_N).

Definition 2.19 [10] A fy. neutrosophic top. space (P, τ_N) is called fy. neutrosophic Baire space if $fn\left(\vee_{i=1}^{\infty}(A_{N_i})\right)^+ = 0_N$, where (A_{N_i})'s are fy. neutrosophic nowh. dense sets in (P, τ_N).

Theorem 2.1 [10] Let (P, τ_N) be a fy. neutrosophic top. space. Then the following are equivalent

(1) (P, τ_N) is a fy. neutrosophic Baire space.
(2) $fn(A_N)^+ = 0_N$, for every fy. neutrosophic one category set A_N in (P, τ_N).
(3) $Fn(B_N)^+ = 1_N$, for every fy. neutrosophic re. set B_N in (P, τ_N).

Definition 2.20 [5] Let (X_N, T_N) be a fuzzy neutrosophic topological space. A fuzzy neutrosophic Set λ_N in (X_N, T_N) is called a fuzzy neutrosophic σ-nowhere dense set if λ_N is a fuzzy neutrosophic F_σ-set in (X_N, T_N) such that $int(\lambda_N) = 0_N$.

Definition 2.21 [5] Let (X_N, T_N) be a fuzzy neutrosophic topological space. A fuzzy neutrosophic set λ_N in (X_N, T_N) is called fuzzy neutrosophic σ- first category set if $\lambda_N = \vee_{i=1}^{\infty}(\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic σ-nowhere dense sets in (X_N, T_N).Any other fuzzy neutrosophic set in (X_N, T_N) is said to be fuzzy neutrosophic σ- second category sets in (X_N, T_N).

Definition 2.22 [5] Let λ_N be a fuzzy neutrosophic σ-first category set in (X_N, T_N). Then $1_N - \lambda_N$ is called a fuzzy neutrosophic σ-residual set in (X_N, T_N).

Definition 2.23 [5] A fuzzy neutrosophic topological space (X_N, T_N) is called fuzzy neutrosophic σ-first category space if the fuzzy neutrosophic set 1_{X_N} is a fuzzy neutrosophic σ-first category set in (X_N, T_N). That is $1_{X_N} = \vee_{i=1}^{\infty}(\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic σ- nowhere dense sets in (X_N, T_N). Otherwise (X_N, T_N) will be called a fuzzy neutrosophic σ-second category space.

Definition 2.24 [5] Let (X_N, T_N) be a fuzzy neutrosophic topological space. Then (X_N, T_N) is called a fuzzy neutrosophic σ-Baire space if $int(\vee_{i=1}^{\infty}(\lambda_{N_i})) = 0_N$, where (λ_{N_i})'s are fuzzy neutrosophic σ- nowhere dense sets in (X_N, T_N).

Theorem 2.2 [5] Let (X_N, T_N) be a fuzzy neutrosophic topological space. Then the following are equivalent

(1) (X_N, T_N) is a fuzzy neutrosophic σ-Baire space.
(2) $int(\lambda_N) = 0_N$, for every fuzzy neutrosophic σ- first category set λ_N in (X_N, T_N).
(3) $cl(\mu_N) = 1_N$, for every fuzzy neutrosophic σ-residual set μ_N in (X_N, T_N).

Definition 2.25 [12] A fuzzy neutrosophic topological space (X, τ_N) is called a fuzzy neutrosophic submaximal space if for each fuzzy neutrosophic set A_N in (X, τ_N) such that $(A_N)^- = 1$ then $A_N \in \tau_N$ in (X, τ_N). That is (X, τ_N) is a fuzzy neutrosophic submaximal space if each fuzzy neutrosophic dense set in (X, τ_N) is a fuzzy neutrosophic open set in (X, τ_N).

Definition 2.26 [12] A fuzzy neutrosophic topological space (X, τ_N) is called a fuzzy neutrosophic resolvable space if there exist a fuzzy neutrosophic dense set A_N in (X, τ_N) such that $(1-A_N)^- = 1$. Otherwise, (X, τ_N) is called a fuzzy neutrosophic irresolvable space.

Definition 2.27 [12] A fuzzy neutrosophic topological space (X, τ_N) is called a fuzzy neutrosophic hyperconnected space if every non-null fuzzy neutrosophic open subset of (X, τ_N) is fuzzy neutrosophic dense in (X, τ_N).

Definition 2.28 [12] A fuzzy neutrosophic topological space (X, τ_N) is called a fuzzy neutrosophic P-space if each fuzzy neutrosophic G_δ-set in (X, τ_N) is fuzzy neutrosophic open set in (X, τ_N).

Definition 2.29 [12] A fuzzy neutrosophic topological space (X, τ_N) is called a fuzzy neutrosophic almost resolvable space if $\vee_{i=1}^{\infty}(A_{N_i}) = 1$, where (A_{N_i})'s in (X, τ_N) are such that $(A_{N_i})^+ = 0$, otherwise, (X, τ_N) is called a fuzzy neutrosophic almost irresolvable space.

Definition 2.30 [4] FNS λ_N in FNTS (X, τ) is called Fuzzy neutrosophic regular-open set (Briefly, FNR-open) if $\lambda_N = FNInt(FNcl(\lambda_N))$

Definition 2.31 [4] FNS λ_N in FNTS (X, τ) is called Fuzzy neutrosophic regular-closed set (Briefly, FNR-closed) if $\lambda_N = FNcl(FNInt(\lambda_N))$.

Definition 2.32 [4] Fuzzy neutrosophic pre-open set (Briefly, FNP-open) if $\lambda_N \subseteq FNInt(FNCl(\lambda_N)$.

Definition 2.33 [4] Fuzzy neutrosophic pre-closed set(Briefly, FNP-closed) if $FNCl(FNInt(\lambda_N)) \wedge (\lambda_N)$.

III Fuzzy Neutrosophic Pre σ—Nowhere Dense sets

Definition 3.1 A fuzzy neutrosophic set λ_N in a fuzzy neutrosophic topological space (X_N, T_N) is called a fuzzy neutrosophic pre F_σ-set in (X_N, T_N) if $\lambda_N = \vee_{i=1}^{\infty}(\lambda_{N_i})$ where (λ_{N_i})'s are fuzzy neutrosophic pre-closed sets in (X_N, T_N).

Definition 3.2 A fuzzy neutrosophic set λ_N in a fuzzy neutrosophic topological space (X_N, T_N) is called a fuzzy neutrosophic pre G_δ-set in (X_N, T_N) if $\lambda_N = \wedge_{i=1}^{\infty}(\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic pre-open sets in (X_N, T_N).

Definition 3.3 A fuzzy neutrosophic set λ_N in a fuzzy neutrosophic topological space (X_N, T_N) is called a fuzzy neutrosophic pre dense if there exist no fuzzy neutrosophic pre-closed set, where μ_N in (X_N, T_N) such that $\lambda_N < \mu_N < 1_N$. That is, $pcl(\lambda_N) = 1_N$ in (X_N, T_N).

Definition 3.4 A fuzzy neutrosophic set λ_N in a fuzzy neutrosophic topological space (X_N, T_N) is called a fuzzy neutrosophic pre σ-nowhere dense set if λ_N is a non-zero fuzzy neutrosophic pre F_σ-set in (X_N, T_N) such that $pint(\lambda_N) = 0_N$.

Example 3.1 Let $X_N = \{a, b, c\}$. The fuzzy neutrosophic sets λ_N, μ_N and γ_N are defined on X_N as follows:

$\lambda_N : X_N \to [0_N, 1_N]$ is defined as,

$\lambda_N = \{(a, (0.7, 0.6, 0.7)), (b, (0.5, 0.6, 0.5)), (c, (0.7, 0.7, 0.6))\}$

$\mu_N : X_N \to [0_N, 1_N]$ is defined as,

$\mu_N = \{(a, (0.5, 0.4, 0.8)), (b, (0.7, 0.6, 0.5)), (c, (0.8, 0.6, 0.6))\}$

$\gamma_N : X_N \to [0_N, 1_N]$ is defined as,

$\gamma_N = \{(a, (0.6, 0.7, 0.5)), (b, (0.5, 0.6, 0.6)), (c, (0.7, 0.6, 0.6))\}$

Then, $T_N = \{0_N, \lambda_N, \mu_N, \gamma_N, \lambda_N \vee \mu_N, \mu_N \vee \gamma_N, \lambda_N \vee \gamma_N, \lambda_N \wedge \mu_N, \mu_N \wedge \gamma_N, \lambda_N \wedge \gamma_N, 1_N\}$ is a fuzzy neutrosophic topology on X_N. The fuzzy neutrosophic set $\beta_N = [(1_N - \lambda_N) \vee (1_N - \mu_N) \vee (1_N - (\lambda_N \vee \mu_N))]$ in (X_N, T_N). Then β_N is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N) and $pint(\beta_N) = 0_N$ and hence β_N is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

Example 3.2 Let $X_N = \{a, b\}$.The fuzzy neutrosophic sets α_N and β_N are defined on X_N as follows:

$\alpha_N : X_N \to [0_N, 1_N]$ is defined as $\alpha_N = \{(a, (0.4, 0.3, 0.4)), (b, (0.5, 0.4, 0.3))\}$

$\beta_N : X_N \to [0_N, 1_N]$ is defined as $\beta_N = \{(a, (0.5, 0.3, 0.3)), (b, (0.4, 0.3, 0.5))\}$

Then, $T_N = \{0_N, \alpha_N, \beta_N, \alpha_N \vee \beta_N, \alpha_N \wedge \beta_N, 1_N\}$ is a fuzzy neutrosophic topology on X_N. Now, consider

$\eta_N = [(1_N - (\alpha_N \vee \beta_N)) \vee (1_N - (\alpha_N \wedge \beta_N))]$

$\eta_N = 1_N - (\alpha_N \wedge \beta_N)$

Therefore η_N is a fuzzy neutrosopic pre F_σ-set in (X_N, T_N). $pint(\eta_N) \neq 0_N$ .Therefore η_N is not a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

Remark 3.1 If λ_N and μ_N are fuzzy neutrosophic pre σ-nowhere dense sets in a fuzzy neutrosophic topological space (X_N, T_N), then $\lambda_N \vee \mu_N$ be a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N). For, consider the following example:

Example 3.3 Let $X_N = \{a, b, c\}$. The fuzzy neutrosophic sets A_N, B_N and C_N are defined X_N as follows:

$A_N : X_N \to [0_N, 1_N]$ is defined as,

$A_N = \{(a, (0.7, 0.6, 0.6)), (b, (0.7, 0.5, 0.6)), (c, (0.5, 0.7, 0.6))\}$

$B_N : X_N \rightarrow [0_N, 1_N]$ is defined as,

$B_N = \{(a, (0.5, 0.4, 0.5)), (b, (0.4, 0.7, 0.6)), (c, (0.8, 0.5, 0.6))\}$

$C_N : X_N \rightarrow [0_N, 1_N]$ is defined as,

$C_N = \{(a, (0.7, 0.6, 0.6)), (b, (0.6, 0.5, 0.5)), (c, (0.5, 0.4, 0.5))\}$

Then, $T_N = \{0_N, A_N, B_N, C_N, A_N \vee B_N, B_N \vee C_N, A_N \vee C_N, A_N \wedge B_N, B_N \wedge C_N, A_N \wedge C_N, A_N \vee B_N \vee C_N, A_N \wedge B_N \wedge C_N, 1_N\}$ is clearly a fuzzy neutrosophic topology on X_N.

Now, consider, $\alpha_N = [(1_N — A_N) \vee (1_N — B_N) \vee (1_N — C_N)] = [1_N — (A_N \wedge B_N)]$

Therefore α_N is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N).

$\beta_N = [(1_N —(\beta_N \vee C_N)) \vee (1_N —(A_N \vee C_N)) \vee (1_N —(A_N \wedge B_N))] = [1_N —(A_N \wedge B_N)]$

$pint(\alpha_N) = 0_N$, which implies that α_N is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N). Then $(1_N — C_N) \vee (1_N — (A_N \vee B_N \vee C_N)) = \gamma_N$ is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N). $pint(\gamma_N) = 0_N$ is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N). $pint(\alpha_N \vee \gamma_N) = 0_N$ is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

Proposition 3.1. A fuzzy neutrosophic set λ_N is a fuzzy neutrosophic pre σ- nowehre dense set in a fuzzy neutrosophic toplogical space (X_N, T_N) if and only if $(1_N — \lambda_N)$ is a fuzzy neutrosophic pre dense and fuzzy neutrosophic pre G_δ-set in (X_N, T_N).

Proof. Let λ_N be a fuzzy neutrosophic pre σ-nowhre dense set (X_N, T_N). Then $\lambda_N = \vee_{i=1}^{\infty}(\lambda_{N_i})$ where (λ_{N_i})'s are fuzzy neutrosophic pre-closed sets in (X_N, T_N) and $pint(\lambda_N) = 0_N$. Then $1_N — pint(\lambda_N) = 1_N — 0_N = 1_N$ and hence $pcl(1_N — \lambda_N) = 1_N$. Also $(1_N — \lambda_N) = 1_N — \vee_{i=1}^{\infty}(\lambda_{N_i}) = \wedge_{i=1}^{\infty}(1_N — \lambda_{N_i})$, where $(1_N — \lambda_{N_i})$'s are fuzzy neutrosophic pre-open sets in (X_N, T_N), implies that $1_N — \lambda_N$ is a fuzzy neutrosophic pre G_δ-set in (X_N, T_N). Hence $(1_N — \lambda_N)$ is a fuzzy neutrosophic pre dense and fuzzy neutrosophic pre G_δ-set in (X_N, T_N).

Conversely,Let λ_N be a fuzzy neutrosophic pre dense and fuzzy neutrosophic pre G_δ-set in (X_N, T_N). Then $\lambda_N = \wedge_{i=1}^{\infty}(\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy pre-open sets in (X_N, T_N). Now, $1_N — \lambda_N = 1_N — \wedge_{i=1}^{\infty}(\lambda_{N_i}) = \vee_{i=1}^{\infty}(1_N — \lambda_{N_i})$, where $(1_N — \lambda_{N_i})$'s are fuzzy neutrosophic pre-closed sets in (X_N, T_N). Hence $(1_N — \lambda_N)$ is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N) and $pint(1_N — \lambda_N) = 1_N — pcl(\lambda_N) = 1_N — 1_N = 0_N$. [Since λ_N is a fuzzy neutrosophic pre dense in (X_N, T_N)]. Therefore $(1_N — \lambda_N)$ is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

Proposition 3.2 If λ_N is a fuzzy neutrosophic pre dense set in a fuzzy neutrosophic topological space (X_N, T_N) such that $\mu_N \leq (1_N — \lambda_N)$, where μ_N is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N), then μ_N is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

Proof. Let λ_N be a fuzzy neutrosophic pre dense set in (X_N, T_N) such that $\mu_N \leq (1_N — \lambda_N)$. Now, $\mu_N \leq (1_N — \lambda_N)$, implies that $pint(\mu_N) \leq pint(1_N — \lambda_N)$. Then $pint(\mu_N) \leq 1_N — pcl(\lambda_N) = 1_N — 1_N = 0_N$ and hence $pint(\mu_N) = 0_N$. Therefore, μ_N is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N) such that $pint(\mu_N) = 0_N$ and hence μ_N is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

Definition 3.5 Let (X_N, T_N) be a fuzzy neutrosophic topological space. A fuzzy neutrosophic set λ_N in (X_N, T_N) is called fuzzy neutrosophic pre σ-first category set if $\lambda_N = \vee_{i=1}^{\infty}(\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic pre σ-nowhere dense sets in (X_N, T_N). Any other fuzzy neutrosophic set in (X_N, T_N) is said to be of fuzzy neutrosophic pre σ-second category set in (X_N, T_N).

Definition 3.6 Let λ_N be a fuzzy neutrosophic pre σ-first category set in (X_N, T_N). Then $(1_N - \lambda_N)$ is called a fuzzy neutrosophic pre σ-residual set in (X_N, T_N).

Definition 3.7 A fuzzy neutrosophic topological space (X_N, T_N) is called fuzzy neutrosophic pre σ-first category space is the fuzzy neutrosphic set 1_{X_N} if a fuzzy neutrosophic pre σ-first category set in (X_N, T_N). That is, $1_{X_N} = \vee_{i=1}^{\infty}(\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic pre σ-nowhere dense sets in (X_N, T_N). Otherwise, (X_N, T_N) will be called a fuzzy neutrosophic pre σ-second category space.

Proposition 3.3 If λ_N is a fuzzy neutrosophic pre σ-first category set in a fuzzy neutrosophic topological space (X_N, T_N), then there is a fuzzy neutrosophic pre F_σ- set δ_N in (X_N, T_N) such that $\lambda_N \le \delta_N$.

Proof. Let λ_N be a fuzzy neutrosophic pre σ-first category set in (X_N, T_N). Then $\lambda_N = \vee_{i=1}^{\infty}(\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic pre σ-nowhere dense sets in (X_N, T_N). Now, $[1_N - pcl(\lambda_{N_i})]'$s arefuzzy neutrosophic pre-open sets in (X_N, T_N). Then $\mu_N = \wedge_{i=1}^{\infty}[1_N - pcl(\lambda_{N_i})]$ is a fuzzy neutrosophic pre $G_\delta -$ set in (X_N, T_N) and $1_N - \mu_N = 1_N - \left[\wedge_{i=1}^{\infty}\left(1_N - pcl(\lambda_{N_i})\right)\right] = \left[\vee_{i=1}^{\infty} pcl(\lambda_{N_i})\right]$. Now, $\lambda_{N_i} \le pcl(\lambda_{N_i})$, implies that $\vee_{i=1}^{\infty}(\lambda_{N_i}) \le \left[\vee_{i=1}^{\infty} pcl(\lambda_{N_i})\right]$. Hence $\lambda_N = \vee_{i=1}^{\infty}(\lambda_{N_i}) \le \left[\vee_{i=1}^{\infty} pcl(\lambda_{N_i})\right] = [1_N - \mu_N]$. That is, $\lambda_N \le [1_N - \mu_N]$and $[1_N - \mu_N]$ is a fuzzy neutrosophic pre $F_\sigma -$ set in (X_N, T_N). Let$\delta_N = [1_N - \mu_N]$. Hence, if λ_N is a fuzzy neutrosophic pre $\sigma -$ first category set in (X_N, T_N), then there is a fuzzy neutrosophic pre $F_\sigma -$ set in δ_N such that (X_N, T_N).

Proposition 3.4 If λ_N is a fuzzy neutrosophic pre σ-first category set in a fuzzy neutrosophic topological space (X_N, T_N), then there is a fuzzy neutrosophic pre F_σ- set δ_N in (X_N, T_N) such that $\lambda_N \le \delta_N \le cl(\lambda_N)$, where δ_N is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N).

Proof. Let λ_N be a fuzzy neutrosophic pre σ-first category set in (X_N, T_N). Then $\lambda_N = \vee_{i=1}^{\infty}(\lambda_{N_i})$, where (λ_{N_i})'s are fuzzy neutrosophic pre σ-nowhere dense sets in (X_N, T_N). Now, $[1_N - pcl(\lambda_{N_i})]$'s (i = 1 to ∞) are fuzzy neutrosophic pre- open sets in (X_N, T_N). Then, $\mu_N = \wedge_{i=1}^{\infty}(1_N - pcl(\lambda_{N_i}))$ is a fuzzy neutrosophic pre G_δ-set in (X_N, T_N) and $1_N - \mu_N = 1_N - [\wedge_{i=1}^{\infty}(1_N - pcl(\lambda_{N_i}))] = [\vee_{i=1}^{\infty} pcl(\lambda_{N_i})]$. Now, $\lambda_N = \vee_{i=1}^{\infty}(\lambda_{N_i}) \le \vee_{i=1}^{\infty} pcl(\lambda_{N_i})] \le \vee_{i=1}^{\infty}(cl(\lambda_{N_i})) \le cl(\vee_{i=1}^{\infty}(\lambda_{N_i}))$. That is, $\lambda_N \le [1_N - \mu_N] \le cl(\lambda_N)$ and $[1_N - \mu_N]$ is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N). Let $\delta_N = [1_N - \mu_N]$. Hence, if λ_N is a fuzzy neutrosophic pre σ-first category set in (X_N, T_N) such that then there is a fuzzy neutrosophic pre F_σ-set δ_N in (X_N, T_N) $\lambda_N \le \delta_N \le cl(\lambda_N)$, where δ_N is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N).

Proposition 3.5 If λ_N is a fuzzy neutrosophic pre-closed set in a fuzzy neutrosophic topological space (X_N, T_N) and if $pint(\lambda_N) = 0_N$, then λ_N is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

Proof. Let λ_N be a fuzzy neutrosophic pre-closed set in (X_N, T_N). Then we have $pcl(\lambda_N) = \lambda_N$. Now, $pint[pcl(\lambda_N)] = pint(\lambda_N)$ and $pint(\lambda_N) = 0_N$, implies that λ_N is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

Proposition 3.6 If λ_N is a fuzzy neutrosophic closed and fuzzy neutrosophic σ-nowehre dense set in a fuzzy neutrosophic topological space (X_N, T_N), then $pint(\lambda_N) = 0_N$ in (X_N, T_N).

Proof. Let λ_N be a fuzzy neutrosophic σ-nowhere dense set in (X_N, T_N). Then λ_N is a fuzzy neutrosophic F_σ-set such that $int(\lambda_N) = 0_N$. We have, $pint(\lambda_N) \leq \lambda_N \wedge intcl(\lambda_N)$. Then, $pint(\lambda_N) \leq \lambda_N \wedge int(\lambda_N)$.[Since λ_N is a fuzzy neutrosophic closed set, $\lambda_N = cl(\lambda_N)$] and hence $pint(\lambda_N) \leq \lambda_N \wedge 0_N$. That is, $pint(\lambda_N) = 0_N$ in (X_N, T_N).

Proposition 3.7 If each fuzzy neutrosophic σ-nowhere dense set λ_N is a fuzzy neutrosophic closed set in a fuzzy neutrosophic topological space (X_N, T_N), then λ_N is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

Proof. Let λ_N be a fuzzy neutrosophic σ-nowhere dense set in (X_N, T_N). Then λ_N is a fuzzy neutrosopohic F_σ-set in (X_N, T_N) such that $int(\lambda_N) = 0_N$. We have, $pint(\lambda_N) \leq \lambda_N \wedge intcl(\lambda_N)$. Since λ_N is a fuzzy neutrosophic closed set in (X_N, T_N), $cl(\lambda_N) = \lambda_N$. Then $pint(\lambda_N) \leq \lambda_N \wedge int(\lambda_N)$. That is, $pint(\lambda_N) \leq \lambda_N \wedge 0_N = 0_N$. Hence, $pint(\lambda_N) = 0_N$ and therefore λ_N is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N).

1V Fuzzy Neutrosophic Pre σ-Baire Spaces

Definition 4.1 Let (X_N, T_N) be a fuzzy neutrosophic topological space. Then (X_N, T_N) is called a fuzzy neutrosophic pre σ-Baire Space if $pint(\vee_{i=1}^{\infty}(\lambda_{N_i}) = 0_N$, where (λ_{N_i})'s are fuzzy neutrosophic pre σ −nowhere dense sets in (X_N, T_N).

Example 4.1 Let $X_N = \{a, b, c\}$. The fuzzy neutrosophic sets A_N, B_N and C_N are defined on X_N as follows:

$A_N : X_N \rightarrow [0_N, 1_N]$ is defined as,

$A_N = \{(a, (0.7, 0.6, 0.5)), (b, (0.5, 0.6, 0.8)), (c, (0.7, 0.5, 0.6))\}$

$B_N : X_N \rightarrow [0_N, 1_N]$ is defined as,
$B_N = \{(a, (0.6, 0.5, 0.7)), (b, (0.6, 0.6, 0.7)),)c, (0.5, 0.7, 0.5))\}$

$C_N : X_N \rightarrow [0_N, 1_N]$ is defined as,

$C_N = \{(a, (0.5, 0.5, 0.7)), (b, (0.7, 0.5, 0.5)), (c, (0.6, 0.5, 0.7))\}$

Then, $T_N = \{0_N, A_N, B_N, C_N, A_N \vee B_N, A_N \vee C_N, B_N \vee C_N, A_N \wedge B_N, A_N \wedge C_N, B_N \wedge C_N, A_N \wedge B_N \wedge C_N, 1_N\}$ is a fuzzy neutrosophic topology on X_N. Now,

$\alpha_N = [(1_N - B_N) \vee (1_N - C_N) \vee (1_N - (A_N \vee B_N))] = [1_N - (B_N \wedge C_N)]$ is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N).

$pint(\alpha_N) = 0_N$, α_N is a fuzzy neutrosophic pre σ-nowhere dense set in (X_N, T_N). $\beta_N = [(1_N - (A_N \wedge B_N))\vee(1_N - A_N \wedge C_N)\vee(1_N - (B_N \wedge C_N))]$ is a fuzzy neutrosophic pre F_σ-set in (X_N, T_N). $pint(\beta_N) = 0_N$, β_N is a fuzzy neutrosopic pre σ-nowhere dense set. $pint(\alpha_N \vee \beta_N) = 0_N$, then (X_N, T_N) is a fuzzy neutrosophic pre σ-Baire space.

Proposition 4.1 Let (X_N , T_N) be a fuzzy neutrosophic topological space. Then the following are equivalent:

(1) (X_N, T_N) is a fuzzy neutrosophic pre σ-Baire Space.
(2) pint(λ_N) = 0_N, for each fuzzy neutrosophic pre σ-first category set in (X_N, T_N).
(3) pcl(μ_N) = 1_N, for each fuzzy neutrosophic pre σ-residual set μ_N in (X_N, T_N).

Proof. (1) ⇒ (2)

Let λ_N be a fuzzy neutrosophic σ-first catergory set in (X_N , T_N). Then λ_N =$\vee_{I=1}^{\infty}$ (λ_{N_i}),
Where $(\lambda_{N_i})'s$ are fuzzy neutrosophic pre σ −nowhere dense sets in (X_N , T_N). Then, pint(λ_N) = $pint\left(\vee_{i=1}^{\infty}(\lambda_{N_i})\right)$. Since ($X_N$, T_N) is a fuzzy neutrosophic σ-Baire space, pint$\left(\vee_{i=1}^{\infty}(\lambda_{N_i})\right)$ = 0_N. Hence, pint(λ_N) = 0_N for a fuzzy neutrosophic pre σ −first category set λ_N in (X_N, T_N).

(2) ⟹ (3)
Let μ_N be a fuzzy neutrosophic pre σ-residual set μ_N in (X_N , T_N). Then (1_N — μ_N) is a fuzzy neutrosophic pre σ-first category set in (X_N , T_N). By hypothesis, pint(1_N — λ_N) = 0_N . Then, 1_N — pcl(μ_N) = 0_N . Hence, pcl(μ_N) = 1_N , for a fuzzy neutrosophic pre σ-residual set μ_N in (X_N, T_N).

(3)⟹ (1)

Let λ_N be a fuzzy neutrosophic σ-first category set in (X_N , T_N). Then λ_N =($\vee_{i=1}^{\infty}$ (λ_{N_i})), where (λ_{N_i})'s are fuzzy neutrosophic pre σ-nowhere dense sets in (X_N , T_N). Now, λ_N is a fuzzy neutrosophic pre σ-first category set in (X_N , T_N), implies that (1_N — λ_N) is a fuzzy neutrosophic pre σ-residual set in (X_N , T_N). By hypothesis, pcl(1_N — λ_N) = 1_N . Then, 1_N — pint(λ_N) = 1_N . Hence pint(λ_N) = 0_N . That is, pint($\vee_{i=}^{\infty}$(λ_{N_i})) = 0_N , where (λ_{N_i})'s are fuzzy neutrosophic pre σ-nowhere dense sets in (X_N , T_N). Hence (X_N , T_N) is a fuzzy neutrosophic pre σ-Baire space.

Proposition 4.2 If the fuzzy neutrosophic topological space (X_N , T_N) is a fuzzy neutrosophic pre σ-Baire space, then (X_N , T_N) is a fuzzy neutrosophic pre σ-second category space.

Proof. Let (X_N , T_N) be a fuzzy neutrosophic pre σ-Baire space. Then, int($\vee_{i=1}^{\infty}$ (λ_{N_i})) = 0_N , where (λ_N)'s are fuzzy neutrosophic pre σ-nowhere dense sets in (X_N , T_N). Then $\vee_{i=1}^{\infty}(\lambda_{N_i}) \neq 1_{X_N}$[Otherwise,$\vee_{i=1}^{\infty}$ (λ_{N_i}) = 1_{X_N} , implies that pint($\vee_{i=1}^{\infty}$ (λ_{N_i})) = pint(1_{X_N}) = 1_{X_N} , which in turn implies that 0_N = 1_N , a contradiction]. Hence (X_N , T_N) is a fuzzy neutrosophic pre σ-second category space.

Proposition 4.3 Let (X_N , T_N) be a fuzzy neutrosophic topological space.If $\wedge_{i=1}^{\infty}$ (λ_{N_i}) ≠0_N , where (λ_{N_i})'s are fuzzy neutrosophic pre dense and fuzzy neutrosophic pre G_δ-sets in (X_N, T_N), then (X_N, T_N) is a fuzzy neutrosophic preσ-second category space.

Proof. Given that $\wedge_{i=1}^{\infty}$ (λ_{N_i})≠0_N , implies that 1_N — $\wedge_{i=1}^{\infty}$ (λ_{N_i})≠1_N — 0_N = 1_N . Then $\vee_{i=1}^{\infty}$(1_N — λ_N) ≠1_N . Since (λ_N)'s are fuzzy neutrosophic pre dense and fuzzy neutrosophic pre G_δ-set in (X_N , T_N), by proposition 3.1., (1_N — λ_{N_i})'s are fuzzy neutrosophic pre σ-nowhere dense sets in

(X_N, T_N). Hence, $(1_N — \lambda_N) \neq 1_N$, where $(1_N — \lambda_{Ni})$'s are fuzzy neutrosophic pre σ-nowhere dense sets in (X_N, T_N). Hence (X_N, T_N) is not a fuzzy neutrosophic pre σ-first category space. Therefore (X_N, T_N) is a fuzzy neutrosophic pre σ-second category space.

Proposition 4.4 If a fuzzy neutrosophic topological space (X_N, T_N) is a fuzzy neutrosophic pre σ-Baire space, then no non-zero fuzzy neutrosophic pre-open set is a fuzzy neutrosophic σ-first category set in (X_N, T_N).

Proof. Let λ_N be a non-zero fuzzy neutrosophic pre-open set in a fuzzy pre σ- Baire space (X_N, T_N). Suppose that $\lambda_N = \vee_{i=1}^{\infty} (\lambda_{N_i})$, where the fuzzy neutrosophic sets (λ_{Ni})'s are fuzzy neutrosophic pre σ-nowhere de sets in (X_N, T_N). Then $\text{pint}(\lambda_N) = \text{pint}(\vee_{i=1}^{\infty} (\lambda_{N_i}))$. Since (X_N, T_N) is a fuzzy neutrosophic pre σ-Baire space, $\text{pint}(\vee_{i=1}^{\infty} (\lambda_{N_i})) = 0_N$. This implies that, $\text{pint}(\lambda_N) = 0_N$. Then we will have $\lambda_N = \text{pint}(\lambda_N) = 0_N$, a contradiction. Since λ_N is a non-zero fuzzy neutrosophic set in (X_N, T_N). Hence no non-zero fuzzy neutrosophic pre-open set is a fuzzy neutrosophic pre σ-first category set in (X_N, T_N).

V. Conclusion

In this study, we have introduced and analyzed the concept of fuzzy neutrosophic pre σ-Baire Spaces, extending classical and fuzzy Baire space theories into the neutrosophic framework. The work lays a foundation for further exploration of fuzzy neutrosophic spaces in advanced topology, particularly in applications.involving decision-making, artificial intelligence and Information systems, where vagueness and indeterminacy play a critical role.

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