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Asymptotic Behavior of Solutions of Nonlinear Neutral Delay Forced Impulsive Differential Equations with Positive and Negative Coefficients

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Abstract:*Sufficient conditions are obtained for every solution of first order nonlinear neutral delay forced impulsive differential equations with positive and negative coefficients tends to a constant as $t \to \infty$.*
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I. Introduction

In recent years, the theory of impulsive differential equations had obtained much attention and a number of papers have been published in this field. This is due to wide possibilities for their applications in Physics, Chemical technology, Biology and Engineering [1,2].The asymptotic behavior of solutions of various impulsive differential equations are systematically studied (see [4,7-10]). In [10] the authors investigated the asymptotic behavior of solutions of following nonlinear impulsive delay differential equations with positive and negative coefficients

$$\left[x(t)+R(t)x(t-\tau)\right]' + P(t)f(x(t-\rho)) - Q(t)\, f(x(t-\sigma)) = 0, \quad t \ge t_0, t \ne t_k$$

$$x(t_k) = b_k x(t_k^-) + (1-b_k)\left(\int_{t_k-\rho}^{t_k} P(s+\rho) f(x(s))ds - \int_{t_k-\sigma}^{t_k} Q(s+\sigma) f(x(s))ds \right) \text{ for } k = 1,2,\ldots$$

In this paper, we adapt the same technique applied in [10] and obtain the asymptotic behavior of solutions of nonlinear neutral delay forced impulsive differential equations with positive and negative coefficients

II. Preliminaries

Consider the nonlinear neutral delay forced impulsive differential equations with positive and negative coefficients

$$\left[x(t)+R(t)x(t-\tau)\right]' + P(t)f(x(t-\rho)) - Q(t)\, f(x(t-\sigma)) = e(t), \quad t \ge t_0, t \ne t_k \tag{1}$$

$$x(t_k) = b_k x(t_k^-) + (1-b_k) \int_{t_k-\rho}^{t_k} P(s+\rho) f(x(s))ds$$

$$- \int_{t_k-\sigma}^{t_k} Q(s+\sigma) f(x(s))ds + (b_k - 1)\int_{t_k}^{\infty} e(s)ds, \quad k = 1,2,... \tag{2}$$

where $\tau > 0$, $\rho \ge \sigma > 0$, $0 \le t_0 < t_1 < \cdots < t_k \cdots \text{with} \lim_{k\to\infty} t_k = +\infty$, $f \in C(R,R)$, $R(t) \in PC([t_0,\infty),R)$, $P(t)$, $Q(t), e(t) \in C([t_0,\infty),[0,\infty))$ and b_k are constants $k = 1,2,3\ldots$, $x(t_k^-)$ denotes the left limit of $x(t)$ at $t = t_k$. With equations (1) and (2), one associates an initial condition of the form

$$x(t_0+s) = \varphi(s),\ s\in[-\delta,0],\ \delta = \max\{\tau,\rho\} \tag{3}$$

where $\varphi\in PC([-\delta,0],R)=\{\varphi : [-\delta,0]\to R$ such that φ is continuous everywhere except at the finite number of points γ and $\varphi(\gamma^+)$ and $\varphi(\gamma^-)$ exist with $\varphi(\gamma^+) = \varphi(\gamma)\}$.

A real valued function $x(t)$ is said to be a solution of the initial value problem (1)-(3) if

(i) $x(t) = \varphi(t-t_0)$ for $t_0-\delta \le t \le t_0$, $x(t)$ is continuous for $t \ge t_0$ and $t \ne t_k$, $k = 1,2,3,\ldots$

(ii) $\left[x(t)+R(t)x(t-\tau)\right]$ is continuously differentiable for $t > t_0$, $t \ne t_k$ $t \ne t_k+\tau$, $t \ne t_k+\rho$, $t \ne t_k+\sigma$, $k = 1,2,3,\ldots$ and satisfies (1).

(iii) for $t = t_k$, $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^+) = x(t_k)$ and satisfies (2).

A solution of (1)-(2) is said to be nonoscillatory if this solution is eventually positive or eventually negative. Otherwise this solution is said to be oscillatory. A more general form of (1)-(2) was considered in [5-6] in which the existence and uniqueness of solutions and stability was studied. The main purpose of this paper is to obtain the sufficient conditions for every solution of (1)-(2) tends to constant as $t\to\infty$. Our results generalize the results of [10].

III. Main results

Theorem 1. Assume that the following conditions hold

(A1) there is a constant C > 0 such that $|x|\le|f(x)|\le C|x|$, $x\in R$, $x f(x) > 0$ for $x\neq 0$; (4)

(A2) $t_k-\tau$ is not an impulsive point, $0< b_k <1$ for k = 1,2,3,… and $\sum_{k=1}^{\infty}(1-b_k) < \infty$;

(A3) $\lim_{t\to\infty}|R(t)| = \mu < 1,\ R(t_k) = b_k R(t_k^-)$ and $\int_t^{\infty} e(s)\,ds \to 0$ as $t\to\infty$; (5)

(A4) $H(t) = P(t) - Q(t + \sigma - \rho) > 0$ for $t \ge t^* = t_0+\rho-\sigma$ (6)

(A5) $\lim_{t\to\infty}\int_{t-\rho}^{t-\sigma} Q(s+\sigma)\,ds = 0$ (7)

(A6) $\lim_{t\to\infty}\sup\left[\int_{t-\rho}^{t+\rho} H(s+\rho)\,ds + \frac{Q(t+\sigma)}{H(t+\rho)}\int_{t-\rho}^{t} H(s+2\rho)du + \mu\left(1+\frac{H(t+\tau+\rho)}{H(t+\rho)}\right)\right] < \frac{2}{C}$ (8)

then every solution of (1)-(2) tends to a constant as $t\to\infty$.

Proof. Let x(t) be any solution of (1)-(2) we shall prove that $\lim_{t\to\infty} x(t)$ exists and is finite. From (1) and (6)

$$\left[x(t)+R(t)x(t-\tau)-\int_{t-\rho}^{t} H(s+\rho)f(x(s))ds - \int_{t-\rho}^{t-\sigma} Q(s+\sigma)f(x(s))ds + \int_t^{\infty} e(s)\,ds\right]' + H(t+\rho)f(x(t)) = 0 \quad (9)$$

From (2)

$$x(t_k) = b_k x(t_k^-) + (1-b_k)\left[\int_{t_k-\rho}^{t_k} P(s+\rho)f(x(s))ds - \left(\int_{t_k-\sigma}^{t_k-\rho} Q(s+\sigma)f(x(s))ds + \int_{t_k-\rho}^{t_k} Q(s+\sigma)f(x(s))ds\right)\right] + (b_k-1)\int_{t_k}^{\infty} e(s)\,ds,$$

$$= b_k x(t_k^-) + (1-b_k)\left[\int_{t_k-\rho}^{t_k-\sigma} Q(s+\sigma)f(x(s))ds + \int_{t_k-\rho}^{t_k} H(s+\rho)f(x(s))ds\right] + (b_k-1)\int_{t_k}^{\infty} e(s)\,ds, \quad k = 1,2,3,..., \quad (10)$$

From (5) and (8) we can choose an $\varepsilon > 0$ sufficiently small that $\mu+\varepsilon < 1$ and

$$\lim_{t\to\infty}\sup\left[\int_{t-\rho}^{t+\rho} H(s+\rho)\,ds + \frac{Q(t+\sigma)}{H(t+\rho)}\int_{t-\rho}^{t} H(s+2\rho)ds + (\mu+\varepsilon)\left(1+\frac{H(t+\tau+\rho)}{H(t+\rho)}\right)\right] < \frac{2}{C} \quad (11)$$

also we select $T > t_0$ sufficiently large such that $|R(t)| \le \mu+\varepsilon$ for $t \ge T$ (12)

From (4) and (12) we have

$|R(t)|\,x^2(t-\tau) \le (\mu+\varepsilon)\, f^2(x(t-\tau))$, $t \ge T$ (13)

Let we introduce three functional

$$W_1(t) = \left[x(t)+R(t)x(t-\tau)-\int_{t-\rho}^{t} H(s+\rho)f(x(s))ds - \int_{t-\rho}^{t-\sigma} Q(s+\sigma)f(x(s))ds + \int_t^{\infty} e(s)\,ds\right]^2$$

$$W_2(t)=\int_{t-\rho}^{t} H(s+2\rho)\int_{s}^{t} Q(u+\sigma)f^2(x(u))du\,ds$$

$$W_3(t)=\int_{t-\rho}^{t} H(s+2\rho)\int_{s}^{t} H(u+\rho)f^2(x(s))\,du\,ds+(\mu+\varepsilon)\int_{t-\tau}^{t} H(s+\tau+\rho)f^2(x(s))\,ds$$
$$-2\int_{t}^{\infty} e(s)\int_{t}^{\infty} H(u+\rho)\,f(x(u))du\,ds,$$

Using (9) and the inequality $2ab\le a^2+b^2$

$$\frac{dW_1}{dt}\le -H(t+\rho)\Bigg[2x(t)\,f(x(t))-|R(t)|x^2(t-\tau)-|R(t)|\,f^2(x(t))-\int_{t-\rho}^{t} H(s+\rho)f^2(x(t))ds$$
$$-\int_{t-\rho}^{t} H(s+\rho)f^2(x(s))ds-\int_{t-\rho}^{t-\sigma} Q(s+\sigma)f^2(x(t))ds-\int_{t-\rho}^{t-\sigma} Q(s+\sigma)f^2(x(s))ds\Bigg]$$
$$-2H(t+\rho)\,f(x(t))\int_{t}^{\infty} e(s)\,ds$$

$$\frac{dW_2}{dt}=Q(t+\sigma)f^2(x(t))\int_{t-\rho}^{t} H(s+2\rho)ds-H(t+\rho)\int_{t-\rho}^{t} Q(s+\sigma)f^2(x(s))ds$$
$$\le Q(t+\sigma)f^2(x(t))\int_{t-\rho}^{t} H(s+2\rho)ds-H(t+\rho)\int_{t-\rho}^{t-\sigma} Q(s+\sigma)f^2(x(s))ds$$

$$\frac{dW_3}{dt}=H(t+\rho)f^2(x(t))\int_{t-\rho}^{t} H(s+2\rho)ds-H(t+\rho)\int_{t-\rho}^{t} H(s+\rho)f^2(x(s))\,ds$$
$$+(\mu+\varepsilon)H(t+\tau+\rho)f^2(x(t))-(\mu+\varepsilon)H(t+\rho)f^2(x(t-\tau))+2\int_{t}^{\infty} e(s)\,H(t+\rho)\,f(x(t))ds.$$

Set $W(t) = W_1(t) + W_2(t) + W_3(t)$; and then using (12) and (13), we obtain

$$\frac{dW}{dt}\le -H(t+\rho)\Bigg[2x(t)\,f(x(t))-(\mu+\varepsilon)f^2(x(t))-f^2(x(t))\int_{t-\rho}^{t} H(s+\rho)ds$$
$$-f^2(x(t))\int_{t-\rho}^{t} H(s+2\rho)ds-f^2(x(t))\int_{t-\rho}^{t-\sigma} Q(s+\sigma)ds$$
$$-\frac{Q(t+\sigma)}{H(t+\rho)}f^2(x(t))\int_{t-\rho}^{t} H(s+2\rho)ds\quad-(\mu+\varepsilon)f^2(x(t))\frac{H(t+\tau+\rho)}{H(t+\rho)}\Bigg]$$
$$=-H(t+\rho)f^2(x(t))\Bigg[\frac{2x(t)}{f(x(t))}-\int_{t-\rho}^{t+\rho} H(s+\rho)ds-\frac{Q(t+\sigma)}{H(t+\rho)}\int_{t-\rho}^{t} H(s+2\rho)ds$$
$$-\int_{t-\rho}^{t-\sigma} Q(s+\sigma)ds-(\mu+\varepsilon)\left(1+\frac{H(t+\tau+\rho)}{H(t+\rho)}\right)\Bigg]$$

From (4) and (7), we have

$$\frac{dW}{dt}\le -H(t+\rho)f^2(x(t))\Bigg[\frac{2}{C}-\Bigg(\int_{t-\rho}^{t+\rho} H(s+\rho)ds-\frac{Q(t+\sigma)}{H(t+\rho)}\int_{t-\rho}^{t} H(s+2\rho)ds$$
$$+(\mu+\varepsilon)\left(1+\frac{H(t+\tau+\rho)}{H(t+\rho)}\right)\Bigg)\Bigg]\le 0\quad t\ne t_k \tag{14}$$

As $t = t_k$ we have

$$W(t_k)=\left[x(t_k)+R(t_k)x(t_k-\tau)-\int_{t_k-\rho}^{t_k} H(s+\rho)f(x(s))ds-\int_{t_k-\rho}^{t_k-\sigma} Q(s+\sigma)f(x(s))ds+\int_{t_k}^{\infty} e(s)ds\right]^2$$

$$+\int_{t_k-\rho}^{t_k} H(s+2\rho)\int_s^{t_k} Q(u+\sigma)f^2(x(u))\,du\,ds+\int_{t_k-\rho}^{t_k} H(s+2\rho)\int_s^{t_k} H(u+\rho)f^2(x(u))\,du\,ds$$

$$+(\mu+\varepsilon)\int_{t_k-\tau}^{t_k} H(s+\tau+\rho)f^2(x(s))\,ds-2\int_{t_k}^{\infty} e(s)\int_{t_k}^{s} H(u+\rho)\,f(x(u))\,du\,ds$$

$$=b_k^{\,2}\left[x(t_k^-)+R(t_k^{\,-})x(t_k^{\,-}-\tau)-\int_{t_k-\rho}^{t_k} H(s+\rho)\,f\,(x(s))\,ds-\int_{t_k-\rho}^{t_k} Q(s+\sigma)\,f\,(x(s))\,ds+\int_{t_k}^{\infty} e(s)\,ds\right]^2$$

$$+\int_{t_k-\rho}^{t_k} H(s+2\rho)\int_s^{t_k} Q(u+\sigma)f^2(x(u))du\,ds+\int_{t_k-\rho}^{t_k} H(s+2\rho)\int_s^{t_k} H(u+\rho)f^2(x(u))\,du\,ds$$

$$+(\mu+\varepsilon)\int_{t_k-\tau}^{t_k} H(s+\tau+\rho)f^2(x(s))\,ds-2\int_{t_k}^{\infty} e(s)\int_{t_k}^{s} H(u+\rho)\,f(x(u))\,du\,ds$$

$$= W(t_k^-) \tag{15}$$

Which together with (7), (8) and (14) we get $H(t+\rho)f^2(x(t))\in L^1(t_0,\infty)$ (16)

and hence for any h ≥ 0 we have $\lim\limits_{t\to\infty}\int_{t-h}^{t} H(s+\rho)f^2(x(s))ds=0$ (17)

On the other hand by (8), (14), (15) we see that W(t) is eventually decreasing.

$$0\le W_2(t)=\int_{t-\rho}^{t}\left(H(s+2\rho)\int_s^{t} Q(u+\sigma)f^2(x(u))du\right)ds$$

$$\le\frac{2}{C}\int_{t-\rho}^{t} H(s+\rho)f^2(x(u))du\ \to 0 \text{ as } t\to\infty$$

Also, $\int_{t-\rho}^{t} H(s+2\rho)\int_s^{t} H(u+\rho)f^2(x(u))\,du\,ds+(\mu+\varepsilon)\int_{t-\tau}^{t} H(s+\tau+\rho)f^2(x(s))\,ds$

$$\le\frac{2}{C}\int_{t-\rho}^{t} H(u+\rho)f^2(x(u))\,du+2\int_{t-\rho}^{t} H(s+\rho)f^2(x(s))\,ds\ \to 0\quad as\quad t\to\infty$$

and $\int_t^{\infty} e(s)\int_t^{s} H(u+\rho)\,f(x(u))\,du\,ds\le\int_t^{\infty} e(s)\int_t^{\infty} H(u+\rho)\,f(x(u))\,du\,ds\to 0\,\text{as}\,t\to\infty$

Hence $\lim\limits_{t\to\infty}W_1(t)=\lim\limits_{t\to\infty}W(t)=\eta$, which is finite. That is

$$\lim_{t\to\infty}\left[x(t)+R(t)x(t-\tau)-\int_{t-\rho}^{t} H(s+\rho)\,f\,(x(s))\,ds-\int_{t-\rho}^{t-\sigma} Q(s+\sigma)\,f\,(x(s))\,ds+\int_t^{\infty} e(s)ds\right]^2=\eta \tag{18}$$

Next we shall prove that

$\lim\limits_{t\to\infty}\left[x(t)+R(t)x(t-\tau)-\int_{t-\rho}^{t} H(s+\rho)\,f\,(x(s))\,ds-\int_{t-\rho}^{t-\sigma} Q(s+\sigma)\,f\,(x(s))\,ds+\int_t^{\infty} e(s)\,ds\right]$ exists and is finite.

Let $y(t)=x(t)+R(t)x(t-\tau)-\int_{t-\rho}^{t} H(s+\rho)\,f\,(x(s))ds-\int_{t-\rho}^{t-\sigma} Q(s+\sigma)\,f\,(x(s))ds+\int_t^{\infty} e(s)\,ds$

From (9) we have $y'(t)+H(t+\rho)\,f(x(t))=0$ and from (10) we have

$$y(t_k)=b_k\left[x(t_k^-)+R(t_k^-)x(t_k-\tau)-\int_{t_k-\rho}^{t_k} H(s+\rho)\,f\,(x(s))\,ds+\int_{t_k-\rho}^{t_k-\sigma} Q(s+\sigma)\,f\,(x(s))\,ds+\int_{t_k}^{\infty} e(s)\,ds\right]$$

$$=b_k\,y(t_k^-)$$

Therefore system (9)-(10) can be rewritten as $y'(t)+H(t+\rho)f(x(t))=0,\ t\ge t_0, t\ne t_k$

$$y(t_k)=b_k y(t_k^-) \qquad k=1,2,\dots \tag{19}$$

In view of (18) we have $\lim_{t\to\infty} y^2(t)=\eta$. (20)

If $\eta = 0$, then $\lim_{t\to\infty} y^2(t)=0$.

If $\eta > 0$, then there exists a sufficiently large T_1 such that $y(t)\ne 0$ for any $t>T_1$.Otherwise there is a sequence $\tau_1,\tau_2,\tau_3,..\tau_k,..$with $\lim_{k\to\infty}\tau_k=+\infty$ such that $y(\tau_k) = 0$ so $y^2(\tau_k)=0$ as $k\to\infty$ .This is contradiction to $\eta > 0$.Therefore for $t_k>T_1$, $t\in[t_k,t_{k+1})$ we have $y(t) > 0$ or $y(t) < 0$ because $y(t)$ is continuous on $[t_k,t_{k+1})$. Without loss of generality we assume that $y(t) > 0$ on $t\in[t_k,t_{k+1})$, it follows that $y(t_{k+1}) = b_{k+1}y(t_{k+1}^-) >0$, thus $y(t) > 0$ on $[t_{k+1},t_{k+2})$. By induction, we can conclude that $y(t) >0$ on $[t_k,\infty)$.

From (20) we have

$$\lim_{t\to\infty} y(t)= \lim_{t\to\infty}\left[x(t)+R(t)x(t-\tau)-\int_{t-\rho}^{t} H(s+\rho)f(x(s))ds-\int_{t-\rho}^{t-\sigma} Q(s+\sigma)f(x(s))ds+\int_{t}^{\infty}e(s)ds\right]=\lambda \tag{21}$$

where $\lambda=\sqrt{\eta}$ and is finite.

In view of (19) we have
$$\int_{t-\rho}^{t} H(s+\rho)f(x(s))ds=y(t-\rho)-y(t)+\sum_{t-\rho<t_k<t}\left[y(t_k)-y(t_k^-)\right]$$
$$=y(t-\rho)-y(t)+\sum_{t-\rho<t_k<t}\left[b_k y(t_k^-)-y(t_k^-)\right]$$
$$=y(t-\rho)-y(t)-\sum_{t-\rho<t_k<t}(1-b_k)y(t_k^-)$$

Let $t\to\infty$ $\sum_{k=1}^{\infty}(1-b_k)<\infty$, we have $\lim_{t\to\infty}\int_{t-\rho}^{t}H(s+\rho)f(x(s))ds=0$ (22)

By condition $\lim_{t\to\infty}\int_{t}^{\infty}e(s)ds=0$ and from (21), we have

$$\lim_{t\to\infty}\left[x(t)+R(t)x(t-\tau)-\int_{t-\rho}^{t-\sigma}Q(s+\sigma)f(x(s))ds\right]=\lambda \tag{23}$$

Next we shall prove that $\lim_{t\to\infty}[x(t)+R(t)x(t-\tau)]$ exists and is finite. To prove this, first prove that $|x(t)|$ is bounded. If $|x(t)|$ is unbounded then there is a sequence $\{s_n\}$ such that $s_n\to\infty$, $|x(s_n^-)|\to\infty$ as $n\to\infty$ and $|x(s_n^-)|=\sup_{t_0\le t\le s_n}|x(t)|$ where if s_n is not an impulsive point then $x(s_n^-)=x(s_n)$. Thus from (7) we have

$$\left|x(s_n^-)+R(s_n^-)x(s_n-\tau)-\int_{s_n-\rho}^{s_n-\sigma}Q(s+\sigma)f(x(s))ds\right|$$
$$\ge|x(s_n^-)|\left(1-(\mu+\varepsilon)-M\int_{s_n-\rho}^{s_n-\sigma}Q(s+\sigma)ds\right)\to\infty \quad as\ n\to\infty$$

which contradicts (23). So $|x(t)|$ is bounded. Also by (7) we obtain

$$0\le\left|\int_{t-\rho}^{t-\sigma}Q(s+\sigma)f(x(s))ds\right|\le\int_{t-\rho}^{t-\sigma}Q(s+\sigma)|f(x(s))|ds\le M\int_{t-\rho}^{t-\sigma}Q(s+\sigma)|x(s)|ds\to 0\quad as\ t\to\infty$$

which together with (23) gives $\lim_{t\to\infty}[x(t)+R(t)x(t-\tau)]=\lambda$ (24)

Next we prove that $\lim_{t\to\infty}x(t)$ exists and finite .

If $\mu = 0$ then $\lim_{t\to\infty} x(t) = \lambda$ which is finite.

If $0 < \mu < 1$ then R(t) is eventually positive or negative and also we can find a sufficiently large T_2 such that $|R(t)| < 1$ for $t > T_2$. Set $\limsup_{t\to\infty} x(t) = L$, $\liminf_{t\to\infty} x(t) = l$, then we can choose two sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to\infty$, $b_n\to\infty$ as $n\to\infty$ and $\lim_{n\to\infty} x(a_n) = L$, $\lim_{n\to\infty} x(b_n) = l$. Since $|R(t)| < 1$ for $t > T_2$, we have the following two possible cases.

Case 1. If $0<R(t)<1$ for $t>T_2$ then $\lim_{n\to\infty} [x(a_n) + R(a_n)x(a_n - \tau)] \geq L + \mu l$ and

$$\lim_{n\to\infty} [x(b_n) + R(b_n)x(b_n - \tau)] \leq l + \mu L$$

Therefore $L+\mu l \leq l+\mu L$. i.e.). $L \leq l$. But $L \geq l$, it follows that $L = l$. Hence $L = l = \lambda/(1+\mu)$. which shows that $\lim_{t\to\infty} x(t)$ exists and finite.

Case 2. If $-1<R(t) < 0$ for $t>T_2$ then $\lim_{n\to\infty} x(a_n) = \lim_{n\to\infty} [x(a_n) + R(a_n)x(a_n - \tau) - R(a_n)x(a_n - \tau)]$

therefore $l = \lambda+\mu l$. i.e. $l = \lambda/(1-\mu)$.

Similarly $\lim_{n\to\infty} x(b_n) = \lim_{n\to\infty} [x(b_n) + R(b_n)x(b_n - \tau) - R(b_n)x(b_n - \tau)]$

Therefore $L = \lambda+\mu L$ i.e.) $L = \lambda/(1-\mu)$.Finally we get, $L= l = \lambda/(1-\mu)$. This shows that $\lim_{t\to\infty} x(t)$ exists and finite.

Theorem 2. Assume that the conditions in theorem (1) hold, then every oscillatory solution of (1)-(2) tends to zero as $t \to \infty$.

Theorem 3. The conditions in theorem (1) together with

(i) for any $\alpha > 0$ there exists $\beta > 0$ such that $|f(x)| \geq \beta$ for $|x| \geq \alpha$ and (25)

(ii) $\int_{t_0}^{\infty} H(t)dt = \infty$

(26) imply that every solution of (1) –(2) tends to zero as $t\to\infty$.

Proof. By theorem (2) we only have to prove that every nonoscillatory solution of (1) tends to zero as $t\to\infty$.Let x(t) be an eventually positive solution of (1). We shall prove $\lim_{t\to\infty} x(t) = 0$. By theorem (1) we rewrite (1)-(2) in the form $y'(t) + H(t+\rho) f(x(t)) = 0$.

Integrating from t_0 to t on both sides we get

$$\int_{t_0}^{t} H(t+\rho) f(x(s))ds = y(t_0) - y(t) - \sum_{t_0<t_k<t} (1-b_k)y(t_k^-)$$

Using $\sum_{k=1}^{\infty} (1-b_k) < \infty$ and (21) we get $\int_{t_0}^{\infty} H(t+\rho) f(x(s))ds < \infty$

which together with (26) we get $\liminf_{t\to\infty} f(x(t)) = 0$.We shall prove that $\liminf_{t\to\infty} x(t) = 0$.Let $\{s_m\}$ be such that $s_m \to \infty$ as $m\to\infty$ and $\liminf_{m\to\infty} f(x(s_m)) = 0$ we must have $\liminf_{m\to\infty} x(s_m) = M = 0$. In fact, if $M > 0$, then there is a subsequence $\{s_{m_k}\}$ such that $x(s_{m_k}) \geq M/2$ for sufficiently large k. By (25) we have $f(x(s_{m_k}) \geq \xi$ for some $\xi > 0$ and sufficiently large k, which yields a contradiction because $\liminf_{k\to\infty} f(x(s_{m_k})) = 0$.Therefore by Theorem 1, $\liminf_{t\to\infty} x(t) = 0$ holds and hence $\lim_{t\to\infty} x(t) = 0$.

Remark 1. When e(t) = 0 the results of this paper reduce to the results of [10].

Remark 2. When R(t) = 0 ,Q(t) = 0 and e(t) = 0 the results of the present paper reduce to the results of [9].

Remark 3. When all $b_k = 1$ and $f(x) = x$, equation (1)-(2) reduced to the differential equation without impulses whose asymptotic behavior of solutions discussed in [3]

IV. Example

Consider the following impulsive differential equation

$$\left[x(t)+\left(\frac{t}{8k}\right)x\left(t-\frac{1}{2}\right)\right]'+\left(\frac{2}{(t-1)^2}\right)[1+\sin^2 x(t-2)]x(t-2)-\frac{1}{t^2}[1+\sin^2 x(t-1)]x(t-1)=e^{-t},\quad t\geq 2, t\neq k$$

$$x(k)=\left(\frac{k^2-1}{k^2}\right)x(t_k^-)+\left(\frac{1}{k^2}\right)\left(\int_{k-2}^{k}\frac{2}{(s+1)^2}[1+\sin^2 x(s)]x(s)ds-\int_{k-1}^{k}\frac{1}{(s+1)^2}[1+\sin^2 x(s)]x(s)ds\right)$$
$$-\frac{1}{k^2}\int_k^\infty e(s)\,ds \text{ for } k=1,2,3,...$$

Here $P(t) = \dfrac{2}{(t-1)^2}$, $Q(t) = \dfrac{1}{t^2}$, $R(t) = \dfrac{t}{8k}$, $f(x) = [1+\sin^2 x]\, x$, $\tau = ½$, $\rho = 2$, $\sigma = 1$, $b_k = \dfrac{k^2-1}{k^2}$

The above equation satisfies all the conditions of Theorem 1. Therefore, every solution of this equation tends to constant as $t\rightarrow\infty$.

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