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CR- Submanifoldsof a Nearly Hyperbolic Cosymplectic Manifold

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Abstract:*In the present paper, we study some properties of CR-submanifolds of a nearly hyperbolic cosymplectic manifold. We also obtain some results on ξ −horizontal and ξ −vertical CR- submanifolds of a nearly hyperbolicсosymplectic manifold.*
Keywords: *CR-submanifolds, nearlyhyperbolic cosymplectic manifold, totally geodesic,parallel distribution.*

I. Introduction

The notion of CR-submanifolds of Kaehler manifold was introduced and studied by A. Bejancu in ([1], [2]). Since then, several papers on Kaehler manifolds were published. CR-submanifolds of Sasakian manifold was studied by C.J. Hsu in [3] and M. Kobayashi in [4].Later, several geometers (see, [5], [6] [7], [8] [9], [10]) enriched the study of CR-submanifolds of almost contact manifolds. On the other hand,almost hyperbolic (f,g,η,ξ)-structure was defined and studied by Upadhyay and Dube in [11]. Dube and Bhatt studied CR-submanifolds of trans-hyperbolic Sasakian manifold in [12]. In this paper, we study some properties of CR-submanifolds of a nearly hyperbolic cosymplectic manifold.

The paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic cosymplectic manifold.In section 3, some properties of CR-submanifolds of nearly hyperbolic cosymplectic manifold are investigated. In section 4, some results on parallel distribution on ξ −horizontal and ξ −vertical CR- submanifolds of a nearly cosymplectic manifold are obtained.

II. Nearly Hyperbolic Cosymplectic manifold

Let$\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric (ϕ,ξ,η,g)- structure, where a tensor ϕ of type $(1,1)$ a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the followings:

$\phi^2X = X + \eta(X)\boldsymbol{\xi}, \quad g(X,\xi) = \eta(X)$,(2.1)

$\eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta o\phi = 0$, (2.2)

$g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y)$ (2.3)

for any X, Y tangent to$\bar{M}$ [11]. In this case

$g(\phi X, Y) = -g(X, \phi Y)$.(2.4)

An almost hyperbolic contact metric (ϕ,ξ,η,g)-structure on$\bar{M}$ is called nearly hyperbolic cosymplecticstructure if and only if

$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = 0$,(2.5)

$\bar{\nabla}_X\xi = 0$(2.6)

for all X, Y tangent to $\bar{M}$ and Riemannian Connection $\bar{\nabla}$.

III. CR-Submanifolds of Nearly Hyperbolic Cosymplectic Manifold

Let M be a submanifold immersed in $\bar{M}$. We assume that the vector field ξ is tangent to M. Then M is called a CR-submanifold [13] of $\bar{M}$ if there exist two orthogonal differentiable distributions D and $D^\perp$ on M satisfying

(i) $TM = D \oplus D^\perp$,

(ii) the distribution D is invariant by ϕ, that is, $\phi D_X = D_X$ for each $X \epsilon M$,

(iii) the distribution $D^\perp$ is anti-invariant by ϕ,that is, $\phi D_X^\perp \subset T_X M^\perp$ for each$X \epsilon M$,

where$TM and T^\perp M$ be the Lie algebra of vector fields tangential to M and normal to M respectively. If $\dim D_x^\perp = 0$ $(resp., \dim D_x = 0)$, then the CR-submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution $D(resp., D^\perp)$ is called the horizontal (resp., vertical)distribution. Also, the pair $(D, D^\perp)$ is called $\xi - horizontal(resp., vertical) if \xi_X \in D_X (resp., \xi_X \in D_X^\perp)$.

Let the Riemannian metric induced on M is denoted by the same symbol g and ∇ be the induced Levi-Civita connection on N, then the Gauss and Weingarten formulas are given respectively by [14]

$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$, (3.1)

$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N$(3.2) for any $X,Y \in TMandN \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M, h$ is the second fundamental form and A_N is the Weingarten map associated with N as

$g(A_N X, Y) = g(h(X,Y), N)$ (3.3)

for any $x \in MandX \in T_x M$. We write

$X = PX + QX$,(3.4)

where$PX \in DandQX \in D^{\perp}$.

Similarly, for N normal to M, we have

$\phi N = BN + CN$,(3.5)

where$BN(resp. CN)$ is the tangential component $(resp. normalcomponent)$ of ϕN.

Lemma 3.1. Let M be a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$.Then

$\phi P(\nabla_X Y) + \phi P(\nabla_Y X) = P\nabla_X(\phi PY) + P\nabla_Y(\phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y$,(3.6)

$2Bh(X,Y) = Q\nabla_X(\phi PY) + Q\nabla_Y(\phi PX) - QA_{\phi QX}Y - QA_{\phi QY}X$,(3.7)

$\phi Q\nabla_X Y + \phi Q\nabla_Y X + 2Ch(X,Y) = h(X,\phi PY) + h(Y,\phi PX) + \nabla_X^{\perp}\phi QY + \nabla_Y^{\perp}\phi QX$(3.8)

$foranyX,Y \in TM$.

Proof. Using (2.4), (2.5) and (2.6), we get

$$(\bar{\nabla}_X\phi)Y + \phi(\nabla_X Y) + \phi h(X,Y) = \nabla_X(\phi PY) + h(X,\phi PY) - A_{\phi QY}X + \nabla_X^{\perp}\phi QY.$$

Interchanging $X\&Y$and adding, we have

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X + \phi(\nabla_X Y) + \phi(\nabla_Y X) + 2\phi h(X,Y) = \nabla_X(\phi PY) + \nabla_Y(\phi PX) + h(X,\phi PY) + h(Y,\phi PX) -A_{\phi QY}X - A_{\phi QX}Y + \nabla_X^{\perp}\phi QY + \nabla_Y^{\perp}\phi QX.$$

Using (2.5) in above equation, we have

$$\phi P(\nabla_X Y) + \phi Q(\nabla_X Y) + \phi P(\nabla_Y X) + \phi Q(\nabla_Y X) + 2Bh(X,Y) +2Ch(X,Y) = P\nabla_X(\phi PY) + Q\nabla_Y(\phi PX) + h(X,\phi PY) +h(Y,\phi PX) - PA_{\phi QY}X - QA_{\phi QY}X - PA_{\phi QX}Y -QA_{\phi QX}Y + \nabla_X^{\perp}\phi QY + \nabla_Y^{\perp}\phi QX.(3.9)$$

Comparing the horizontal, vertical and normal components, we get (3.6) — (3.8).

Hence the Lemma is proved. □

Lemma 3.2. Let M be a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then

$2(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y - \bar{\nabla}_Y\phi X + h(X,\phi Y) - \nabla_Y\phi X - \phi[X,Y]$(3.10)

$foranyX,Y \in D$.

Proof. From Gauss formula (3.1), we have

$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X)$.(3.11)

Also, we have

$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y]$. (3.12)

From (3.11) and (3.12), we get

$$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]. \quad (3.13)$$

Adding (3.15) and (2.5), we obtain

$$2(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y + h(X,\phi Y) - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y].$$

Hence the Lemma is proved. □

Lemma 3.3. Let M be a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then

$2(\bar{\nabla}_X\phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y^{\perp}\phi X - \phi[X,Y]$(3.14)

$foranyX,Y \in D^{\perp}$.

Proof. From Weingarten formula (3.2), we have

$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y^{\perp}\phi X$.(3.15)

Also,

$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y]$.(3.16)

From (3.15) and (3.16), we get

$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y^{\perp}\phi X - \phi[X,Y]$. (3.17)

Adding (3.17) and (2.5), we obtain

$$2(\bar{\nabla}_X\phi)Y = A_{\phi X}Y - A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y^{\perp}\phi X - \phi[X,Y].$$

Hence the Lemma is proved. □

Lemma 3.4. Let M be a CR- submanifold of a nearly hyperbolic cosymplectic manifold $\bar{M}$. Then

$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$ (3.18)

$foranyX \in DandY \in D^{\perp}$.

Proof. Using Gauss and Weingarten formula for $\in D and Y \in D^{\perp}$, we have

$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \nabla_Y \phi X + h(Y, \phi X)$. (3.19)

Also, we have

$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi)Y - (\overline{\nabla}_Y \phi)X + \phi[X, Y]$.(3.20)

By virtue of (3.19) and (3.20), we get

$(\overline{\nabla}_X \phi)Y - (\overline{\nabla}_Y \phi)X = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \nabla_Y \phi X + h(Y, \phi X) - \phi[X, Y]$.(3.21)

Adding (3.21) and (2.5), we obtain

$2(\overline{\nabla}_X \phi)Y = -A_{\phi Y} X + \nabla_X^{\perp} \phi Y - \nabla_Y \phi X + h(Y, \phi X) - \phi[X, Y]$.

Hence the Lemma is proved. □

IV. Parallel Distribution

Definition4.1. The horizontal (resp., vertical) distribution $D(resp., D^{\perp})$ is said to be parallel [13] with respect to the connectionon $M if \nabla_X Y \in D$ $(resp., \nabla_Z W \in D^{\perp})$ for any vector field $X, Y \in D(resp., W, Z \in D^{\perp})$.

Theorem 4.2. Let M be a$\xi - vertical$ CR-submanifold of a nearly hyperbolic cosymplecticmanifold$\overline{M}$. If the horizontal distribution D is parallel,then

$h(X, \phi Y) = h(Y, \phi X)$. (4.1)

$for any X, Y \in D$.

Proof. Using parallelism of horizontal distribution D, we have

$\nabla_X(\phi Y) \in D and \nabla_Y \phi X \in D for any X, Y \in D$.(4.2)

Now, by virtue of (3.7), we have

$Bh(X, Y) = 0$.(4.3)

From (3.5) and (4.3), we get

$\phi h(X, Y) = Ch(X, Y)$(4.4)

$for any X, Y \in D$.

From (3.8), we have

$h(X, \phi Y) + h(Y, \phi X) = 2Ch(X, Y)$(4.5)

$for any X, Y \in D$.

Replacing $X by \phi X in$ (4.5) and using (4.4), we have

$h(\phi X, \phi Y) + h(Y, X) = \phi h(\phi X, Y)$. (4.6)

Now, replacing $Y by \phi Y$ in (4.6), we get

$h(X, Y) + h(\phi Y, \phi X) = \phi h(X, \phi Y)$.(4.7)

Thus from (4.6) and (4.7), we find

$h(X, \phi Y) = h(Y, \phi X)$.

Hence the Theorem is proved. □

Theorem 4.3. Let$M$ be aCR-submanifold of a nearly hyperbolic cosymplecticmanifold$\overline{M}$. If the distribution $D^{\perp}$ is parallel with respect to the connection on M, then

$$A_{\phi Y} Z + A_{\phi Z} Y \in D^{\perp}$$
$$for any Y, Z \in D^{\perp}.$$

Proof. Let $Y, Z \in D^{\perp}$, then using (3.1) and (3.2), we have

$-A_{\phi Z} Y - A_{\phi Y} Z + \nabla_Y^{\perp} \phi Z + \nabla_Z^{\perp} \phi Y = \phi(\nabla_Y Z) + \phi \nabla_Z Y + 2\phi h(Y, Z)$.(4.8)

Taking inner product with $X \in D in (4.8)$, we get

$g(A_{\phi Y} Z + A_{\phi Z} Y) = 0$

which is equivalent to

$(A_{\phi Y} Z + A_{\phi Z} Y) \in D^{\perp}$

$$for any Y, Z \in D^{\perp}.$$

Definition 4.4.A CR-submanifold is said to be mixed-totally geodesic if$h(X, Z) = 0$ $for all X \in D and Z \in D^{\perp}$.

Lemma 4.5. Let M be a CR-submanifold of a nearly hyperbolic cosymplecticmanifold $\overline{M}$. Then $M$is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.

Definition 4.6.A Normal vector field $N \neq 0$ is called$D - parallel$ normal section if $\nabla_X^{\perp} N = 0$for all $X \in D$.

Theorem 4.7. Let M be a mixed totally geodesic CR-submanifold of a nearly hyperbolic cosymplecticmanifold$\overline{M}$. Then the normal section$N \in \phi D^{\perp}$ is $D - parallel$ if and only if $\nabla_X \phi N \in D for all X \in D$.

Proof. Let $N \in \phi D^{\perp}$, then from (3.7), we have

$Q \nabla_Y \phi X = 0$.

In particular, we have $Q \nabla_Y \phi X = 0$. Using it in (3.8), we have

$\phi Q \nabla_X \phi N = \nabla_X^{\perp} N$. (4.9)

Thus, if the normal section $N \neq 0$ is D-parallel, then using ‘definition 4.6’and (4.9), we get

$$\phi\nabla_X(\phi N) = 0$$

which is equivalent to $\nabla_X(\phi N) \in 0 for all X \in D$.

The converse part easily follows from (4.9). This completes the proof of the theorem.

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