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Combination of Cubic and Quartic Plane Curve

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Abstract

- *The set of complex eigenvalues of unistochastic matrices of order three forms a deltoid.*
- *A cross-section of the set of unistochastic matrices of order three forms a deltoid.*
- *The set of possible traces of unitary matrices belonging to the group SU(3) forms a deltoid.*
- *The intersection of two deltoids parametrizes a family of Complex Hadamard matrices of order six.*
- *The set of all Simson lines of given triangle, form an envelope in the shape of a deltoid. This is known as the Steiner deltoid or Steiner's hypocycloid after Jakob Steiner who described the shape and symmetry of the curve in 1856.*
- *The envelope of the area bisectors of a triangle is a deltoid (in the broader sense defined above) with vertices at the midpoints of the medians. The sides of the deltoid are arcs of hyperbolas that are asymptotic to the triangle's sides.*

I. Introduction

Various combinations of coefficients in the above equation give rise to various important families of curves as listed below.

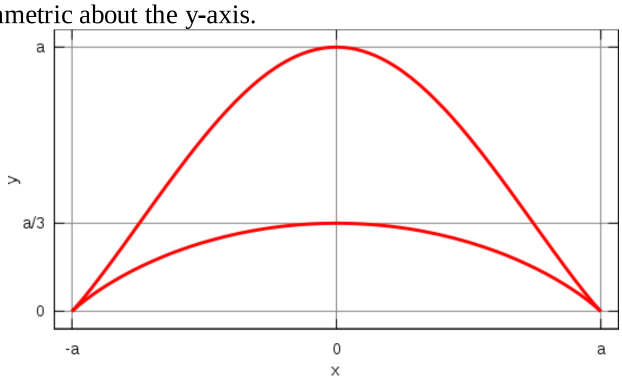
1. Bicorn curve
2. Klein quartic
3. Bullet-nose curve
4. Lemniscate of Bernoulli
5. Cartesian oval
6. Lemniscate of Gerono
7. Cassini oval
8. Lüroth quartic
9. Deltoid curve
10. Spiric section
11. Hippopede
12. Toric section
13. Kampyle of Eudoxus
14. Trott curve

II. Bicorn curve

In geometry, the bicorn, also known as a cocked hat curve due to its resemblance to a bicorne, is a rational quartic curve defined by the equation

$$y^2(a^2 - x^2) = (x^2 + 2ay - a^2)^2.$$

It has two cusps and is symmetric about the y-axis.



The bicorn is a plane algebraic curve of degree four and genus zero. It has two cusp singularities in the real plane, and a double point in the complex projective plane at x=0, z=0 . If we move x=0 and z=0 to the origin substituting and perform an imaginary rotation on x bu substituting ix/z for x and 1/z for y in the bicorn curve, we obtain

$$(x^2 - 2az + a^2z^2)^2 = x^2 + a^2z^2.$$

This curve, a limaçon, has an ordinary double point at the origin, and two nodes in the complex plane, at x = ± i and z=1.

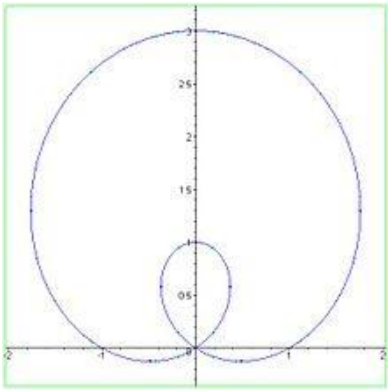


Figure A transformed bicorn with *a* = 1

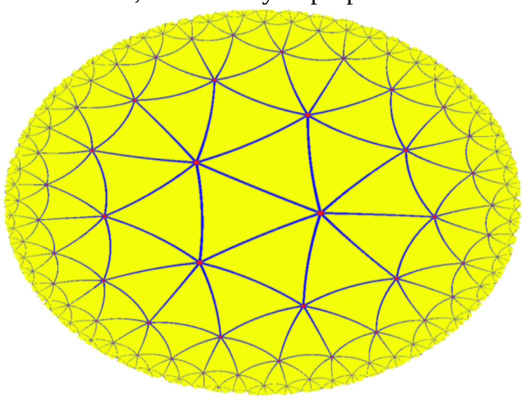
The parametric equations of a bicorn curve are:

$x = a\sin(\theta)$ and $y = \dfrac{\cos^2(\theta)\,(2+\cos(\theta))}{3+\sin^2(\theta)}$ with $-\pi \le \theta \le \pi$

III. Klein quartic

In hyperbolic geometry, the Klein quartic, named after Felix Klein, is a compact Riemann surface of genus 3 with the highest possible order automorphism group for this genus, namely order 168 orientation-preserving automorphisms, and 336 automorphisms if orientation may be reversed. As such, the Klein quartic is the Hurwitz surface of lowest possible genus; given Hurwitz's automorphisms theorem. Its (orientation-preserving) automorphism group is isomorphic to PSL(2,7), the second-smallest non-abelian simple group. The quartic was first described in (Klein 1878b).

Klein's quartic occurs in many branches of mathematics, in contexts including representation theory, homology theory, octonion multiplication, Fermat's last theorem, and the Stark–Heegner theorem on imaginary quadratic number fields of class number one; for a survey of properties.



IV. Bullet-nose curve

In mathematics, a bullet-nose curve is a unicursal quartic curve with three inflection points, given by the equation

$$a^2y^2 - b^2x^2 = x^2y^2$$

The bullet curve has three double points in the real projective plane, at x=0 and y=0, x=0 and z=0, and y=0 and z=0, and is therefore a unicursal (rational) curve of genus zero.

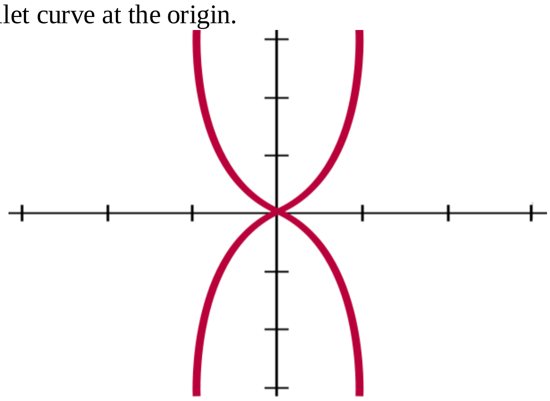
If

$$f(z) = \sum_{n=0}^{\infty} \binom{2n}{n} z^{2n+1} = z + 2z^3 + 6z^5 + 20z^7 + \cdots$$

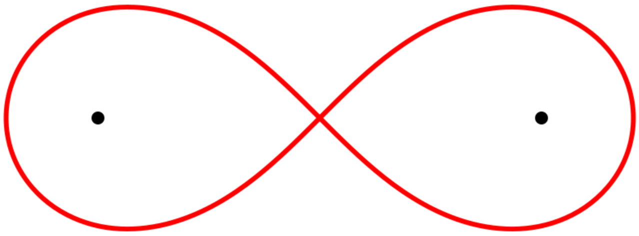
then

$$y = f\left(\frac{x}{2a}\right) \pm 2b$$

are the two branches of the bullet curve at the origin.



V. Lemniscate of Bernoulli



In geometry, the lemniscateof Bernoulli is a plane curve defined from two given points F_1 and F_2, known as **foci**, at distance $2a$ from each other as the locus of points P so that $PF_1 \cdot PF_2 = a^2$. The curve has a shape similar to the numeral 8 and to the ∞ symbol. Its name is from *lemniscus*, which is Latin for "pendant ribbon". It is a special case of the Cassini oval and is a rational algebraic curve of degree 4.

The lemniscate was first described in 1694 by Jakob Bernoulli as a modification of an ellipse, which is the locus of points for which the sum of the distances to each of two fixed *focal points* is a constant. A Cassini oval, by contrast, is the locus of points for which the *product* of these distances is constant. In the case where the curve passes through the point midway between the foci, the oval is a lemniscate of Bernoulli.

This curve can be obtained as the inverse transform of a hyperbola, with the inversion circle centered at the center of the hyperbola (bisector of its two foci). It may also be drawn by a mechanical linkage in the form of Watt's linkage, with the lengths of the three bars of the linkage and the distance between its endpoints chosen to form a crossed square.[

- Its Cartesian equation is (up to translation and rotation):

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$$

- In polar coordinates:

$$r^2 = 2a^2 \cos 2\theta$$

- As parametric equation:

$$x = \frac{a\sqrt{2}\cos(t)}{\sin(t)^2 + 1}; \qquad y = \frac{a\sqrt{2}\cos(t)\sin(t)}{\sin(t)^2 + 1}$$

In two-center bipolar coordinates:

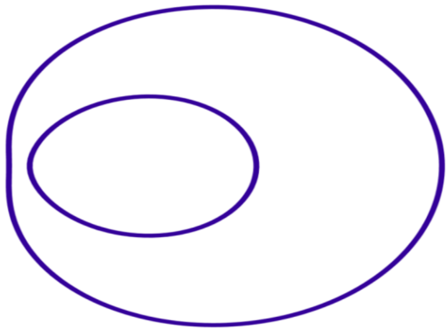
$$rr' = a^2$$

In rational polar coordinates:

$$Q = 2s - 1$$

VI. Cartesian oval

In geometry, a Cartesian oval, named after René Descartes, is a plane curve, the set of points that have the same linear combination of distances from two fixed points.



Let P and Q be fixed points in the plane, and let d(P,S) and d(Q,S) denote the Euclidean distances from these points to a third variable point S. Let m and a be arbitrary real numbers. Then the Cartesian oval is the locus of points S satisfying d(P,S) + m d(Q,S) = a. The two ovals formed by the four equations d(P,S) + m d(Q,S) = ± a and d(P,S) − m d(Q,S) = ± a are closely related; together they form a quartic plane curve called the ovals of Descartes.

The ovals of Descartes were first studied by René Descartes in 1637, in connection with their applications in optics.

These curves were also studied by Newton beginning in 1664. One method of drawing certain specific Cartesian ovals, already used by Descartes, is analogous to a standard construction of an ellipse by stretched thread. If one stretches a thread from a pin at one focus to wrap around a pin at a second focus, and ties the free end of the thread to a pen, the path taken by the pen, when the thread is stretched tight, forms a Cartesian oval with a 2:1 ratio between the distances from the two foci. However, Newton rejected such constructions as insufficiently rigorous. He defined the oval as the solution to a differential equation, constructed its subnormals, and again investigated its optical properties.

The French mathematician Michel Chasles discovered in the 19th century that, if a Cartesian oval is defined by two points P and Q, then there is in general a third point R on the same line such that the same oval is also defined by any pair of these three points.

The set of points (x,y) satisfying the quartic polynomial equation

$[(1 - m^2)(x^2 + y^2) + 2m^2cx + a^2 - m^2c^2]^2 = 4a^2(x^2 + y^2)$,

where c is the distance $\mathrm{d}(P, Q)$between the two fixed foci P = (0, 0) and Q = (c, 0), forms two ovals, the sets of points satisfying the two of the four equations

d(P,S) ± m d(Q,S) = a,

d(P,S) ± m d(Q,S) = −a

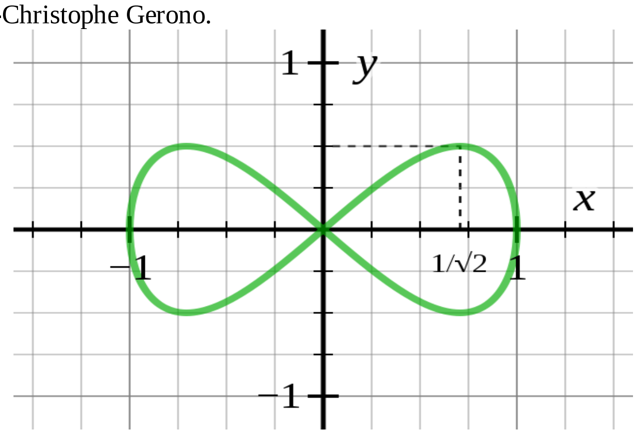
that have real solutions. The two ovals are generally disjoint, except in the case that P or Q belongs to them. At least one of the two perpendiculars to PQ through points P and Q cuts this quartic curve in four real points; it follows from this that they are necessarily nested, with at least one of the two points P and Q contained in the interiors of both of them. For a different parametrization and resulting quartic,

VII. Lemniscate of Gerono

In algebraic geometry, the lemniscateof Gerono, or lemnicate of Huygens, or figure-eight curve, is a plane algebraic curve of degree four and genus zero shaped like an ∞symbol, or figure eight. It has equation

$$x^4 - x^2 + y^2 = 0.$$

It was studied by Camille-Christophe Gerono.



Because the curve is of genus zero, it can be parametrized by rational functions; one means of doing that is

$$x = \frac{t^2 - 1}{t^2 + 1}, \ y = \frac{2t(t^2 - 1)}{(t^2 + 1)^2}.$$

Another representation is

$$x = \cos\varphi, \ y = \sin\varphi \ \cos\varphi = \sin(2\varphi)/2$$

which reveals that this lemniscate is a special case of a lissajous figure.

The dual curve, pictured below, has therefore a somewhat different character. Its equation is

$$(x^2 - y^2)^3 + 8y^4 + 20x^2y^2 - x^4 - 16y^2 = 0.$$

VIII. Cassini oval

A Cassini oval is a quartic plane curve defined as the set (or locus) of points in the plane such that the product of the distances to two fixed points is constant. This is related to an ellipse, for which the *sum* of the distances is constant, rather than the product. They are the special case of polynomial lemniscates when the polynomial used has degree 2.

Cassini ovals are named after the astronomer Giovanni Domenico Cassini who studied them in 1680. Other names include Cassinian ovals, Cassinian curves and ovals of Cassini.

Let q_1 and q_2 be two fixed points in the plane and let b be a constant. Then a Cassini oval with foci q_1 and q_2 is defined to be the locus of points p so that the product of the distance from p to q_1 and the distance from p to q_2 is b^2. That is, if we define the function dist(x,y) to be the distance from a point x to a point y, then all points p on a Cassini oval satisfy the equation

$$\operatorname{dist}(q_1, p) \cdot \operatorname{dist}(q_2, p) = b^2.$$

If the foci are (a, 0) and ($-a$, 0), then the equation of the curve is

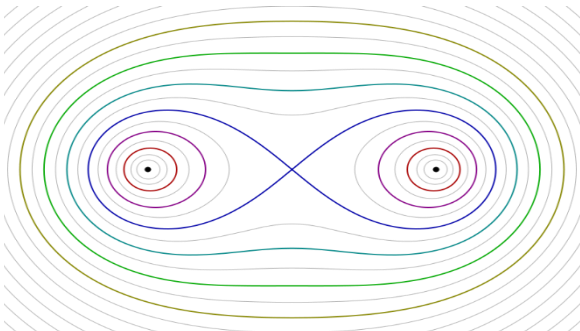
$$((x - a)^2 + y^2)((x + a)^2 + y^2) = b^4.$$

When expanded this becomes

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 = b^4.$$

The equivalent polar equation is

$$r^4 - 2a^2r^2\cos 2\theta = b^4 - a^4.$$



Form of the curve

The shape of the curve depends, up to similarity, on $e=b/a$. When $e>1$, the curve is a single, connected loop enclosing both foci. When $e<1$, the curve consists of two disconnected loops, each of which contains a focus. When $e=1$, the curve is the lemniscate of Bernoulli having the shape of a sideways figure eight with a double point (specifically, a crunode) at the origin. The limiting case of $a \to 0$ (hence $e \to \infty$), in which case the foci coincide with each other, is a circle.

The curve always has x-intercepts at $\pm c$ where $c^2=a^2+b^2$. When $e<1$ there are two additional real x-intercepts and when $e>1$ there are two real y-intercepts, all other x and y-intercepts being imaginary.

The curve has double points at the circular points at infinity, in other words the curve is bicircular. These points are biflecnodes, meaning that the curve has two distinct tangents at these points and each branch of the curve has a point of inflection there. From this information and Plücker's formulas it is possible to deduce the Plücker numbers for the case $e\neq1$: Degree = 4, Class = 8, Number of nodes = 2, Number of cusps = 0, Number of double tangents = 8, Number of points of inflection = 12, Genus = 1.

The tangents at the circular points are given by $x\pm iy=\pm a$ which have real points of intersection at ($\pm a$, 0). So the foci are, in fact, foci in the sense defined by Plücker.[6] The circular points are points of inflection so these are triple foci. When $e\neq1$ the curve has class eight, which implies that there should be at total of eight real foci. Six of these have been accounted for in the two triple foci and the remaining two are at

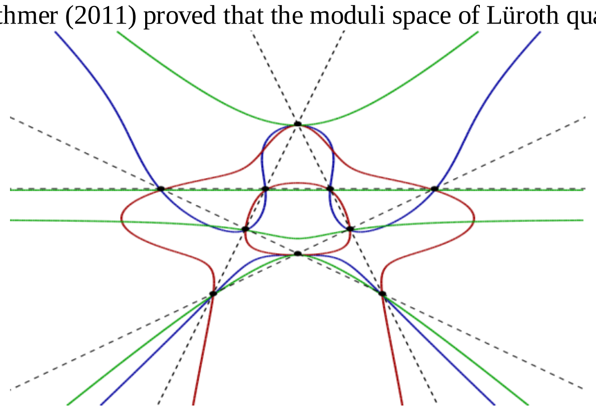
$$(\pm a\sqrt{1-e^4}, 0) \quad (e<0)$$
$$(0, \pm a\sqrt{e^4-1}) \quad (e>0).$$

So the additional foci are on the *x*-axis when the curve has two loops and on the *y*-axis when the curve has a single loop.

Curves orthogonal to the Cassini ovals: Formed when the foci of the Cassini ovals are the points (a,0) and (-a,0), equilateral hyperbolas centered at (0,0) after a rotation around (0,0) are made to pass through the foci.

IX. Lüroth quartic

In mathematics, a Lüroth quartic is a nonsingular quartic plane curve containing the 10 vertices of a complete pentalateral. They were introduced by Jacob Lüroth (1869). Morley (1919) showed that the Lüroth quartics form an open subset of a degree 54 hypersurface, called the Lüroth hypersurface, in the space $\mathbf{P}^{14}$ of all quartics. Böhning & von Bothmer (2011) proved that the moduli space of Lüroth quartics is rational.



X. Deltoid curve

In geometry, a deltoid, also known as a tricuspoid or Steiner curve, is a hypocycloid of three cusps. In other words, it is the roulette created by a point on the circumference of a circle as it rolls without slipping along the inside of a circle with three times its radius. It can also be defined as a similar roulette where the radius of the outer circle is $\frac{3}{2}$ times that of the rolling circle. It is named after the Greek letter delta which it resembles.

More broadly, a deltoid can refer to any closed figure with three vertices connected by curves that are concave to the exterior, making the interior points a non-convex set.

A deltoid can be represented (up to rotation and translation) by the following parametric equations

$$x = 2a\cos(t) + a\cos(2t)$$
$$y = 2a\sin(t) - a\sin(2t)$$

where a is the radius of the rolling circle.

In complex coordinates this becomes

$$z = 2ae^{it} + ae^{-2it}.$$

The variable t can be eliminated from these equations to give the Cartesian equation

$$(x^2+y^2)^2 + 18a^2(x^2+y^2) - 27a^4 = 8a(x^3-3xy^2)$$

and is therefore a plane algebraic curve of degree four. In polar coordinates this becomes

$$r^4 + 18a^2r^2 - 27a^4 = 8ar^3\cos 3\theta.$$

The curve has three singularities, cusps corresponding to $t = 0, \pm\frac{2\pi}{3}$. The parameterization above implies that the curve is rational which implies it has genus zero.

A line segment can slide with each end on the deltoid and remain tangent to the deltoid. The point of tangency travels around the deltoid twice while each end travels around it once.

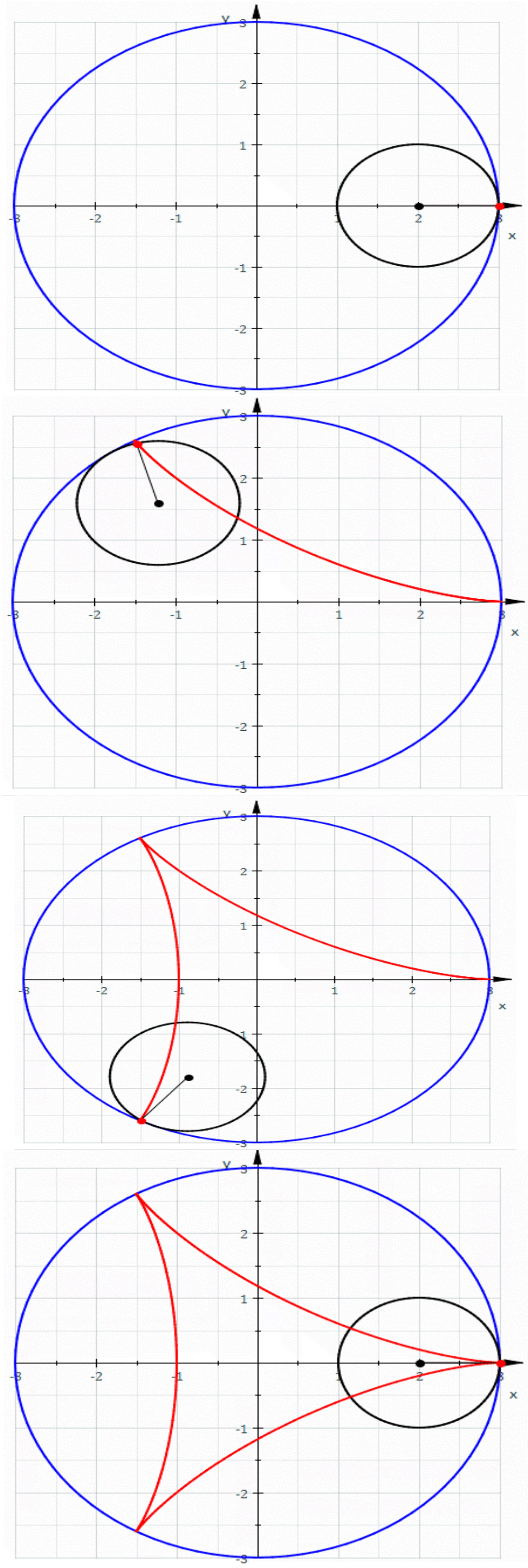
The dual curve of the deltoid is

$$x^3 - x^2 - (3x+1)y^2 = 0,$$

which has a double point at the origin which can be made visible for plotting by an imaginary rotation y ↦ iy, giving the curve

$$x^3 - x^2 + (3x+1)y^2 = 0$$

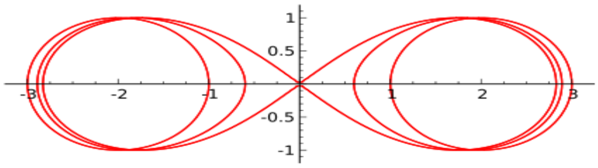
with a double point at the origin of the real plane.



XI. Spiric section

In geometry, a **spiric section**, sometimes called a **spiric of Perseus**, is a quartic plane curve defined by equations of the form

$$(x^2 + y^2)^2 = dx^2 + ey^2 + f.$$

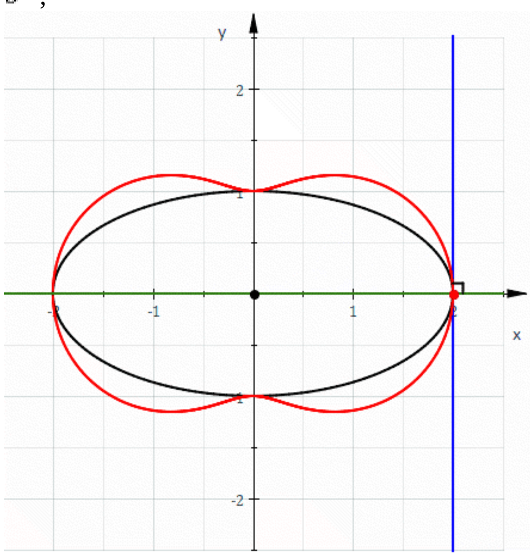


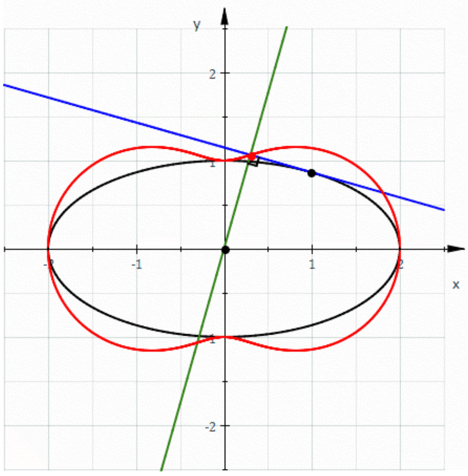
Equivalently, spiric sections can be defined as bicircular quartic curves that are symmetric with respect to the *x* and *y*-axes. Spiric sections are included in the family of toric sections and include the family of hippopedes and the family of Cassini ovals. The name is from σπειρα meaning torus in ancient Greek.
A spiric section is sometimes defined as the curve of intersection of a torus and a plane parallel to its rotational symmetry axis. However, this definition does not include all of the curves given by the previous definition unless imaginary planes are allowed.
Spiric sections were first described by the ancient Greek geometer Perseus in roughly 150 BC, and are assumed to be the first toric sections to be described.

Equations
Start with the usual equation for the torus:
$$(x^2 + y^2 + z^2 + b^2 - a^2)^2 = 4b^2(x^2 + y^2).$$
Interchanging *y* and *z* so that the axis of revolution is now on the *xy*-plane, and setting *z*=*c* to find the curve of intersection gives
$$(x^2 + y^2 - a^2 + b^2 + c^2)^2 = 4b^2(x^2 + c^2).$$
In this formula, the torus is formed by rotating a circle of radius *a* with its center following another circle of radius *b* (not necessarily larger than *a*, self-intersection is permitted). The parameter *c* is the distance from the intersecting plane to the axis of revolution. There are no spiric sections with *c*>*b* + *a*, since there is no intersection; the plane is too far away from the torus to intersect it.
Expanding the equation gives the form seen in the definition
$$(x^2 + y^2)^2 = dx^2 + ey^2 + f$$
where
$d = 2(a^2+b^2-c^2),\ e = 2(a^2-b^2-c^2),\ f = -(a+b+c)(a+b-c)(a-b+c)(a-b-c).$
In polar coordinates this becomes
$$(r^2 - a^2 + b^2 + c^2)^2 = 4b^2(r^2\cos^2\theta + c^2)$$
or
$$r^4 = dr^2\cos^2\theta + er^2\sin^2\theta + f$$

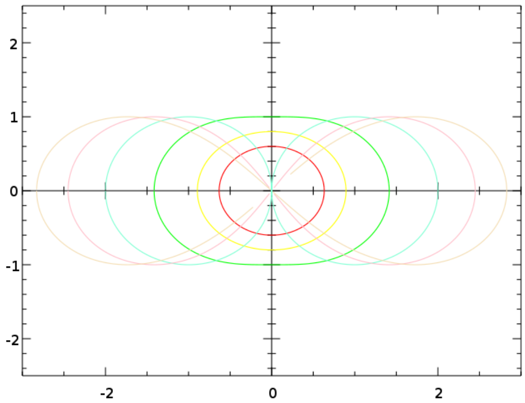
XII. Hippopede

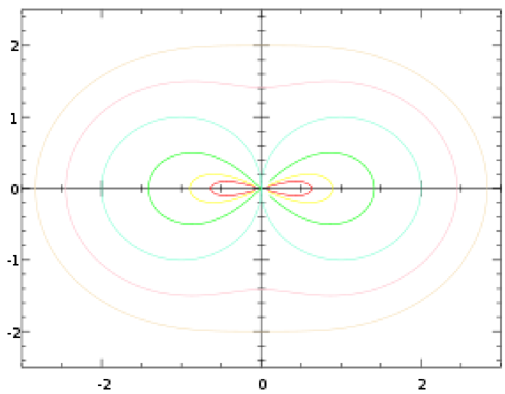
In geometry, a hippopede (from ἱπποπέδη meaning "horse fetter" in ancient Greek) is a plane curve determined by an equation of the form
$$(x^2 + y^2)^2 = cx^2 + dy^2,$$





where it is assumed that $c>0$ and $c>d$ since the remaining cases either reduce to a single point or can be put into the given form with a rotation. Hippopedes are bicircular rational algebraic curves of degree 4 and symmetric with respect to both the x and y axes. When $d>0$ the curve has an oval form and is often known as an oval of Booth, and when $d<0$ the curve resembles a sideways figure eight, or lemniscate, and is often known as a lemniscate of Booth, after James Booth (1810–1878) who studied them. Hippopedes were also investigated by Proclus (for whom they are sometimes called Hippopedes of Proclus) and Eudoxus. For $d = -c$, the hippopede corresponds to the lemniscate of Bernoulli.





Definition

Hippopedes can be defined as the curve formed by the intersection of a torus and a plane, where the plane is parallel to the axis of the torus and tangent to it on the interior circle. Thus it is a spiric section which in turn is a type of toric section.

If a circle with radius *a* is rotated about an axis at distance *b* from its center, then the equation of the resulting hippopede in polar coordinates

$$r^2 = 4b(a - b\sin^2\theta)$$

or in Cartesian coordinates

$$(x^2+y^2)^2 + 4b(b-a)(x^2+y^2) = 4b^2x^2.$$

Note that when *a*>*b* the torus intersects itself, so it does not resemble the usual picture of a torus.

XIII. Toric section

A toric section is an intersection of a plane with a torus, just as a conic section is the intersection of a plane with a cone.

In general, toric sections are fourth-order (quartic) plane curves of the form

$$\left(x^2+y^2\right)^2 + ax^2 + by^2 + cx + dy + e = 0.$$

Spiric sections

A special case of a toric section is the spiric section, in which the intersecting plane is parallel to the rotational symmetry axis of the torus. They were discovered by the ancient Greek geometer Perseus in roughly 150 BC. Well-known examples include the hippopede and the Cassini oval and their relatives, such as the lemniscate of Bernoulli.

Villarceau circles

Another special case is the Villarceau circles, in which the intersection is a circle despite the lack of any of the obvious sorts of symmetry that would entail a circular cross-section.

General toric sections

More complicated figures such as an annulus can be created when the intersecting plane is perpendicular or oblique to the rotational symmetry axis.

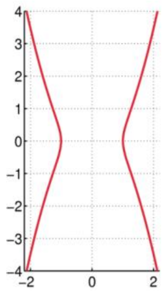
XIV. Kampyle of Eudoxus

The Kampyle of Eudoxus (Greek: καμπύλη [γραμμή], meaning simply "curved [line], curve") is a curve, with a Cartesian equation of

$$x^4 = x^2 + y^2$$

from which the solution $x = y = 0$ should be excluded, or, in polar coordinates,

$$r = \sec^2\theta\,.$$



This quartic curve was studied by the Greek astronomer and mathematician Eudoxus of Cnidus (c. 408 BC – c.347 BC) in relation to the classical problem of doubling the cube.

The Kampyle is symmetric about both the x- and y-axes. It crosses the x-axis at $(-1,0)$and $(1,0)$. It has inflection points at

$$(\pm\sqrt{3/2}, \pm\sqrt{3}/2)$$

(four inflections, one in each quadrant). The top half of the curve is asymptotic to $x^2 - \frac{1}{2}$as $x \to \infty$, and in fact can be written as

$$y = x^2\sqrt{1-x^{-2}} = x^2 - \frac{1}{2}\sum_{n\geq 0} C_n(2x)^{-2n}$$

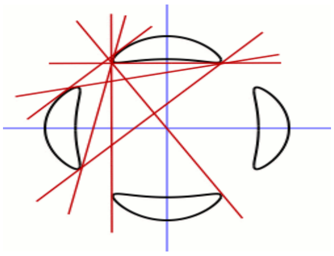
where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the nth Catalan number.

XV. Trott curve

In real algebraic geometry, a general quartic plane curve has 28 bitangent lines, lines that are tangent to the curve in two places. These lines exist in the complex projective plane, but it is possible to define curves for which all 28 of these lines have real numbers as their coordinates and therefore belong to the Euclidean plane.



An explicit quartic with twenty-eight real bitangents was first given by Plücker (1839) As Plücker showed, the number of real bitangents of any quartic must be 28, 16, or a number less than 9. Another quartic with 28 real bitangents can be formed by the locus of centers of ellipses with fixed axis lengths, tangent to two non-parallel lines Shioda (1995) gave a different construction of a quartic with twenty-eight bitangents, formed by projecting a cubic surface; twenty-seven of the bitangents to Shioda's curve are real while the twenty-eighth is the line at infinity in the projective plane.

The Trott curve, another curve with 28 real bitangents, is the set of points (x,y) satisfying the degree four polynomial equation

$$144(x^4 + y^4) - 225(x^2 + y^2) + 350x^2y^2 + 81 = 0.$$

These points form a nonsingular quartic curve that has genus three and that has twenty-eight real bitangents.

Like the examples of Plücker and of Blum and Guinand, the Trott curve has four separated ovals, the maximum number for a curve of degree four, and hence is an M-curve. The four ovals can be grouped into six different pairs of ovals; for each pair of ovals there are four bitangents touching both ovals in the pair, two that separate the two ovals, and two that do not. Additionally, each oval bounds a nonconvex region of the plane and has one bitangent spanning the nonconvex portion of its boundary.

XVI. Conclusion

The dual curve to a quartic curve has 28 real ordinary double points, dual to the 28 bitangents of the primal curve.

The 28 bitangents of a quartic may also be placed in correspondence with symbols of the form

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

where a, b, c, d, e and f are all zero or one and where

$ad + be + ef = 1 \pmod 2$.

There are 64 choices for a, b, c, d, e and f, but only 28 of these choices produce an odd sum. One may also interpret a, b, and c as the homogeneous coordinates of a point of the Fano plane and d, e, and f as the coordinates of a line in the same finite projective plane; the condition that the sum is odd is equivalent to requiring that the point and the line do not touch each other, and there are 28 different pairs of a point and a line that do not touch.

The points and lines of the Fano plane that are disjoint from a non-incident point-line pair form a triangle, and the bitangents of a quartic have been considered as being in correspondence with the 28 triangles of the Fano plane. The Levi graph of the Fano plane is the Heawood graph, in which the triangles of the Fano plane are represented by 6-cycles. The 28 6-cycles of the Heawood graph in turn correspond to the 28 vertices of the Coxeter graph

The 28 bitangents of a quartic also correspond to pairs of the 56 lines on a degree-2 del Pezzo surface, and to the 28 odd theta characteristics.

The 27 lines on the cubic and the 28 bitangents on a quartic, together with the 120 tritangent planes of a canonic sextic curve of genus 4, form a "trinity" in the sense of Vladimir Arnold, specifically a form of McKay correspondence, and can be related to many further objects, including E_7 and E_8, as discussed at *trinities*.